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The Lie-Santilli admissible hyperalgebras of type A_n

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Abstract

The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v-structures introduced in 1990 and they proved to have a lot of applications on several applied sciences. In this paper we present a construction of the hyperstructures used in the Lie-Santilli admissible theory on square matrices.

Key words: hyperstructures, H_v-structures, hopes, weak hopes, ∂-hopes, e-hyperstructures, admissible Lie-algebras.

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1 Introduction

We deal with hyperstructures called H_v*-structures* introduced in 1990 [30], which satisfy the weak axioms where the non-empty intersection replaces the equality.

Some basic definitions are the following:

In a set H equipped with a hyperoperation (abbreviation *hyperoperation = hope)*

$$\cdot : H \times H \to P(H) - \{\varnothing\},$$

we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing, \forall x, y, z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \varnothing, \forall x, y \in H$.

The hyperstructure $(H, \cdot)$ is called an H_v-*semigroup* if it is *WASS*, it is called H_v-*group* if it is reproductive H_v-semigroup, i.e.,

$$xH = Hx = H, \forall x \in H.$$

The hyperstructure $(R, +, \cdot)$ is called an H_v-*ring* if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing,\ (x+y)z \cap (xz+yz) \neq \varnothing,\ \forall x, y, z \in R.$$

Motivations. The motivation for H_v-structures is the following: We know that the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition (or equivalently to any equivalence relation) is an H_v-group. This is the motivation to introduce the H_v-structures [24].

In an H_v-semigroup the powers of an element $h \in H$ are defined as follows:

$$h^1 = \{h\}, h^2 = h \cdot h, ..., h^n = h \circ h \circ ... \circ h,$$

where $(\circ)$ denotes the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is called *cyclic of period s*, if there exists an element h, called *generator*, and a natural number s, the minimum one, such that

$$H = h^1 \cup h^2 ... \cup h^s.$$

Analogously the cyclicity for the infinite period is defined [23]. If there is an element h and a natural number s, the minimum one, such that $H = h^s$, then $(H, \cdot)$ is called *single-power cyclic of period s.*

For more definitions and applications on H_v-structures, see the books [2],[8],[24],[4],[1] and papers as [3],[28],[21],[22],[26],[9],[14],[13].

The main tool to study hyperstructures are the *fundamental relations* β^*, γ^* and ϵ^*, which are defined in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. These relations were introduced by T. Vougiouklis [30],[24],[29]. A way to find the fundamental classes is given by theorems as the following [24],[21],[25],[22],[7],[9],[20]:

Theorem 1.1. *Let* $(\boldsymbol{H}, \cdot)$ *be an* H_v*-group and denote by* $\boldsymbol{U}$ *the set of all finite products of elements of H. We define the relation* β *in H by setting* $x\beta y$ *iff* $\{x, y\} \subset \boldsymbol{u}$ *where* $\boldsymbol{u} \in \boldsymbol{U}$*. Then* β^* *is the transitive closure of* β*.*

Analogous theorems for the relations γ^* in H_v-rings, ϵ^* in H_v-modules and H_v-vector spaces, are also proved. An element is called *single* if its fundamental class is singleton [24].

Fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called H_v*-field* if R/γ^* is a field.

Let $(H,\cdot),(H,*)$ be H_v-semigroups defined on the same set H. The hope $(\cdot)$ is called *smaller* than the hope $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f \in Aut(H,*) \text{ such that } xy \subset f(x*y),\ \forall x,y \in H.$$

Then we write $\cdot \leq *$ and we say that $(H,*)$ contains $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called H_b-structure and $(*)$ is called *b-hope*.

Theorem 1.2. *(The Little Theorem). Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.*

Definition 1.1. *[20],[25] Let $(H,\cdot)$ be hypergroupoid. We remove $h \in H$, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h} \in H$ absorbs $h \in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. $\underline{h} \in H$ merges with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$, therefore, h does not appear in the hyperstructure.*

For several definitions and applications of hyperstructures in mathematics or in sciences and social sciences one can see [11],[15],[13],[3].

