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Contributions in Mathematics: Hyperstructures of Professor Thomas Vougiouklis

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Abstract

After presenting some basic notions of hyperstructures and their applications, I shall point out on the contribution of Professor Thomas Vougiouklis to this field of research: algebraic hyperstructures.
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1 Hyperstructures and applications

Theory of hyperstructures is a field of algebra, around 80 years old and very rich in applications, for instance in geometry, fuzzy and rough sets, automata, cryptography, codes, probabilities, graphs and hypergraphs (see [2], [3]).

Some basic definitions:

A *hyperoperation* on a nonempty set H is a map $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H.

For subsets A, B of H, set $A \circ B = \bigcup_{a \in A; b \in B} a \circ b$, and for $h \in H$ write $h \circ A$ and $A \circ h$ for $\{h\} \circ A$ and $A \circ \{h\}$.

The pair $(H, \circ)$ is a *hypergroup* if for all a, b, c of H we have

$$(a \circ b) \circ c = a \circ (b \circ c) \text{ and } a \circ H = H \circ a = H.$$

If only the associativity is satisfied then $(H, \circ)$ is a *semihypergroup*. The condition $a \circ H = H \circ a = H$ for all a of H is called the *reproductive law*.

A nonempty subset K of H is a *subhypergroup* if $K \circ K \subseteq K$ and for all $a \in K$, $K \circ a = K = a \circ K$.

A commutative hypergroup $(H, \circ)$ is a *join space* iff the following implication holds: for all a, b, c, d, x of H,

$$a \in b \circ x, c \in d \circ x \Rightarrow a \circ d \cap b \circ c \neq \emptyset.$$

A *semijoin space* is a commutative semihypergroup satisfying the join condition.

Hypergroups have been introduced by Marty [5] and join spaces by Prenowitz [6]. Join spaces are an important tool in the study of graphs and hypergraphs, binary relations, fuzzy and rough sets and in the reconstruction of several types of noneuclidean geometries, such as the descriptive, spherical and projective geometries [3], [6]. Several interesting books have been writen on hyperstructures [2], [3], [4], [6], [8].

2 Emeritus Professor Thomas Vougiouklis and his contribution to hyperstructures

Professor Thomas Vougiouklis is an author of more than 150 research papers and seven text books in mathematics. He have over 3000 references. He also wrote eight books on poetry, one CD music and lyrics.

He participated in Congresses (invited) about 60 congresses, over 20 countries. His monograph:

Hyperstructures and their representations, Hadronic Press monograph in Mathematics, USA (1994)

is an important book on the theory of algebraic hyperstructures.

Let us mention here some of his main contributions in Hyperstructures, especially H_v -Structures, Lie Algebras of infinite dimension, ring theory, Mathematical Models.

He first introduced and studied:

- The term hope=hyperoperation (2008)
- P-hypergroups, single-power cyclicity (1981).
- Fundamental relations in hyper-rings (γ^*-relation) and Representations of hypergroups by generalized permutations and hypermatrices (1985).
- Very Thin hyperstructures, S-construction (1988).
- Uniting elements procedure (1989), with P.Corsini.
- General hyperring, hyperfield (1990).
- The weak properties and the Hv-structures (1990).
- General Hypermodules, hypervector spaces(1990).
- Representation Theory by Hv-matrices (1990).
- Fundamental relations in hyper-modulus and hyper-vector spaces (ε^* - relation) (1994).
- The e-hyperstructures, Hv-Lie algebras (1996).
- The h/v-structures (1998).
- ∂ - operations (2005),
- The helix hyperoperations, with S. Vougiouklis,
- n-ary hypergroups (2006), with B.Davvaz.,
- Bar instead of scale, (2008), with P. Vougioukli, etc

Let us present here some of these notions.

H_V- structures

These notions were introduced in 1990 and they satisfy the weak axioms, where the non-empty intersection replaces the equality.

WASS means *weak associativity*:

$$\forall x, y, z \in H,\ (xy)z \cap x(yz) \neq \emptyset.$$

COW means *weak commutativity*:

$$\forall x, y \in H,\ xy \cap yx \neq \emptyset.$$

A hyperstructure $(H, \cdot)$ is called *H_V-semigroup* if it is WASS and it is called *H_V-group* if it is a reproductive H_V-semigroup, i.e.

$$xH = Hx = H,\ \forall x \in H.$$

Similarly, H_V-vector spaces, H_V-algebras and H_V-Lie algebras are defined and their applications are mentioned in the above books.

