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On Fuzzy Gamma Hypermodules

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Abstract

Let R be a Γ-hyperring and M be an Γ-hypermodule over R. We introduce and study fuzzy R_Γ-hypermodules. Also, we associate a Γ-hypermodule to every fuzzy Γ-hypermodule and investigate its basic properties.

Key words: Γ-hyperring, Γ-hypermodule, fundamental relation, fuzzy Γ-hypermodule.

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1 Introduction

Hyperstructure theory was born in 1934 when Marty [13] defined hypergroups, began to analysis their properties and applied them to groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. Zadeh [18] introduced the notion of a fuzzy subset of a non-empty set X, as a function from X to $[0, 1]$. Rosenfeld [15] defined the concept of fuzzy group. Since then many papers have been published in the field of fuzzy algebra. In [16], Sen, Ameri and Chowdhury introduced the notions of fuzzy hypersemigroups and obtained a characterization of them. Then in [10], Leoreanu-Fotea and Davvaz introduced and analyzed the fuzzy hyperring notion and in [11], Leoreanu-Fotea introduced the fuzzy hypermodule notion and obtained a connection between hypermodules and fuzzy hypermodules (for more information about fuzzy hypersrtuctures see [1]-[6]). The notion of a Γ-ring was introduced by N. Nobusawa in [14]. Recently, W.E. Barnes [7], J. Luh [12], W.E. Coppage studied the structure of Γ-rings and obtained various generalization analogous of corresponding parts in ring theory. In [3] Ameri, Sadeghi introduced the notion of Γ-module over a Γ-ring.

Now in this paper we introduced and study fuzzy Γ-hypermodules as generalization of Γ-hypermodule as well as fuzzy modules. The paper has been prepared in 5 sections. In section 2, we introduce some definitions and results of Γ-hypermodules and fuzzy sets which we need to developing our paper. In section 3, we introduced and study fuzzy Γ-hypermodules and obtain its basic results. In section 4, we study fundamental relation of fuzzy Γ-hypermodules.

2 Preliminaries

In this section, we present some definitions which need to developing our paper. As it is well known a *hypergroupoid* is a set together with a function $\circ : H \times H \longrightarrow P^{\star}(H)$, which is called a hyperoperation, where $P^{\star}(H)$ denotes the set of all nonempty subsets of H. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in H$ is called a *semihypergroup*. A *hypergroup* is a semihypergroup such that for all $x \in H$ we have $x \circ H = H = H \circ x$ (called the *reproduction axiom*). We say that a hypergroup H is *canonical hypergroup* if it is *commutative*, it has a *scalar identity*, every element has a unique *inverse* and it is *reversible* (for more details of hypergroups see [9]).

Definition 2.1. *The triple $(R, +, .)$ is a hyperring (in the sense of Krasner) if the following hold: (i) $(R, +)$ is a commutative hypergroup;*
(ii) $(R, .)$ is a semihypergroup;
(iii) the hyperoperation "." is distributive over the hyperoperation "+", which means that for all r, s, t of R we have: $r.(s + t) = r.s + r.t$ and $(r + s).t = r.t + s.t$ (for more about hyperrings see [9] and [11]).

Definition 2.2. *Let $(R, \uplus, \circ)$ be a hyperring. A nonempty set M, endowed with two hyperoperations $\oplus, \odot$ is called a left hypermodule over $(R, \uplus, \circ)$ if the following conditions hold:*
(1) $(M, \oplus)$ *is a commutative hypergroup;*
(2) $\odot : R \times M \longrightarrow P^{*}(M)$ *is such that for all $a, b \in M$ and $r, s \in R$ we have*
(i) $r \odot (a \oplus b) = (r \odot a) \oplus (r \odot b)$;
(ii) $(r \uplus s) \odot a = (r \odot a) \oplus (s \odot a)$;
(iii) $(r \circ s) \odot a = r \odot (s \odot a)$.

For more details about hypermodules see [8], [9], [?] and [18]).

Definition 2.3. *([7]) Let R and Γ be additive abelian groups. We say that R is a $\Gamma-$ring if there exists a mapping*

$$\cdot : R \times \Gamma \times R \longrightarrow R$$
$$(r, \gamma, r') \longmapsto \quad r.\gamma.r' \ (= r\gamma r')$$

such that for every $a, b, c \in R$ and $\alpha, \beta \in \Gamma$, the following conditions hold:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$;
$a(\alpha+\beta)c = a\alpha c + a\beta c$;
$a\alpha(b+c) = a\alpha b + a\alpha c$;
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Definition 2.4. *Let R be a Γ-ring. A (left)gamma module over R is an additive abelian group M together with a mapping $. : R \times \Gamma \times M \longrightarrow M$ (the image of (r, γ, m) being denoted by $r\gamma m$), such that for all $m, m_1, m_2 \in M$ and $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $r, r_1, r_2 \in R$ the following conditions are satisfied:*
(GM_1) $r.\gamma.(m_1 + m_2) = r.\gamma.m_1 + r.\gamma.m_2$;
(GM_2) $(r_1 + r_2).\gamma.m = r_1.\gamma.m + r_2.\gamma.m$;
(GM_3) $r.(\gamma_1 + \gamma_2).m = r.\gamma_1.m + r.\gamma_2.m$;
(GM_4) $r_1.\gamma_1.(r_2.\gamma_2.m) = (r_1.\gamma_1.r_2).\gamma_2.m$.
A right gamma module over R is defined in analogous manner. In this case we say that M is a left(or right) R_Γ-module (for more details about gamma modules see [2]).
Let $(H, \circ)$ be a hypergroupoid. If $\{A, B\} \subseteq P^*(H)$ and ρ is an equivalence relation on H, then we denote $A\bar{\rho}B$ if

$$\forall a \in A, \quad \exists b \in B : a\rho b, \quad and, \quad \forall b \in B, \quad \exists a \in A : a\rho b.$$

We denote $A \ \bar{\bar{\rho}} \ B$ if $\forall a \in A, \ \ \forall b \in B$ we have $a\rho b$.

An equivalence relation ρ on H is called regular (strongly regular) if for all a, a', b, b' of H. The following implication holds:

$$a\rho b, a'\rho b' \implies (a \circ a')\bar{\rho}(b \circ b')$$

$$(a\rho b, a'\rho b' \implies (a \circ a')\bar{\bar{\rho}}(b \circ b')).$$

Theorem 2.1. *([17]) Let $(M, +, .)$ be a hypermodule over a hyperring R, let δ be an equivalence relation on M and let ρ be an strongly regular relation on R. The following statements hold:*
(1) *if δ is strongly regular on M and $\forall x, y \in M$ and $\forall r \in R$ the hyperoperations:*

$$\delta(x) \oplus \delta(y) = \{\delta(z) \mid z \in x + y\} \quad and \quad \rho(r) \odot \delta(x) = \{\delta(z) \mid z \in r.x\},$$

is define a module structure on M/δ over R/ρ;
(2) *if $(M/\delta, \oplus, \odot)$ is a module over R/ρ, then δ is strongly regular on M. The relation δ^* is the smallest strongly regular relation on the hypermodule $(M, +, .)$ such that $(M/\delta, \oplus, \odot)$ the quotient structure $(M/\delta, \oplus, \odot)$ is a module over the ring R/ρ, and it is called the fundamental relation over hypermodule M.*