2 The theta (∂) hopes

In [19],[32],[11],[15] a hope, in a groupoid with a map f on it, denoted ∂_f, is introduced. Since there is no confusion, we write simply *theta* ∂. The symbol "∂" appears in Greek papyrus to represent the letter "theta "usually in middle rather than the beginning of the words.

Definition 2.1. *Let H be a set equipped with n operations (or hopes) $\otimes_1,...,\otimes_n$ and a map (or multivalued map) $f: H \to H$ (or $f: H \to P(H)-\emptyset$, respectively), then n hopes $\partial_1,\partial_2,...,\partial_n$ on H can be defined, called theta-operations (we rename here theta-hopes and we write ∂-hope) by putting*

$$x\partial_i y = \{f(x)\otimes_i y, x\otimes_i f(y)\}, \forall x,y \in H \text{ and } i \in \{1,2,...,n\}$$

or, in case where $\otimes_i$ is hope or f is multivalued map, we have

$$x\partial_i y = (f(x) \otimes_i y) \cup (x \otimes_i f(y)), \forall x, y \in H \text{ and } i \in \{1, 2, ..., n\}$$

if $\otimes_i$ is associative then ∂_i is WASS.

Analogously one can use several maps f, instead than only one.

Let $(G, \cdot)$ be a groupoid and $f_i : G \rightarrow G, i \in I$, be a set of maps on G. Take the map $f_\cup : G \rightarrow \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x)|i \in I\}$ and we call it *the union* of the $f_i(x)$. We call *union ∂-hopes*, on G if we consider the map $f_\cup(x)$. A special case is to take the union of f with the identity, i.e. $\underline{f} = f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$, which is called *b-$\partial$-hope*. We denote the *b-∂-hope* by $(\underline{\partial})$, so

$$x\underline{\partial}y = \{xy, f(x) \cdot y, x \cdot f(y)\}, \forall x, y \in G$$

This hope contains the operation $(\cdot)$ so it is a b-hope. If $f : G \rightarrow P(G) - \{\emptyset\}$, then the b-$\partial$-hope is defined by using the map $\underline{f}(x) = \{x\} \cup f(x), \forall x \in G$.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions s(x), t(x), we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

For several results one can see [19],[32].

Examples. (a) Taking the application on the derivative, consider all polynomials of up to first degree $g_i(x) = a_i x + b_i$. We have

$$g_1\partial g_2 = \{a_1a_2x + a_1b_2, a_1a_2x + b_1a_2\},$$

so this is a hope in the first degree polynomials. Remark that all polynomials x+c, where c be a constant, are units.

(b) *The constant map.* Let $(G, \cdot)$ be group and $f(x) = a$, thus $x\partial y = \{ay, xa\}, \forall x, y \in G$. If $f(x) = e$, then we obtain $x\partial y = \{x, y\}$, the smallest incidence hope.

Properties. If $(G, \cdot)$ is a semigroup then:

(a) For every f, the ∂-hope is WASS.

(b) For every f, the b-∂-hope $(\underline{\partial})$ is WASS.

(c) If f is homomorphism and projection, then (∂) is associative.

Properties.

Reproductivity. If $(\cdot)$ is reproductive then (∂) is also reproductive.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre of G, then (∂) is commutative. If $(\cdot)$ is COW then, (∂) is COW.

Unit elements. The elements of the kernel of f, are the units of (G,∂).

Inverse elements. For given x, the elements $x' = (f(x))^{-1}u$ and $x' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Proposition. Let $(G,\cdot)$ be a group then, for all maps $f : G \to G$, the hyperstructure (G,∂) is an H_v-group.