Fundamental relations

The fundamental relations β^*, γ^* and ϵ^* are defined in H_V-groups, H_V-rings and H_V-vector spaces being the smallest equivalences, such that the quotient structures are a group, a ring or a vector space respectively.

The following theorem holds:

Theorem. Let $(H, \cdot)$ be an H_V-group and denote by U the set of all finite products of elements of H. We define the relation β in H as follows:

$$x\beta y \Leftrightarrow \exists u \in U : \{x, y\} \subseteq u$$

Then β^* is the transitive closure of β.

In a similar way, relation γ^* is defined in an H_V-ring and relation ϵ^* is defined in an H_V-vector space.

An H_V-ring $(R, +, \cdot)$ is called an H_V-field if R/γ^* is a field.

If $(H, \cdot)$, $(H, *)$ are H_V-semigroups defined on the same set H, then the hyperoperation $(\cdot)$ is *smaller* than $(*)$ (and $(*)$ is *greater* than $(\cdot)$ if there exists an $f \in Aut(H, *)$, such that

$$x \cdot y \subseteq f(x * y).$$

Theorem. Greater hopes than the ones which are WASS or COW are also WASS or COW, respectively.

This theorem leads to a partial order on H_V-structures and mainly to a correspondence between hyperstructures and posets.

The determination of all H_V-groups and H_V-rings is very interesting, but difficult. There are many results of R. Bayon and N. Lygeros in this direction.

In paper [1] one can see how many H_V-groups and H_V-rings there exist, up to isomorphism, for several chasses of hyperstructures of two, three or four elements.

∂- operations

The hyperoperations, called theta-operations, are motivated from the usual property, which the derivative has on the derivation of a product of functions.

If H is a set endowed with n operations (or hyperoperations) $\circ_1, \circ_2, ..., \circ_n$ and with one map or multivalued map $f : H \to H$ (or $f : H \to \mathcal{P}(H)$ respectively), then n hyperoperations $\partial_1, \partial_2, ..., \partial_n$ on H can be defined as follows:

$\forall x, y \in H, \forall i \in \{1, 2, ..., n\}$,

$$x \partial_i y = \{f(x) \circ_i y,\ x \circ_i f(y)\}$$

or in the case $\circ_i$ is a hyperoperation or f is a multivalued map, we have

$\forall x, y \in H, \forall i \in \{1, 2, ..., n\}$,

$$x \partial_i y = (f(x) \circ_i y) \cup (x \circ_i f(y)).$$

If $\circ_i$ is WASS, then ∂_i is WASS too.

n-ary hypergroups

A mapping $f : \underbrace{H \times \cdots \times H}_{n} \longrightarrow \mathcal{P}^*(H)$ is called an *n-ary hyperoperation*, where $\mathcal{P}^*(H)$ is the set of all the nonempty subsets of H. An algebraic system (H, f), where f is an n-ary hyperoperation defined on H, is called an *n-ary hypergroupoid*.

We shall use the following abbreviated notation:
The sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty symbol. When $y_{i+1} = \cdots = y_j = y$ the last expression will be written in the form $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$.

For nonempty subsets $A_1, ..., A_n$ of H we define

$$f(A_1^n) = f(A_1, ..., A_n) =$$

$$\bigcup\{f(x_1^n) \mid x_i \in A_i, i = 1, ..., n\}.$$

An n-ary hyperoperation f is called *associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) =$$
$$f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}),$$

hold for every $1 \leq i < j \leq n$ and all $x_1, x_2, ..., x_{2n-1} \in H$. An n-ary hypergroupoid with the associative n-ary hyperoperation is called an
n-ary semihypergroup.

An n-ary hypergroupoid (H, f) in which the equation $b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$ has a solution $x_i \in H$ for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an
n-ary quasihypergroup.

Moreover, if (H, f) is an n-ary semihypergroup, (H, f) is called an
n-ary hypergroup.

An n-ary hypergroupoid (H, f) is
commutative if for all $\sigma \in S_n$ and for every $a_1^n \in H$ we have
$f(a_1, ..., a_n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)})$.

Let (H, f) be an n-ary hypergroup and B be a non-empty subset of H. B is called an *n-ary subhypergroup* of (H, f), if $f(x_1^n) \subseteq B$ for $x_1^n \in B$, and the equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has a solution $x_i \in B$ for every $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

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