Hence, δ^* is the smallest equivalence relation on M, such that M/δ^* is a module over the ring R/ρ^*, where ρ^* is fundamental relation on R. If we denote by $\mathcal{U}$ the set of all expressions consisting of finite hyperoperations either on R and M or the external hyperoperation applied on finite sets of elements of R and M, then we have

$$x\delta y \Longleftrightarrow \exists u \in \mathcal{U}, \text{ such that } \{x, y\} \subset u.$$

δ^* is the transitive closure of δ. In the fundamental module $(M/\delta^*, \oplus, \odot)$ over R/ρ^*, the hyperoperations $\oplus$ and $\odot$ are defined as follows:
$\forall x, y \in M$ and $\forall z \in \delta^*(x) \oplus \delta^*(y)$, we have $\delta^*(x) \odot \delta^*(y) = \delta^*(z)$; $\forall r \in R$, $\forall x \in M$ and $\forall z \in \delta^*(r).\delta^*(x)$, we have $\rho^*(r) \odot \delta^*(x) = \delta^*(z)$, (for more details about the fundamental relation on hyperstructures see [8] and [9]).

Definition 2.5. *A multivalued system $(R, +, .)$ is a Γ-hyperring if the following hold:*
(i) $(R, +)$ and Γ are canonical hypergroups;
(ii) $(R, .)$ is semihypergroup.
(iii) $(.)$ is distributive with respect to $(+)$, i.e., for all x, y, z in R we have $x.(y + z) = (x.y) + (x.z)$ and $(x + y).z = (x.z) + (y + z)$.

Definition 2.6. *Let $(R, \uplus, \circ)$ be a Γ-hyperring and $(\Gamma, *)$ be a canonical hypergroup. We say that $(M, +, .)$ is a left $\Gamma -$ hypermodule over R, if $(M, +)$ be a canonical hypergroup and there exists a mapping*

$$\begin{aligned} \cdot : R \times \Gamma \times M &\longrightarrow P^\star(M) \\ (r, \gamma, m) &\longmapsto \quad r \cdot \gamma \cdot m \end{aligned}$$

such that for every $r, s \in R$ and $\alpha, \beta \in \Gamma$ and $a, b \in M$, the following conditions are satisfied:
(GHM_1) *(i)* $(r \uplus s).\alpha.a = r.\alpha.a + s.\alpha.a$;
(ii) $r.(\alpha * \beta).a = r.\alpha.a + r.\beta.a$;
(iii) $r.\alpha.(a + b) = r.\alpha.a + r.\alpha.b$;
(GHM_2) $(r \circ \alpha \circ s).\beta.a = r.\alpha.(s.\beta.a)$.

A right Γ-hypermodule of R is defined in a similar way. In this case we say that M is a R_Γ-hypermodule.

3 Fuzzy Gamma Subhypermodules

In the sequel R is a Γ-hyperring and all gamma hypermodules are considered over R. In [16] M.K. Sen, R. Ameri, G. Chowdhury introduced the notion of fuzzy semihypergroups, in [10] V. Leoreanu-Fotea, B. Davvaz study fuzzy hyperrings and V. Leoreanu-Fotea in [11] studied fuzzy hypermodules. Now in this section we follows these and introduce and studied fuzzy gamma hypermodules.

Let S and Γ be two nonempty sets. $F^*(S)$ denotes the set H of all nonzero fuzzy subset of S. A *Fuzzy $\Gamma-$ hyperoperation* on S is a map $\circ : S \times \Gamma \times S \longrightarrow F^*(S)$, which associates a nonzero subset $a \circ \gamma \circ b$ for all $a, b \in S$ and $\gamma \in \Gamma$. $(S, \circ)$ is called a *Fuzzy $\Gamma-$ hypergroupoid*.

A fuzzy Γ-*hypergroupoid* $(S, \circ)$ is called a *fuzzy* Γ-*hypersemigroup* if for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, we have $a \circ \alpha \circ (b \circ \beta \circ c) = (a \circ \alpha \circ b) \circ \beta \circ c$, where for any $\mu \in F^*(S)$, we have $(a \circ \gamma \circ \mu)(r) = \bigvee_{t \in S}((a \circ \gamma \circ t)(r) \wedge \mu(t))$ and $(\mu \circ \gamma \circ a)(r) = \bigvee_{t \in S}(\mu(t) \wedge (t \circ \gamma \circ a)(r))$ for all $r \in S, \gamma \in \Gamma$.

If A is a nonempty subset of S and $x \in S$, then for all $r \in S, \gamma \in \Gamma$ we have:

$$(x \circ \gamma \circ A)(r) = \bigvee_{a \in A}(x \circ \gamma \circ a)(r),$$

and

$$(A \circ \gamma \circ x)(r) = \bigvee_{a \in A}(a \circ \gamma \circ x)(r).$$

A fuzzy Γ-*hypersemigroup* $(S, \circ)$ is called a *fuzzy* Γ-*hypergroup* if for all $a \in S$ and $\gamma \in \Gamma$, we have $a \circ \gamma \circ S = S \circ \gamma \circ a = \chi_S$. We say that an element e of $(S, \circ)$ is *identity (resp. scalar identity)* if for all $s, r \in S, \gamma \in \Gamma$, we have

$$(e \circ \gamma \circ r)(r) > 0, \quad \text{and} \quad (r \circ \gamma \circ e)(r) > 0,$$

$$((e \circ \gamma \circ r)(s) > 0, \text{ and } \ (r \circ \gamma \circ e)(s) > 0 \ \ it follows r = s).$$

Let $(S, \circ)$ be a fuzzy hypergroup, endowed with at least an identity. An element $a' \in S$ is called an *inverse* of $a \in S$ if there is an identity $e \in S$, such that

$$(a \circ a')(e) > 0, \text{ and } (a' \circ a)(e) > 0.$$

Definition 3.1. *A fuzzy hypergroup S is regular if it has at least one identity and each element has at least one inverse.*
A regular fuzzy hypergroup $(S, \circ)$ is called reversible if for any $x, y, a \in S$, it satisfies the following conditions:
(1) *if $(a \circ x)(y) > 0$, then there exists an inverse a_1 of a, such that $(a_1 \circ y)(x) > 0$;*
(2) *if $(x \circ a)(y) > 0$, then there exists an inverse a_2 of a, such that $(y \circ a_2)(x) > 0$.*

Definition 3.2. *We say that a fuzzy hypergroup S is a fuzzy canonical if*
(1) *it is commutative;*
(2) *it has an scalar identity;*
(3) *every element has a unique inverse;*
(4) *it is reversible.*
Let μ and ν be two nonzero fuzzy subsets of a fuzzy Γ-hypergroupoid $(S, \circ)$. We define

$$(\mu \circ \gamma \circ \nu)(t) = \bigvee_{p,q \in S} (\mu(p) \wedge (p \circ \gamma \circ q)(t) \wedge \nu(q), \forall t \in S, \gamma \in \Gamma.$$

In the following we introduce and study *fuzzy gamma hyperrings* .