Definition 2.2. *Let $(R,+,\cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be two maps. We define two hopes (∂_+) and (∂_-), called both theta-hopes, on R as follows*

$$x\partial_+ y = \{f(x) + y, x + f(y)\} \text{ and } x\partial_. y = \{g(x)\cdot y, x\cdot g(y)\}, \forall x, y \in G.$$

A hyperstructure $(R,+,\cdot)$, where $(+)$, $(\cdot)$ are hopes which satisfy all H_v-ring axioms, except the weak distributivity, will be called H_v-near-ring.

Propositions.

(a) Let $(R,+,\cdot)$ be a ring and $f : R \to R$, $g : R \to R$ be maps. The $(R,\partial)_+,\partial_.)$, called *theta*, is an H_v-near-ring. Moreover (∂_+) is commutative.

(b) Let $(R,+,\cdot)$ be a ring and $f : R \to R$, $g : R \to R$ maps, then $(R,\underline{\partial}_+,\partial_.)$, is an H_v-ring.

Properties. *(Special classes).* The theta hyperstructure $(R,\partial_+,\partial_.)$ takes a new form and has some properties in several cases as the following ones:

(a) If f is a homomorphism and projection, then

$$x\partial_.(y\partial_+ z)\cap(x\partial_. y)\partial_+(x\partial_. z) = \{f(x)f(y)+f(x)z, f(x)y+f(x)f(z)\} \neq \emptyset.$$

Therefore, $(R,\partial)_+,\partial_.)$ is an H_v-ring.

(b) If $f(x) = x, \forall x \in R$, then $(R,+,\partial_.)$ becomes a multiplicative H_v-ring:

$$x\partial_.(y+z)\cap(x\partial_. y)+(x\partial_. z) = \{g(x)y + g(x)z\} \neq \emptyset.$$

If, moreover, f is a homomorphism, then we have a "more" strong distributivity:

$$x\partial_.(y+z)\cap((x\partial_. y)+(x\partial_. z)) = \{g(x)y + g(x)z, xg(y) + xg(z)\} \neq \emptyset.$$

Now we can see theta hopes in H_v-vector spaces and H_v-Lie algebras:

Theorem 2.1. *Let $(V, +, \cdot)$ be an algebra over the field $(F, +, \cdot)$ and $f : V \to V$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(V, +, \partial)$ is an H_v-algebra over F, where the related properties are weak. If, moreover f is linear then we have*

$$\lambda(x\partial y) = (\lambda x)\partial y = x\partial(\lambda y).$$

Another well known and large class of hopes is given as follows [23],[24]:

Let $(G, \cdot)$ be a groupoid then for every $P \subset G$, $P \neq \varnothing$, we define the following hopes called P-*hopes*: for all $x, y \in G$

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \quad \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy).$$

The $(G, \underline{P})$,$(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called P-*hyperstructures*. The most usual case is if $(G, \cdot)$ is semigroup, then $x\underline{P}y = (xP)y \cup x(Py) = xPy$ and $(G, \underline{P})$ is a semihypergroup but we do not know about $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$. In some cases, depending on the choice of P, the $(G, \underline{P}_r)$ and $(G, \underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes, introduced by Davvaz, Santilli, Vougiouklis in [7],[6] is the following:

Construction 2.1. *Let $(G, \cdot)$ be an abelian group and P any subset of G with more than one elements. We define the hope $\times P$ as follows:*

$$x \times_p y = \begin{cases} x \times_P y = x \cdot P \cdot y = \{x \cdot h \cdot y | h \in P\} & \textit{if } x \neq e \textit{ and } c \neq e \\ x \cdot y & \textit{if } x = e \textit{ and } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_p)$ is an abelian H_v-group.

Matrix Representations

H_v-structures are used in Representation Theory of H_v-groups which can be achieved either by generalized permutations or by H_v-matrices [28],[24]. Representations by generalized permutations can be faced by translations. In this theory the single elements are playing a crucial role. H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices is defined in a usual manner. In representations of H_v-groups by H_v-matrices, there are two difficulties: To find an H_v-ring and an appropriate set of H_v-matrices.

Most of H_v-structures are used in Representation (abbreviate by *rep*) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [24]. Reps by generalized permutations can be achieved by using translations. In the rep theory the singles are playing a crucial role.