Definition 3.3. *Let R, Γ be two nonempty sets and $\boxplus, \boxdot$ be two fuzzy hyperoperations on R and $\otimes$ be a fuzzy hyperoperation on Γ. Let $(R, \boxplus)$ and $(\Gamma, \otimes)$ be two canonical fuzzy hypergroups. R is called a fuzzy Γ-hyperring if there exists the mapping:*

$$\begin{gathered} \boxdot : R \times \Gamma \times R \longrightarrow F^*(R) \\ (r, \gamma, s) \longmapsto r \boxdot \gamma \boxdot s, \end{gathered}$$

such that for all $r, s, t \in R, \alpha, \beta \in \Gamma$, the following conditions are satisfied:
(*i*) $r \boxdot \alpha \boxdot (s \boxplus t) = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \alpha \boxdot t)$*;*
(*ii*) $r \boxdot (\alpha \otimes \beta) \boxdot s = (r \boxdot \alpha \boxdot s) \boxplus (r \boxdot \beta \boxdot s)$;
(*iii*) $(r \boxplus s) \boxdot \alpha \boxdot t = (r \boxdot \alpha \boxdot t) \boxplus (s \boxdot \alpha \boxdot t)$;
(*iv*) $r \boxdot \alpha \boxdot (s \boxdot \beta \boxdot t) = (r \boxdot \alpha \boxdot s) \boxdot \beta \boxdot t$.

Definition 3.4. *Let $(\Gamma, \otimes)$ be a fuzzy canonical hypergroups. Let $(R, \boxplus, \boxdot)$ be a fuzzy Γ-hyperring. A nonempty set M, endowed with two fuzzy Γ-hyperoperation $\oplus, \odot$ is called a left fuzzy Γ-hypermodule over $(R, \boxplus . \boxdot)$ if the following conditions hold:*

(1) $(M, \oplus)$ is a canonical fuzzy Γ-hypergroup;
(2) $\odot : R \times \Gamma \times M \longrightarrow F^*(M)$ is such that for all $a, b \in M, r, s \in R$ and $\alpha, \beta \in \Gamma$ we have
(i) $r \odot \alpha \odot (a \oplus b) = (r \odot \alpha \odot a) \oplus (r \odot \alpha \odot b)$;
(ii) $(r \boxplus s) \odot \alpha \odot a = (r \odot \alpha \odot a) \oplus (s \odot \alpha \odot a)$;
(iii) $r \odot (\alpha \otimes \beta) \odot a = (r \odot \alpha \odot a) \oplus (r \odot \beta \odot a)$;
(iv) $r \odot \alpha \odot (s \odot \beta \odot a) = (r \cdot \alpha \cdot s) \odot \beta \odot a$.
If both $(R, \boxplus), (\Gamma, \otimes)$ and $(M, \oplus)$ have scaler identities, denoted by $0_R, 0_\Gamma$ and 0_M, then the fuzzy Γ-hypermodule $(M, \oplus, \odot)$ also satisfies the condition:

$$0_R \odot \gamma \odot a = \chi_{0_M},$$

$$r \odot 0_\Gamma \odot a = \chi_{0_\Gamma},$$

$$r \odot \gamma \odot 0_M = \chi_{0_M},$$

for all $r \in R, \gamma \in \Gamma, a \in A$. Moreover, if $(R, \boxdot)$ has an identity, say 1, then the fuzzy Γ-hypermodule $(M, \oplus, \odot)$ is called unitary if it satisfies the condition:
for all a of M, we have $1 \odot \gamma \odot a = \chi_a$.

Clearly, any fuzzy Γ-hyperring is a fuzzy Γ-hypermodule over itself.
Proposition 3.5. Let $(M, +, .)$ be a module over a ring $(R, \uplus, \circ)$ and $\Gamma = R$. We define the following fuzzy Γ-hyperoperations:
for a, b of M, $a \oplus b = \chi_{\{a,b\}}$,
for all a of M and $r \in R, \gamma \in \Gamma$, $r \odot \gamma \odot a = \chi_{\{r.\gamma.a\}}$,
for all r, s of R, $r \boxplus s = \chi_{\{r,s\}}$ and $r \boxdot \gamma \boxdot s = \chi_{\{r \circ \gamma \circ s\}}$.
Then $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over the fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$.
Note that the last theorem is satisfied, when M is a Γ-module over a Γ-ring R, such that $\Gamma \neq R$.
Proposition 3.6. Let $(R, \circ)$ and $(S, \bullet)$ be two fuzzy Γ-hyperrings. Let $(M, \oplus, \odot)$ be a left fuzzy Γ-hypermodule over R and a right fuzzy Γ-hypermodule over S. Then

$A = \{\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \mid r \in R, s \in S, m \in M\}$ is a fuzzy Γ-hyperring and fuzzy Γ-hypermodule over A, under the mappings

$$\star : A \times \Gamma \times A \longrightarrow F^*(A)$$
$$(\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}, \gamma, \begin{pmatrix} r_1 & m_1 \\ 0 & s_1 \end{pmatrix}) \longmapsto$$
$$\begin{pmatrix} r \circ \gamma \circ r_1 & r \odot \gamma \odot m_1 \oplus m \odot \gamma \odot s_1 \\ 0 & s \bullet \gamma \bullet s_1 \end{pmatrix}.$$

such that

$$\begin{pmatrix} r\circ\gamma\circ r_1 & r\odot\gamma\odot m_1\oplus m\odot\gamma\odot s_1 \\ 0 & s\bullet\gamma\bullet s_1 \end{pmatrix}\begin{pmatrix} r_2 & m_2 \\ 0 & s_2 \end{pmatrix} =$$
$$\begin{pmatrix} (r\circ\gamma\circ r_1)(r_2) & (r\odot\gamma\odot m_1\oplus m\odot\gamma\odot s_1)(m_2) \\ 0 & (s\bullet\gamma\bullet s_1)(s_2) \end{pmatrix} =$$
$$\begin{cases} 1, & r_2, m_2, s_2 \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$
.

Proof. Straightforward.□

Example 3.7. Let R be a Γ-ring and $(M,+,.)$ a Γ-module. Consider the mapping $\alpha \quad : \quad M \longrightarrow \quad R$. Then M is an fuzzy Γ-hypermodule over M, under the following operations:

$$m \oplus n = m+n. \text{ and } \circ : M\times\Gamma\times M \longrightarrow F^*(M)(m,\gamma,n) \longmapsto m\circ\gamma\circ n = \chi_{\alpha(m).\gamma.n},$$

for all $m, n \in M, \gamma \in \Gamma$.