The rep problem by H_v-matrices is the following:

H_v-matrix is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices **A**$= (a_{ij})$ and **B**$= (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is a set of $m \times r$ H_v-matrices, defined in a usual manner:

$$A \cdot B = (a_{ij}) \cdot (b_{ij}) = \{C = (c_{ij}) | (c_{ij}) \in \oplus \sum a_{ik} \cdot b_{kj}\},$$

where ($\oplus$) denotes the *n-ary circle hope* on the hyperaddition.

Definition 2.3. *Let $(H, \cdot)$ be an H_v-group,$(R, +, \cdot)$ be an H_v-ring R and consider a set $M_R = \{(a_{ij}) | a_{ij} \in R\}$ then any map*

$$T : H \to M_R : h \mapsto T(h) \text{ with } T(h_1h_2) \cap T(h_1)T(h_2) \neq \varnothing, \forall h_1, h_2 \in H.$$

is called H_v-matrix rep. If $T(h_1h_2) \subset T(h_1)T(h_2)$, then $\boldsymbol{T}$ is an inclusion rep, if $T(h_1h_2) = T(h_1)T(h_2)$, then $\boldsymbol{T}$ is a good rep.

3 The general H_v-Lie Algebra

Definition 3.1. *Let $(F, +, \cdot)$ be an H_v-field, $(V, +)$ be a COW H_v-group and there exists an external hope*

$$\cdot : F \times V \to P(V) - \{\emptyset\} : (a, x) \to zx$$

such that, for all a,b in F and x,y in V we have

$$a(x + y) \cap (ax + ay) \neq \emptyset, (a + b)x \cap (ax + bx) \neq \emptyset, (ab)x \cap a(bx) \neq \emptyset,$$

then V is called an H_v-vector space over F. In the case of an H_v-ring instead of an H_v-field then the H_v-modulo is defined. In these cases the fundamental relation ϵ^* is the smallest equivalence relation such that the quotient V/ϵ^* is a vector space over the fundamental field F/γ^*.

The general definition of an H_v-Lie algebra was given in [31] as follows:

Definition 3.2. *Let $(L, +)$ be an H_v-vector space over the H_v-field $(F, +, \cdot)$, $\phi : F \to F/\gamma^*$ the canonical map and $\omega_F = \{x \in F : \phi(x) = 0\}$, where 0 is the zero of the fundamental field F/γ. Similarly, let ω_L be the core of the* *canonical map $\phi' : L \to L/\epsilon^*$ and denote by the same symbol 0 the zero of L/ϵ^*. Consider the bracket (commutator) hope:*

$$[,] : L \times L \to P(L) : (x, y) \to [x, y]$$

then $\boldsymbol{L}$ is an H_v-Lie algebra over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $\forall x, x_1, x_2, y, y_1, y_2 \in L, \lambda_1, \lambda_2 \in F$

$$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1 [x_1, y] + \lambda_2 [x_2, y]) \neq \varnothing$$

$$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1 [x, y_1] + \lambda_2 [x, y_2]) \neq \varnothing,$$

(L2) $[x, x] \cap \omega_L \neq \varnothing, \quad \forall x \in L$

(L3) $([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \varnothing, \quad \forall x, y \in L$

Definition 3.3. *Let $(\mathbf{A}, +, \cdot)$ be an algebra over the field F. Take any map $f : A \to A$, then the ∂-hope on the Lie bracket $[x, y] = xy - yx$, is defined as follows*

$$x \partial y = \{f(x)y - f(y)x, f(x)y - yf(x), xf(y) - f(y)x, xf(y) - yf(x)\}.$$

Remark that if we take the identity map $f(x) = x, \forall x \in A$, then $x\partial y = \{xy - yx\}$, thus we have not a hope and remains the same operation.

Proposition. Let $(A, +, \cdot)$ be an algebra F and $f : A \to A$ be a linear map. Consider the $\partial - hope$ defined only on the multiplication of the vectors $(\cdot)$, then $(A, +, \cdot)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Proposition. Let $(A, +, \cdot)$ be an algebra and $f : A \to A : f(x) = a$ be a constant map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(A, +, \partial)$ is an H_v-Lie algebra over F.

In the above theorem if one take a=e, the unit element of the multiplication, then the properties become more strong.

4 Santilli's admissibility

The Lie-Santilli isotopies born to solve Hadronic Mechanics problems. Santilli proposed [16] a "lifting" of the trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit.

The isofields needed correspond to H_v-structures called e-hyperfields which are used in physics or biology. Definition: Let $(H_o, +, \cdot)$ be the attached H_v-field of the H_v-semigroup $(H, \cdot)$. If $(H, \cdot)$ has a left and right scalar unit e then $(H_o, +, \cdot)$ is e-hyperfield, the attached H_v*-field* of $(H, \cdot)$.

The Lie-Santilli theory on isotopies was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a "lifting" of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [5],[17] and they are called *e-hyperfields.* The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the H_v-structures.

Definition 4.1. *A hyperstructure* $(H, \cdot)$ *which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse* x^{-1}*, i.e.* $e \in x \cdot x^{-1} \cap x^{-1} \cdot x$. *Remark that the inverses are not necessarily unique.*

Definition 4.2. *A hyperstructure* $(F, +, \cdot)$*, where* $(+)$ *is an operation and* $(\cdot)$ *is a hope, is called e-hyperfield if the following axioms are valid:*

1. $(F, +)$ *is an abelian group with the additive unit 0,*
2. $(\cdot)$ *is WASS,*
3. $(\cdot)$ *is weak distributive with respect to* $(+)$*,*
4. *0 is absorbing element:* $0 \cdot x = x \cdot 0 = 0, \forall x \in F$*,*
5. *exist a multiplicative scalar unit 1, i.e.* $1 \cdot x = x \cdot 1 = x, \forall x \in F$*,*
6. *for every* $x \in F$ *there exists a unique inverse* x^{-1}*, such that* $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1 = x \cdot x^{-1} = x^{-1} \cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Now we present a general construction which is based on the partial ordering of the H_v-structures and on the Little Theorem.

Definition 4.3. *The Main e-Construction. Given a group* $(G, \cdot)$*, where e is the unit, then we define in G, a large number of hopes* $(\otimes)$ *as follows:*

$$x \otimes y = \{xy, g_1, g_2, ...\}, \forall x, y \in G - \{e\}, \text{ and } g_1, g_2, ... \in G - \{e\}$$

$g_1, g_2, ...$ *are not necessarily the same for each pair (x,y). Then* $(G, \otimes)$ *becomes an* H_v*-group, actually is an* H_b*-group which contains the* $(G, \cdot)$*. The* H_v*-group* $(G, \otimes)$ *is an e-hypergroup. Moreover, if for each x,y such that* $xy = e$*, so we have* $x \otimes y = xy$*, then* $(G, \otimes)$ *becomes a strong e-hypergroup*

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$ and if this condition is valid then we have $1 = x \cdot x^{-1} = x^{-1} \cdot x$. So the hyperstructure $(G, \otimes)$ is a strong e-hypergroup.

5 Mathematical Realisation of type A_n

The representation theory by matrices gives to researchers a flexible tool to see and handle algebraic structures. This is the reason to see Lie-Santilli's admissibility using matrices or hypermatrices to study the multivalued (hyper) case. Using the well known P-hyperoperations we extend the Lie-Santilli's admissibility into the hyperstructure case. We present the problem and we give the basic definitions on the topic which cover the four following cases:

Construction 5.1. *[18] Suppose R, S be sets of square matrices (or hypermatrices). We can define the hyper-Lie bracket in one of the following ways:*

1. $[x, y]_{RS} = xRy - ySx$ *(General Case)*
2. $[x, y]_R = xRy - yx$
3. $[x, y]_S = xy - ySx$
4. $[x, y]_{RR} = xRy - yRx$