Proposition 3.8. Let $(M,+,.)$ be a Γ-module over Γ-ring R and ν be a nonzero fuzzy Γ-semigroup on M. Let μ and ρ be two nonzero fuzzy Γ-semigroups on R. For $r \in R$, $a, b \in M$ and $\gamma \in \Gamma$, define a fuzzy Γ-hyperoperation $\odot$ on M by

$$(r\odot\gamma\odot a)(t) = \begin{cases} \mu(r)\wedge\rho(\gamma)\wedge\nu(a), & if\ t = r.\gamma.a \\ 0 \qquad , & otherwise. \end{cases}$$

Also, $a \oplus b = \chi_{\{a+b\}}$. It is easy to verify that $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule.

Let S, Γ be nonempty sets, and S endowed with a fuzzy Γ-hyperoperation $\circ$. For all $a, b \in S, \gamma \in \Gamma$ and $p \in [0,1]$ consider the p-cuts:

$$(a\circ\gamma\circ b)_p = \{t \in S : \ (a\circ\gamma\circ b)(t) \geq p\}$$

of $a\circ\gamma\circ b$, where $p \in [0,1]$.

For all $p \in [0,1]$, we define the following crisp Γ-hyperoperation on S:

$$a \circ_p \gamma \circ_p b = (a\circ\gamma\circ b)_p.$$

Example 3.9. Let $R = \Gamma = \mathbb{Z}$ and $M = \mathbb{Z}_n$ for $n \in \mathbb{N}$. Define following fuzzy Γ-hyperoperations for all $a, b \in M, \gamma \in \Gamma$:

$$a \oplus b = \chi_{\{a,b\}}, \forall a \in M, \forall r \in R, \gamma \in \Gamma,$$

$$r \odot \gamma \odot a = \chi_{\{\overline{r\gamma a}\}}, \quad \forall r, s \in R, \forall \gamma \in \Gamma,$$

$$r.\gamma.s = \chi_{\{\overline{r\gamma s}\}} \quad \text{and} \quad r + s = \chi_{\{r,s\}}, \quad \text{for all} \quad \alpha, \beta \in \Gamma,$$

and

$$\alpha \boxplus \beta = \chi_{\{\alpha,\beta\}},$$

such that $\overline{x}$ is denote a typical element in $\mathbb{Z}_n$. Then it is easy to verify that $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over fuzzy Γ-hyperring R and canonical fuzzy hypergroup $(\Gamma, \boxplus)$.

Proposition 3.10. Let $(M, \circ)$ be a fuzzy Γ-hyperoperation. For all $a, b, c, u \in M$ and $\alpha, \beta \in \Gamma$ and for all $p \in [0, 1]$ the following equivalence holds:

$$(a \circ \alpha \circ (b \circ \beta \circ c)) \geq p \Longleftrightarrow u \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c).$$
$$((a \circ \alpha \circ b) \circ \beta \circ c) \geq p \Longleftrightarrow u \in (a \circ_p \alpha \circ_p b) \circ_p \beta \circ_p c.)$$

Proof. Clearly,

$$(a \circ \alpha \circ (b \circ \beta \circ c))(u) = \bigvee_{t \in M} (a \circ \alpha \circ t)(u) \wedge (b \circ \beta \circ c)(t) \geq p,$$

if and only if there exists $t_0 \in M$, such that $(a \circ \alpha \circ t_0)(u) \geq p$ and $(b \circ \beta \circ c)(t_0) \geq p$, which means that $u \in a \circ_p \alpha \circ_p t_0, t_0 \in b \circ_p \beta \circ_p c$. Therefore, $u \in a \circ_p \alpha \circ_p (b \circ_p \beta \circ_p c)$.□

Proposition 3.11. Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$. Then for all $a \in M, r \in R, \gamma \in \Gamma$, conditions are equivalence:

(1) $a \oplus M = \chi_M \Longleftrightarrow \forall p \in [0, 1],\ a \oplus_P M = M$;

(2) $r \odot \gamma \odot M = \chi_M \Longleftrightarrow \forall p \in [0, 1],\ r \odot_p \gamma \odot_p M = M$.

Proof. We only proof (2). Let $r \odot \gamma \odot M = \chi_M$. Then for all $t \in M$ and $p \in [0, 1]$, we have $\bigvee_{u \in M} (r \odot \gamma \odot u)(t) = 1 \geq p$, whence there exists $m \in M$, such that $(r \odot \gamma \odot m)(t) \geq p$, which means that $t \in r \odot_p \gamma \odot_p m$. Hence, $\forall p \in [0, 1],\ r \odot_p \gamma \odot_p M = M$. Conversely, for $p = 1$ we have $r \odot_1 \gamma \odot_1 M = M$, whence for all $t \in M$, there exists $u \in M$, such that $t \in r \odot_1 \gamma \odot_1 u$, which means that $(r \odot \gamma \odot u)(t) = 1$. In other words, $r \odot \gamma \odot M = \chi_M$.□

Proposition 3.12. The structure $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ if and only if $\forall p \in [0, 1]$, $(M, \oplus_p, \odot_p)$ is a Γ-hypermodule over the hyperring $(R, \boxplus_p, \boxdot_p)$.

Proof. It is straightforward.□

Consider $(M, \oplus, \odot)$ as a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, \otimes)$. Now we follow [8], and define a new types of Γ-hyperoperations on M, R, Γ, as follows:

$$\forall a, b \in M, \ a + b = \{x \in M | (a \oplus b)(x) > 0\}, \quad \forall r, s \in R,$$

$$r \uplus s = \{t \in R \mid (r \boxplus s)(t) > 0\}, forall \alpha, \beta \in \Gamma,$$

$$\alpha * \beta = \{\gamma \in \Gamma \mid (\alpha * \beta)(\gamma) > 0\}, \quad \forall a \in M, \quad \forall r \in R, \forall \gamma \in \Gamma,$$

$$r.\gamma.a = \{b \in M \mid (r \odot \gamma \odot a)(b) > 0\}, \quad \forall r, s \in R, \quad \forall \gamma \in \Gamma,$$

$$r \circ \gamma \circ s = \{t \in R \mid (r \boxdot \gamma \boxdot s)(t) > 0\}.$$

Proposition 3.13. If $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, \otimes)$, then $(M, +, .)$ is a Γ-hypermodule over the Γ-hyperring $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, \star)$.

Proof. By [10], it is obtained that $(R, \uplus)$, $(\Gamma, *)$ and $(M, +)$ are canonical hypergroups. It is sufficient to verify $(M, .)$ is a Γ-hypermodule. We consider the following cases:

Case: (i)

$$(r \uplus s).\gamma.a = (r.\gamma.a) + (s.\gamma.a), \quad \text{for all} \quad r, s \in R, \gamma \in \Gamma, a \in M.$$