The question is when the conditions, for all square matrices (or hypermatrices) x, y, z,

$$[x, x]_{RS} \ni 0$$

$$[x, [y, z]_{RS}]_{RS} + [y, [z, x]_{RS}]_{RS} + [z, [x, y]_{RS}]_{RS} \ni 0$$

of a hyper-Lie algebra are satisfied [18].
We apply this generalization on the Lie algebras of the type A_n*.*

We deal with Lie-Algebra of type A_n, of traceless matrices M (Tr(M)=0), which is a graded algebra, using the principal realisation used in Infinite Dimensional Kac Moody Lie Algebras introduced in 1981[10] by Lepowsky and Wilson, Kac [12]. In this special algebra examples on the above described hyperstructure theory are being presented.

Denote as

$$E_{ij}(i,j=1,...,n)$$

the $n \times n$ matrix which is 1 in the ij-entry and 0 everywhere else and by

$$e_i = E_{ii} - E_{i+1,i+1}, i = 1,...,n-1$$

The Simple base of the above type is the following:

Base of Level 0 : e_i, $i = 1,2,...,n-1$

Base of Level 1 : $E_{i,i+1}$, $i = 1,2,...,n$

Base of Level 2 : $E_{i,i+2}$, $i = 1,2,...,n$

...

Base of Level n-1 : $E_{i,i+(n-1)}$, $i = 1,2,...,n$

Denote that all the subscripts are mod n.

Therefore the levels are in bold as follows:

Level 0 :

$$\begin{pmatrix} \boldsymbol{a_{11}} & 0 & 0 & \dots & 0 \\ 0 & \boldsymbol{a_{22}} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{a_{nn}} \end{pmatrix}$$

Level 1 :

$$\begin{pmatrix} 0 & \boldsymbol{a_{12}} & 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{a_{23}} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \boldsymbol{a_{n-1,n}} \\ \boldsymbol{a_{n1}} & 0 & 0 & \dots & 0 \end{pmatrix}$$

Level 2 :

$$\begin{pmatrix} 0 & 0 & \mathbf{a_{13}} & \dots & 0 \\ 0 & 0 & 0 & \dots & \mathbf{a_{2n}} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{a_{n-1,1}} & 0 & 0 & \dots & 0 \\ 0 & \mathbf{a_{n2}} & 0 & \dots & 0 \end{pmatrix}$$

....

Level n-1 :

$$\begin{pmatrix} 0 & 0 & 0 & \dots & \mathbf{a_{1n}} \\ \mathbf{a_{21}} & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \mathbf{a_{n,n-1}} & 0 \end{pmatrix}$$

For our examples the Konstant's Cyclic Element E is being used as the sum of First Level's Simple Base [10].

$$E = E_{12} + E_{23} + E_{34} + ... + E_{n-1,n} + E_{n1}$$

This element is shifting every element of level L to the next level $L + 1$ [10],[27]. The base of the first level as well as for every level, except zero, has n elements. Level 0 has a $n - 1$ dimension because of the limitation of the zero trace. The cyclic element gets different element from the base and goes to different ellement of the next level, creating an 1-1 correspondance. The element E shifts level $n - 1$ to the Level 0 and because, as already remarked, Level 0 has $n - 1$ elements, contrary with every other level, the 1-1 correspondance is being corrupted.

To summarize, according to the related theory, removing from every level (except Level-0), all the powers of E until $n - 1$ $(E, E^2, ..., E^{n-1})$, an one to one complete correspondance between all levels, Level-0 included, is being created.

We denote the first power :

$$[E, E_{n1}]^1 = E \cdot E_{n1} - E_{n1} \cdot E = A_1$$

the second power:

$$[E, E_{n1}]^2 = [E, A_1] = A_2$$

....

and inductively by the n-power:

$$[E, E_{n1}]^n = [E, A_{n-1}] = A_n$$

One can prove the following:

Theorem 5.1.

$$[E, E_{n1}]^n =$$
$$= diag(\binom{n-1}{0}, (-1)^1\binom{n-1}{1}, (-1)^2\binom{n-1}{2}, ..., (-1)^{n-2}\binom{n-1}{n-2}, (-1)^{n-1}\binom{n-1}{n-1})$$

The above theorem helps as to find the basic element of first Level's base and based on this theorem all the n^{th} powers of the elements of the first level can also be found.

Theorem 5.2. *Based on this theory and P-hyperstructures a set P with two elements can be used, either from zero or first level, but only with two elements. In this case the shift is depending on the level, so if we take P from Level-0, the result will not change, although the result will be multivalued. In case of different level insted, the shift will be analogous to the level of P.*

In the general case in Construction 5.1(1), one can notice the possible cardinality of the result, checking the Jacoby identity is very big. Even in the small case when $|R| = |S| = |P| = 2$ in the anticommutativity $xPx - xPx$ could have cardinality 4 and the left side of the Jacoby identity is

$$(xP(yPz - zPy) - (yPz - zPy)Px) + (yP(zPx - xPz)-$$
$$-(zPx - xPz)Py) + (zP(xPy - yPx) - (xPy - yPx)Pz)$$

could have cardinality 2^{18}. The number is reduced in special cases.

Theorem 5.3. *In the case of the Lie-algebra of type A_n, of traceless matrices M, we can define a hyper-Lie-Santilli-admissible bracket hope as follows:*

$$[xy]p = xPy - yPx$$

where $P = \{p, q\}$, with p,q elements of the zero level. Then we obtain a hyper-Lie-Santilli-algebra.

Proof

We need only to proof the anticommutativity and the Jacobi identity as in the hyperstructure case. Therefore we have

(a) $[xy]p = xPy - yPx = \{0, xpx - xqx, xqx - xpx\} \ni 0$, so the "weak" anticommutativity is valid, and

(b) $[x, [y, z]p]p + [y, [z, x]p]p + [z, [x, y]p]p =$
$(xP(yPz - zPy) - (yPz - zPy)Px) + (yP(zPx - xPz)-$
$-(zPx - xPz)Py) + (zP(xPy - yPx) - (xPy - yPx)Pz)$.

But this set contains the element

$$xpypz - xpzpy - ypzpx + zpypx + ypzpx - ypxpz-$$
$$-zpxpy + xpzpy + zpxpy - zpypx - xpypz + ypxpz = 0$$

So the "weak" Jacobi identity is valid.

Thus, zero belongs to the above results, as it has to be, but there are more elements because it is a multivalued operation.

References

[1] P. Corsini, *Prolegomena of hypergroup theory*, Aviani, 1994.

[2] P. Corsini, V. Leoreanu, *Applications of Hyperstructure Theory*, Kluwer Academic Publ.,2003.

[3] B. Davvaz, *A brief survey of the theory of H_v-structures*, 8^{th} AHA Congress, Spanidis Press (2003), 39-70.

[4] B. Davvaz, *Polygroup theory and related systems*, World Sci., (2013).

[5] B. Davvaz, R.M Santilli, T. Vougiouklis, *Studies of multivalued hyperstructures for the characterization of matter-antimatter systems and their extension*, Algebras, Groups and Geometries, 28(1) (2011), 105-116.

[6] B. Davvaz, R.M. Santilli, T. Vougiouklis, *Multi-valued Hypermathematics for characterization of matter and antimatter systems*, Journal of Computational Methods in Sciences and Engineering (JCMSE) 13, (2013), 37-50.

[7] B. Davvaz, S. Vougiouklis, T. Vougiouklis, *On the multiplicative H_v-rings derived from helix hyperoperations*,Util. Math., Vol. **84** (2011), 53-63.

[8] B. Davvaz, V. Leoreanu, *Hyperring Theory and Applications*, Int. Academic Press, 2007.