Suppose that $x \in (r \uplus s).\gamma.a = \bigcup_{y \in r \uplus s} \ y \odot \gamma \odot a$. Then $(y \odot \gamma \odot a)(x) > 0$ and $(r \boxplus s)(y) > 0$, for some $y \in r \uplus s$, and hence $\vee_{p \in M} \ ((r \boxplus s)(p) \wedge (p \odot \gamma \odot a)(x) > 0$. Thus $((r \boxplus s) \odot \gamma \odot a)(x) > 0$, which implies that $((r \odot \gamma \odot a) \oplus (s \odot \gamma \odot a))(x) > 0$. Thus there exist $z, t \in M$, such that $(z \oplus t)(x) > 0$, $(r \odot \gamma \odot a)(z) > 0$ and $(s \odot \gamma \odot a)(t) > 0$ i.e., $x \in z + t, z \in r.\gamma.a$ and $t \in s.\gamma.a$ and hence $x \in (r.\gamma.a) + (s.\gamma.a)$. Therefore, $(r \uplus s).\gamma.a \subseteq (r.\gamma.a) + (s.\gamma.a)$. Similarly, we can show that $(r.\gamma.a) + (s.\gamma.a)t \subseteq (r \uplus s).\gamma.a$. Therefore, $(r \uplus s).\gamma.a = (r.\gamma.a) + (s.\gamma.a)$. The other conditions are verified similarly and omitted. □

On the other hands, if $(M,+,.)$ is a Γ-hypermodule over a Γ-hyperring $(R,\uplus,\circ)$, then we define the following fuzzy Γ-hyperoperations:

$$\begin{aligned} a\oplus b &= \chi_{\{a+b\}}, \forall a,b\in M, r\boxplus s\\ &= \chi_{\{r\uplus s\}}, \forall r,s\in R, \gamma\in\Gamma, r\odot\gamma\odot a\\ &= \chi_{\{r.\gamma.a\}}, \forall a\in M, r\in R, r\boxdot\gamma\boxdot s\\ &= \chi_{\{r\circ\gamma\circ s\}}, \forall r,s\in R, \forall\gamma\in\Gamma, \beta\\ &= \chi_{\{\alpha*\beta\}}\forall\alpha,\beta\in\Gamma, \alpha\otimes\beta. \end{aligned}$$

The next result is immediately follows from above discussion and [14].

Proposition 3.14. For every hypergroup $(M,+)$, there is an associated fuzzy hypergroup.

Proposition 3.15. Let $(M,+,.)$ be a Γ-hypermodule over a Γ-hyperring. Let $(R,\uplus,\circ)$ be a canonical hypergroup $(\Gamma,\star)$. Then $(M,\oplus,\odot)$ is a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R,\boxplus,\boxdot)$ and canonical fuzzy hypergroup $(\Gamma,\otimes)$, where the fuzzy hyperoperations $\oplus,\odot,\boxplus,\boxdot$ and $\otimes$ are defined above.

Proof. By Proposition 3.14, $(M,\oplus)$ is a commutative fuzzy Γ-hypergroup. We show that $(M,\oplus)$ is canonical. Since $(M,+)$ is canonical Γ-hypergroup, then there exists $e\in M, \forall a\in M,\ a=e+a=a+e \implies (e\oplus a)(a)=\chi_{\{e+a\}}(a)>0, (a\oplus e)(a)=\chi_{\{e+a\}}(a)>0$ and because for all $a\in M$ there exists $b\in M$, such that $e\in a+b\cap b+a$, $b)$ is the inverse of a with respect to $+$). Then

$$(a\oplus b)(e)=\chi_{\{a+b\}}(e)=\chi_{\{b+a\}}(e)=(b\oplus a)(e)>0.$$

Let $(a\oplus x)(y)=\chi_{\{a+x\}}(y)>0 \implies y\in a+x \implies \exists\, b$ (the inverse of a such that $x\in b+y \implies (b\oplus y)(x)=\chi_{\{b+y\}}(x)>0$. The other cases is can be proved in a similar way and omitted. Then $(M,\oplus)$ is a canonical fuzzy Γ-hypergroup. Now, we show that $(M,\oplus,\odot)$ is a fuzzy Γ-hypermodule. We investigate only the condition (iv) of Definition 3.4.

First , we show that that for all $r,s\in R,\alpha,\beta\in\Gamma, a\in M$, we have

$$(r\odot\alpha\odot(s\odot\beta\odot a))=(r\boxdot\alpha\boxdot s)\odot\beta\odot a,\quad \forall t\in M.$$

Then

$$(r\odot\alpha\odot(s\odot\beta\odot a))(t)=\bigvee_{p\in M},$$

$$[(r\odot\alpha\odot p)(t)\wedge(s\odot\beta\odot a)(p)]=\bigvee_{p\in M}\ [\chi_{r.\alpha.p}(t)\wedge\chi_{s.\beta a}(p)]=$$

$\left\{\begin{array}{ll} 1, & t \in r.\alpha.(s.\beta.a) \\ 0, & otherwise \end{array}\right. = \left\{\begin{array}{ll} 1, & t \in (r.\alpha.s).\beta.a \\ 0, & otherwise \end{array}\right.$
$= ((r \boxdot \alpha \boxdot s) \odot \beta \odot a)(t)$, for all $t \in M$.

It is easy to verify that the other conditions of Definition 3.4 can be obtained in a similar way.□

Proposition 3.16. Let M an R_Γ-module and μ be a fuzzy Γ-module of M. Then the set M will be a fuzzy Γ-hypermodule.

Proof. Let $(\Gamma, *)$ be an abelian group and $(M, +, .)$ be a Γ-module over Γ-ring $(R, \uplus, \circ)$. We define fuzzy Γ-hyperoperations on M as follows:

$$(a \oplus b)(t) = \chi_{\{a+b\}},\ (r \odot \gamma \odot a)(t) = \mu(r.\gamma.a - t),$$

$$(\alpha \otimes \beta)(\gamma) = \chi_{\{\alpha*\beta\}}(\gamma) = \chi_{\{r\uplus s\}} r \boxplus s)(z)(r \boxdot \alpha \boxdot s)(z) = \chi_{\{r\circ\alpha\circ s\}}(z),$$

$\forall a, b, t \in M, r, s, z \in R, \alpha, \beta, \gamma \in \Gamma$.

It is easy to verify that $(M, \oplus)$ is a canonical fuzzy hypergroup. Now, we show $(M, \oplus, \odot)$ is a fuzzy Γ-hypermodule with $\mu(0) = 1$.

(i)

$$\begin{array}{rcl} ((r \boxplus s) \odot \gamma \odot a)(t) & = & \vee_{p\in R}(r \boxplus s)(p) \wedge (p \odot \gamma \odot a)(t) \\ & = & \vee_{p\in R}\chi_{r\uplus s}(p) \wedge \mu(p.\gamma.a - t) \\ & = & \mu((r \uplus s).\gamma.a - t) \quad \text{if} \quad p = r \uplus s. \end{array}$$

Also, $((r \odot \gamma \odot a) \oplus (s \odot \gamma \odot a))(t) =$

$$\begin{array}{cl} = & \vee_{p,q\in M}(r \odot \gamma \odot a)(p) \wedge (p \oplus q)(t) \wedge (s \odot \gamma \odot a)(q) \\ = & \vee_{p,q\in M}\mu(r.\gamma.a - p) \wedge \chi_{\{p+q\}}(t) \wedge \mu(s.\gamma.a - q) \\ = & \vee_{p,q\in M,t=p+q}\mu(r.\gamma.a - p) \wedge \mu(s.\gamma.a - q) \\ \leq & \mu(r.\gamma.a - p + s.\gamma.a - q) \\ = & \mu((r \uplus s).\gamma.a - (p + q)), \end{array}$$

On the other hands, if $q = s.\gamma.a,\ p = t - s.\gamma.a.$, then

$$\begin{array}{rcl} \vee_{p,q\in M,t=p+q}\mu(r.\gamma.a - p) \wedge \mu(r.\gamma.a - q) & \geq & \vee_{p\in M}\mu(r.\gamma.a - p) \\ & \geq & \mu(r.\gamma.a - t + s.\gamma.a) \\ & = & \mu((r \uplus s).\gamma.q - t). \end{array}$$

(ii)

$$\begin{array}{rcl} (r \odot (\alpha \otimes \beta) \odot a)(t) & = & \vee_{\gamma\in\Gamma}[(r \odot \gamma \odot a)(t) \wedge (\alpha \otimes \beta)(\gamma)] \\ & = & \vee\mu(r.\gamma.a - t) \wedge \chi_{\{\alpha*\beta\}}(\gamma) \\ & = & \mu(r.(\alpha * \beta).a - t). \end{array}$$

Also, $((r \odot \alpha \odot a) \oplus (r \odot \beta \odot a))(t) =$

$$\begin{aligned}
&= \vee_{p,q\in M}[(r \odot \alpha \odot a)(p) \wedge (p \oplus q)(t) \wedge (r \odot \beta \odot a)(q)\\
&= \vee_{p,q\in M}[\mu(r.\alpha.a - p) \wedge \chi_{\{p+q\}}(t) \wedge \mu(r.\beta.a - q)]\\
&= \vee_{t=p+q}\ \mu(r.\alpha.a - p) \wedge \mu(r.\beta a - q)\\
&\leq \mu(r.\alpha a - p + r.\beta a - q)\\
&= \mu(r.(\alpha * \beta).a - (p+q)).
\end{aligned}$$

On the other hands, suppose that $q = r.\beta.a$, then for $p = t - r.\beta.a$ we have

$$\begin{aligned}
\vee_{t=p+q}\mu(r.\alpha.a - p) \wedge \mu(r.\beta a - q) &= \vee_{p\in M}\mu(r.\alpha a - p)\\
&\geq \mu(r.\alpha a - (t - r\beta a))\\
&= \mu(r.(\alpha * \beta).a - (p+q)),
\end{aligned}$$

(iii)

$$\begin{aligned}
r \odot \gamma \odot (a \oplus b) &= \vee_{p\in M}(r \odot \gamma \odot p)(t) \wedge (a \oplus b)(p)\\
&= \vee_{p\in M}\mu(r.\gamma.p - t) \wedge \chi_{\{a+b\}}(p)\\
&= \mu(r.\gamma.(a+b) - t) \quad \text{and} \quad ((r \odot \gamma \odot a) \oplus (r \odot \gamma \odot b))(t)\\
&= \vee_{p,q\in M}(r \odot \gamma \odot a)(p) \wedge (p \oplus q)(t) \wedge (r \odot \gamma \odot b)(q)\\
&= \vee_{p,q\in M}\mu(r.\gamma.a - p) \wedge \chi_{\{p+q\}}(t) \wedge \mu(r.\gamma.b - q)\\
&= \vee_{p,q\in M,t=p+q}\mu(r.\gamma.a - p) \wedge \mu(r.\gamma.b - q)\\
&\leq \mu(r.\gamma.a - p + r.\gamma.b - q) = \mu(r.\gamma.(a+b) - t).
\end{aligned}$$

On the other hands, for $q = r.\gamma.b, p = t - r.\gamma.b.$ we have

$$\begin{aligned}
&\vee_{p,q\in M,t=p+q}\mu(r.\gamma.a - p) \wedge \mu(r.\gamma.b - q)\\
&\qquad \geq \vee_{p\in M}\mu(r.\gamma.a - p)\\
&\qquad \geq \mu(r.\gamma(a+b) - t).
\end{aligned}$$

(iv)

$$\begin{aligned}
(r \odot \alpha \odot (s \odot \beta \odot a))(t) &= \vee_{p\in M}(r \odot \alpha \odot p)(t) \wedge (s \odot \beta \odot a)(p)\\
&= \vee_{p\in M}\mu((r.\alpha.p) - t) \wedge \mu((s.\beta.a) - p)\\
&= \mu(r.\alpha.(s.\beta.a) - t), \quad \text{and} \quad ((r \boxdot \alpha \boxdot s) \odot \beta \odot a)(t)\\
&= \vee_{p\in R}(r \odot \alpha \odot s)(p) \wedge (p \odot \beta \odot a)(t)\\
&= \vee_{p\in R}\chi_{\{r\circ\alpha\circ s\}}(p) \wedge \mu(p.\beta.a - t)\\
&= \mu(r \circ \alpha \circ s \cdot (\beta \cdot a) - t) \quad \text{if} \quad p = r \circ \alpha \circ s.
\end{aligned}$$

□

Remark. Let $\mathbb{H} = \langle H, (\beta_i : i \in I) \rangle$ be a fuzzy hyperalgebra. Denote by $F^*(H)$ the set of the nonzero fuzzy subsets of H. Then $\mathbb{H}$ can be organized as a universal algebra under the following operations:

$$\beta_i(\mu_1, ..., \mu_{n_i})(t) = \bigvee_{(x_1,...,x_{n_i}) \in H^{n_i}} [(\mu_1(x_1) \bigwedge ...\mu_{n_i}(x_{n_i}) \bigwedge \beta_i(x_1, ..., x_{n_i})(t))],$$

for every $i \in I, \mu_1, ..., \mu_{n_i} \in F * (H)$ and $t \in H$. We denote this algebra by $F^*(\mathbb{H})$.

Proposition 3.17. If $(M, \oplus, \diamondsuit)$ is a fuzzy Γ-hypermodule, then $(F^*(M), *, \bigcirc)$ is a Γ-module.

Proof. We define operations $*, \diamondsuit$ on $F^*(M)$ by $\mu * \nu = \mu \oplus \nu$, and $r \diamondsuit \gamma \diamondsuit \mu = r \odot \gamma \odot \mu$ for all $\mu, \nu \in F^*(M), r \in R, \gamma \in \Gamma$. It is easy to see that $(F^*(M), *)$ is a group. Clearly, $(F^*(M), \oplus)$ is a semigroup.

(i) Identity: we must prove that there exists a $\nu \in F^*(M)$ such that ,$\mu * \nu = \mu$. We have

$$\begin{array}{rcl}(\mu * \nu)(t) & = & (\mu \oplus \nu)(t) \\ & = & \vee_{p,q \in M} \mu(p) \wedge (p \oplus q)(t) \wedge \nu(q) \\ & = & \vee_{p \in M} \mu(p) \wedge (p \oplus e)(t) \\ & = & \mu(t) \oplus \quad \text{if} \end{array}$$

$$q = e; \nu(q) = 1, p = t.$$

Thus it is enough we choose $\nu = \chi_e$.