[9] S. Hoskova, *H_v-structure is fifteen*, Proc. of 4^{th} International mathematical workshop FAST VUT Brno, Czech Republic, (2005), 55-57,

http://math.fce.vutbr.cz/pribyl/workshop2005/prispevky/Hoskova.pdf

[10] V.G Kac, *Infinite-dimensional Lie algebras*, Vol. 44. Cambridge University Press, 1994.

[11] P. Kampaki-Vougioukli, *Bar in SILL Questionnaire for multiple results processing: Users' frequency and confidence*, Sino-US English Teaching, Vol. 10 N.3, March 2013 (Serial N.111), 184-199.

[12] J. Lepowski, R.L. Wilson, *Costruction of the Affine Lie Algebra* $A_1^{(1)}$, Comm. Math. Phys., 62, (1978), 43-53.

[13] N. Lygeros, T. Vougiouklis, *The LV-hyper structures*, Ratio Mathematica, 25, (2013), 59-66.

[14] A. Maturo, B. Ferri, *On some applications of fuzzy sets and commutative hypergroups to evaluation in architectyre and town-planning*, Ratio Mathematica, 13, (1999), 51-60.

[15] P. Nikolaidou, T. Vougiouklis, *H_v-structures and the Bar in questionnaires*, Italian J. Pure Applied Math 29, (2012), 341-350.

[16] R.M. Santilli, *Hadronic Mathematics, Mechanics and Chemistry, Vol. I, II, III, IV and V*, International Academic Press, USA, 2008.

[17] R.M. Santilli, T. Vougiouklis, *Isotopies, Genotopies, Hyperstructures and their Applications*, Proc. New Frontiers in Hyperstructures and Related Algebras, Hadronic Press, (1996), 177-188.

[18] R.M. Santilli, T. Vougiouklis, *Lie-admissible hyperalgebras*, Italian Journal of Pure and Applied Mathematics,31,(2013), 239-254.

[19] T. Vougiouklis, *A hyperoperation defined on a groupoid equipped with a map*, Ratio Mathematica on line, N.1 (2005), 25-36.

[20] T. Vougiouklis, *Construction of H_v-structures with desired fundamental structures*, New frontiers in Hyperstructures, Hadronic, (1996), 177-188.

[21] T. Vougiouklis, *Enlarging H_v-structures, Algebras and Combinatorics*, ICAC' 97, Hong Kong, Springer - Verlag (1999), 455-463.

[22] T. Vougiouklis, *Finite H_v-structures and their representations*, Rend. Sem. Mat. Messina S.II, V.9 (2003), 245-265.

[23] T. Vougiouklis, *G*eneralization of P-hypergroups, Rend. Circ. Mat. Palermo, S.II,, 36 (1987), 114-121.

[24] T. Vougiouklis, *Hyperstructures and their Representations*, Monographs in Mathematics, Hadronic Press, 1994.

[25] T. Vougiouklis, *On H_v-rings and H_v-representations,* Discrete Math., 208/209, (1999), 615-620.

[26] T. Vougiouklis, *On a matrix hyperalgebra*, J. Basic Science V.3, N.1, (2006), 95-102.

[27] T. Vougiouklis, *On affine Kac-Moody Lie algebras*, Commentationes Mathematicae Universitatis Carolinae 26.2 (1985), 387-395.

[28] T. Vougiouklis, *Representations of hypergroups by hypermatrices*, Rivista Mat. Pura ed Appl., N2(1987), 7-19.

[29] T. Vougiouklis, *Some remarks on hyperstructures*, Contemp. Math., 184, (1995), 427-431.

[30] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, 4^{th} AHA Congress, World Scientific,(1991),203-211.

[31] T. Vougiouklis, *The Lie-hyperalgebras and their fundamental relations*, Southeast Asian Bulletin of Mathematics, 37(4),(2013),601-614.

[32] T. Vougiouklis, *∂-operations and H_v-fields*, Acta Math. Sinica, (Engl. Ser.), Vol. 24, N.7 (2008), 1067-1078.