(ii) Inverse: it must prove that for $\mu \in F^*(M)$, there exists a $\nu \in F^*(M)$, such that $\mu * \nu = \chi_e$. It is sufficient to consider $\nu = -\mu$, then we have

$$\begin{array}{rcl}(\mu * \nu)(t) & = & (\mu \oplus \nu)(t) \\ & = & \vee_{p,q \in M} \mu(p) \wedge (p \oplus q)(t) \wedge (-\mu)(q) \\ & = & \vee_{p,q \in M} \mu(p) \wedge (p \oplus q)(t) \wedge \mu(-q) \\ & \leq & \mu(p - (-q)) \wedge (p \oplus q)(t) \leq (p \oplus q)(t) \\ & = & \chi_e(t) \;\; \text{where,} \;\; p \;\; \text{is inverse of} \;\; q. \end{array}$$

On the other hands, we have

$$\begin{array}{rcl}\vee_{p,q \in M} \mu(p) \wedge (p \oplus q)(t) \wedge \mu(-q) & \geq & \vee_{p \in M} \mu(p) \wedge (p \oplus -p)(t) \\ & \geq & (p \oplus -p)(t) \\ & = & \chi_e(t). \end{array}$$

Other cases are easy and omitted. □

Definition 3.18. Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$. A nonempty subset N of M is called a *subfuzzy* Γ-hypermodule if for all $x, y \in N, r \in R$ and $\gamma \in \Gamma$, the following conditions hold:

(1) $(x \oplus y)(t) > 0 \quad \Rightarrow \quad t \in N$;

(2) $x \oplus N = \chi_N$;

(3) $(r \odot \gamma \odot x)(t) > 0 \quad \Rightarrow \quad t \in N$.

Proposition 3.19. (i) If $(N, \oplus, \odot)$ is a subfuzzy Γ-hypermodule of $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$, then the associated Γ-hypermodule $(N, +, .)$ is a Γ-hypersubmodule of $(M, +, .)$ over $(R, \uplus, \circ)$;

(ii) $(N, +, .)$ is a Γ-hypersubmodule of $(M, +, .)$ over $(R, \uplus, \circ)$ if and only if $(N, \oplus, \odot)$ is a subfuzzy Γ-hypermodule of $(M, \oplus, \odot)$ over $(R, \boxplus, \boxdot)$.

4 Fundamental Relation of Fuzzy Γ-hypermodule

In [14], fuzzy regular relations are introduced in the context of fuzzy hypersemigroups. In this section we extend this notion to fuzzy Γ-hypermodules. Let ρ be an equivalence relation on a fuzzy Γ-hypersemigroup $(M, \circ)$ and μ, ν be two fuzzy subsets on M. We say that $\mu\rho\nu$ if the following conditions hold:

(1) if $\mu(a) > 0$, then there exists $b \in M$, such that $\nu(b) > 0$ and $a\rho b$ and;

(2) if $\nu(x) > 0$, then there exists $y \in M$, such that $\mu(y) > 0$ and $x\rho y$.

An equivalence relation ρ on a fuzzy Γ-hypersemigroup $(M, \circ)$ is called a *fuzzy regular relation* (or *fuzzy hypercongruence*) on $(M, \circ)$ if, for all $a, b, c \in M, \gamma \in \Gamma$, the following implication holds:

$$a\rho b \Longrightarrow (a \circ \gamma \circ c)\ \rho\ (b \circ \gamma \circ c) \text{ and } (c \circ \gamma \circ a)\ \rho\ (c \circ \gamma \circ b).$$

This condition is equivalent to

$$a\rho a', b\rho b' \quad \Rightarrow \quad (a \circ \gamma \circ b)\rho(a' \circ \gamma \circ b'), \forall a, b, a', b' \in M, \gamma \in \Gamma.$$

Definition 4.1. An equivalence relation ρ on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and a canonical fuzzy hypergroup $(\Gamma, \otimes)$ is called a *fuzzy regular relation* on $(M, \oplus, \odot)$ if it is a fuzzy regular relation on $(M, \oplus)$ and for all $x, y \in M, r \in R, \gamma \in \Gamma$, the following implication holds:

$$x\rho y \Longrightarrow (r \odot \gamma \odot x)\rho(r \odot \gamma \odot y).$$

Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and a canonical fuzzy hypergroup $(\Gamma, \otimes)$. Suppose $(M, +, .)$ is the associated Γ-hypermodule over the Γ-hyperring $(R, \uplus, \circ)$ and the canonical hypergroup $(\Gamma, *)$. Then we have the next result.

Theorem 4.2. An equivalence relation ρ is a fuzzy regular relation on $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, \otimes)$ if and only if ρ is a regular relation on $(M, +, .)$ over the Γ-hyperring $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, *)$.

Proof. Letting $x\rho y$ and $x'\rho y'$, where $x, x', y, y' \in M$. We have $(x \oplus x')\rho(y+y')$ if and only if the following conditions hold:

$$(x \oplus x')(u) > 0, \Rightarrow \exists v \in M : (y \oplus y')(v) > 0 \ \text{ and } \ u\rho v,$$

and

$$(y \oplus y')(t) > 0 \quad \Rightarrow \quad \exists w \in M : (x \oplus x')(w) > 0 \ \text{ and } \ at\rho w.$$

These are equivalent to:
if $u \in x + x'$, then there exists $v \in y + y'$, such that $u\rho v$;
if $t \in y + y'$, then there exists $w \in x + x'$, such that $t\rho w$;
which mean that $(x + x')\bar{\rho}(y + y')$. Hence ρ is fuzzy regular on $(M, \oplus)$ if and only if ρ is regular on $(M, +)$.
On the other hands, if $x\rho y$ and $r \in R, \gamma \in \Gamma$. We have $(r \odot \gamma \odot x)\rho(r \odot \gamma \odot y)$ if and only if the next conditions hold:
if $(r \odot \gamma \odot x)(u) > 0$, then there exists $v \in M$, such that $(r \odot \gamma \odot y)(v) > 0$ and $u\rho v$;
if $(r \odot \gamma \odot y)(t) > 0$, then there exists $w \in M$, such that $(r \odot \gamma \odot x)(w) > 0$ and $t\rho w$.
These are equivalent to:
if $u \in r.\gamma.x$, then there exists $v \in r.\gamma.y$, such that $u\rho v$;
if $t \in r.\gamma.y$, then there exists $w \in r.\gamma.x$, such that $t\rho w$;
which means that $(r.\gamma.x)\rho(r.\gamma.y)$.□

Definition 4.3. An equivalence relation ρ on a fuzzy Γ-hypersemigroup $(M, \circ)$ is called a *fuzzy strongly regular relation* on $(M, \circ)$ if, for all a, a', b, b' of M and for all $\gamma \in \Gamma$, such that $a\rho b$ and $a'\rho b'$, the following condition holds:

$$(a \circ \gamma \circ c)(x) > 0, (b \circ \gamma \circ d)(y) > 0 \quad \Rightarrow \quad x\rho y,$$

for all $x, y \in M$. Note that if ρ is a fuzzy strongly relation on a fuzzy Γ-hypersemigroup $(M, \circ)$, then it is a fuzzy regular on $(M, \circ)$. An equivalence relation ρ on a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ is called a *fuzzy strongly regular* *relation* on $(R, \boxplus, \boxdot)$ if it is a fuzzy strongly regular relation both on $(R, \boxplus)$ and on $(R, \boxdot)$.

Definition 4.4. Let ρ be a fuzzy strongly regular relation on a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and θ be a fuzzy strongly regular relation on a canonical fuzzy Γ-hypergroup $(\Gamma, *)$. An equivalence relation δ on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy Γ-hypergroup $(\Gamma, \otimes)$ is called a *fuzzy strongly regular relation* on $(M, \oplus, \odot)$ if it is a fuzzy strongly regular relation on $(M, \oplus)$ and if $x\delta y$, $r\rho s$ and $\alpha\theta\beta$, then the next condition holds:
for all $u \in M$, such that $(r \odot \alpha \odot x)(u) > 0$ and for all $v \in M$, such that $(s \odot \beta \odot y)(v) > 0$, we have $u\delta v$.

Theorem 4.5. An equivalence relation δ is a fuzzy strongly regular relation on $(M, \oplus, \odot)$ if and only if δ is a strongly regular relation on $(M, +, .)$.

Proof. Set $x\delta y$ and $x'\delta y'$, where $x, x', y, y' \in M$ and set $r\rho s$, where $r, s \in R$ and $\alpha\theta\beta$, where $\alpha, \beta \in \Gamma$. The relation δ is strongly regular on $(M, \oplus, \odot)$ if and only if the following conditions are satisfied:
$\forall u \in M$, such that $(x \oplus x')(u) > 0$ and $\forall v \in M$, such that $(y \oplus y')(v) > 0$, we have $u\delta v$;
$\forall t \in M$, such that $(r\odot\alpha\odot x)(t) > 0$ and $\forall w \in M$, such that $(s\odot\beta\odot y)(w) > 0$, we have $t\delta w$.
These conditions are equivalent to the following ones:
$\forall u \in M$, such that $u \in x + x'$ and $\forall v \in M$, such that $v \in y + y'$, we have $u\delta v$;
$\forall t \in M$, such that $t \in r.\alpha.x$ and $\forall w \in M$, such that $w \in s.\beta.y$, we have $t\delta w$, which mean that $(x + x')\bar{\bar{\delta}}(y + y')$ and $(r.\alpha.x)\bar{\bar{\delta}}(s.\beta.y)$. Hence δ is strongly regular on $(M, \oplus, \odot)$ if and only if δ is strongly regular on $(M, +, .)$.
Now, Let δ be a fuzzy regular relation on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy Γ-hypergroup $(\Gamma, \otimes)$ and ρ, θ be fuzzy strongly regular relations on the Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy Γ-hypergroup. $(\Gamma, \otimes)$.
We consider the following Γ-hyperoperations on the quotient set M/δ:

$$\bar{x} \star \bar{y} = \{\bar{z} \mid z \in x + y\} = \{\bar{z} \mid (x \oplus y)(z) > 0\},$$

$$\bar{r} \circledcirc \bar{\alpha} \circledcirc \bar{x} = \{\bar{z} \mid z \in r.\alpha.x\} = \{\bar{z} \mid (r \odot \alpha \odot x)(z) > 0\}.$$

Theorem 4.6. Let $(M, \oplus, \odot)$ be a fuzzy Γ-hypermodule over a fuzzy Γ-hyperring $(R, \boxplus, \boxdot)$ and canonical fuzzy hypergroup $(\Gamma, *)$. Let $(M, +, .)$ be the associated Γ-hypermodule over the corresponding Γ-hypergroup $(R, \uplus, \circ)$ and canonical hypergroup $(\Gamma, *)$. Then we have:

(i) The relation δ is a fuzzy regular relation on $(M, \oplus, \odot)$ if and only if $(M/\delta, \star, \circledcirc)$ is a Γ-hypermodule over $(R, \uplus, \circ)$ and $(\Gamma, *)$.
(ii) The relation δ is a fuzzy strongly regular relation on $(M, \oplus, \odot)$ over $(R, \boxplus, \boxdot)$ and $(\Gamma, \otimes)$ if and only if $(M/\delta, \star, \circledcirc)$ is a Γ-module over R/ρ and Γ/θ.
If we denote by $\mathfrak{U}$ the set of all expressions consisting of finite fuzzy Γ-hyperoperations either on R, Γ and M or the external fuzzy Γ-hyperoperations applied on finite sets of elements of R, Γ and M, then we have
$x\epsilon y \Longleftrightarrow \exists u \in \mathfrak{U} \ : \ \{x, y\} \subset u.$
Now, we introduced *fundamental relation* on fuzzy Γ-hypemodules.
Definition 4.7. An equivalence relation ϵ^* is called *fundamental relation* on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ if ϵ^* is fundamental relation on the associated Γ-hypermodule $(M, +, .)$.
Hence, ϵ^* is fundamental relation on a fuzzy Γ-hypermodule $(M, \oplus, \odot)$ if and only if ϵ^* is the smallest fuzzy strongly equivalence relation on $(M, \oplus, \odot)$.
Denote by $\mathfrak{UF}$ the set of all expressions consisting of finite fuzzy Γ-hyperoperations either on R, Γ and M or the external fuzzy Γ-hyperoperation applied on finite sets of elements of R, Γ and M. We obtain

$$x\epsilon y \Longleftrightarrow \exists \ \mu_f \in \mathfrak{UF} : \ \{x, y\} \subseteq \mu_{f\gamma} \Longleftrightarrow \mu_{f\gamma}(x) > 0 \text{ and } \mu_{f\gamma}(y) > 0.$$

The relation ϵ^* is the transitive closure of ϵ.
Denote by $\sum_{\oplus}^*$ any finite fuzzy hypersum and by $\prod_{\odot}^*$ any finite fuzzy Γ-hyperproduct of the fuzzy Γ-hypemodule $(M, \oplus, \odot)$. As above, we obtain that

$$(\textstyle\sum_{i\oplus}^* \prod_{j \ \odot}^* a_{ji})(p) > 0 \text{ if and only if } p \in \sum_{i\oplus}^* \prod_{j\odot}^* a_{ji}.$$

Hence, $\{x, y\} \subset \ \sum_{i\oplus}^* \prod_{j \ \odot}^* a_{ji}$ if and only if $(\sum_{i\oplus}^* \prod_{j \ \odot}^* a_{ji})(x) > 0$ and $(\sum_{i\oplus}^* \prod_{j \ \odot}^* a_{ji})(y) > 0$. Therefore, we obtain $x\epsilon y \Longleftrightarrow \exists \mu_{f\gamma} \in \mathfrak{UF}$ such that $\mu_{f\gamma}(x) > 0$ and $\mu_{f\gamma}(y) > 0$.
So, in order to obtain a fuzzy Γ-module starting from a fuzzy Γ-hypermodule, we consider first the relation ϵ, then the transitive closure ϵ^* of ϵ and finally the quotient structure $(M/\epsilon^*, \star, \circledcirc)$ of the fuzzy Γ-hypermodule $(M, \oplus, \odot)$.

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