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A Brief Survey on the two Different Approaches of Fundamental Equivalence Relations on Hyperstructures

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This paper is dedicated to Prof. Thomas Vougiouklis lifetime work.

Abstract

Fundamental structures are the main tools in the study of hyperstructures. Fundamental equivalence relations link hyperstructure theory to the theory of corresponding classical structures. They also introduce new hyperstructure classes.The present paper is a brief reference to the two different approaches to the notion of the fundamental relation in hyperstructures. The first one belongs to Koskas, who introduced the β^* - relation in hyperstructures and the second approach to Vougiouklis, who gave the name fundamental to the resulting quotient sets. The two approaches, the necessary definitions and the theorems for the introduction of the fundamental equivalence relation in hyperstructures, are presented.
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1 Introduction

Dealing with classical algebraic structures often leads to the study of the behaviours of the elements of these sets with respect to the introduced operation(s). This study focuses, very often, on looking for elements with similar behaviour. Therefore, the use of the quotient set is intertwined with the search for regularity and symmetry between elements of algebraic structures and 'similar' algebraic structures too.
It is well known that "... the most powerful tool in order to obtain a stricter structure from a given one is the quotient out procedure. To use this method in ordinary algebraic domains, one needs special equivalence relations. If one suggests a method that can applied for every equivalence relation, has to use the hyperstructures" [6].
In the commutative algebra, many problems of algebraic structures are not always visible, resulting in a large number of questions and obstacles appearing in the non-commutative algebra. For example, in classical theory if G is a group and $H \subseteq G$ is a subgroup, then G/H quotient is a group only when H is a normal subgroup. This obstacle [21] is overcome by the definition of Fr. Marty (1934) [17], since

"If G is a group and $H \subseteq G$ is a subgroup of it, then the quotient G/H is a hypergroup."

The previous proposition is generalized by the definition [26] of the weak hyperstructures by Th. Vougiouklis (1990), as follows:

"If G is a group and S is any partition of G, then the quotient G/H is a H_v-group".

In these cases, the quotient set functioned as a process that led to 'looser' structures than classic algebraic ones, but increased complexity.
The utility of outmost importance of the quotient set in hyperstructures is its use as a bridge between classical structures and hyperstructures. In 1970, this connection was achieved by M. Koskas [16] using the β - relation and its transitive closure. Observing the similar behaviour of elements belonging to the same hyperproduct leads to the introduction of the β - relation which, clearly, is reflective, symmetric but not always transitive. The next step is to use the transitive closure of β to obtain equivalence relation and partition in equivalence classes. Using the usual definition of operations between classes, we return to classical algebraic structures. This relation studied mainly by Corsini [5], Vougiouklis [25], Davvaz [8], Leoreanou-Fotea [7], Freni [12], Migliorato [19] and many others.
The quotient set not only links the hyperstructures with the classical structures

as a bridge, but also enhances the view of hyperstructures as a generalization of the corresponding classical algebraic structures. In this way, as reported in [22], the algebraic structures are contained in the corresponding hyperstructures as subcases. It seems that Fr. Marty defined the hypergroup replacing the axiom of the existence of a unitary and inverse element with the axiom of reproduction because he had "sensed" this connection and chose the widest possible generalization in order to create space for the introduction of new types of hypergroups.
In 1988 at a congress in Italy, Th. Vougiouklis presents a paper titled "How a hypergroup hides a group" [22], [27] and finds out that
a) Vougiouklis, eighteen years after Koskas worked on the same subject without having knowledge of his work.
b) Much of the study was completed with the work of Koskas and especially P. Corsini and his school.
c) Vougiouklis approach was different from that of Koskas and the others.
In the following we will present the two different approaches.

2 Preliminaries

In a set $H \neq \emptyset$ equipped with a hyperoperation $(\cdot) : H \times H \to \wp^*(H)$ we abbreviate by

WASS the weak associativity: $x \cdot (y \cdot z) \cap (x \cdot y) \cdot z \neq \emptyset, \forall (x, y, z) \in H^3$.

COW the weak commutativity: $x \cdot y \cap y \cdot x \neq \emptyset, \forall (x, y) \in H^2$.

Definition 2.1. *The hyperstructure $(H, \cdot)$ is called H_v-semigroup if it is WASS and it is called H_v-group if it is reproductive H_v-semigroup, that is $xH = Hx = H, \forall x \in H$.*

Definition 2.2. *The hyperstructure $(H, \cdot)$ is called semihypergroup if $x \cdot (y \cdot z) = (x \cdot y) \cdot z, \forall (x, y, z) \in H^3$ and it is called hypergroup if it is reproductive semihypergroup.*

Definition 2.3. *A H_v-group is called H_b-group if its hyperoperation contains operation which define a group. We define analogously H_b-ring, H_b-vector space.*

Definition 2.4. *Let $(H, \cdot)$ be a H_v - structure. An element $e \in H$ is called identity if $x \in ex \cap xe, \forall x \in H$. We define analogously the left (right) identity.*

Definition 2.5. *Let $\phi : H \to H/\beta^*$ be the fundamental map of a H_v-group then, the kernel of ϕ is called core and it is denoted by ω_H.*

Definition 2.6. *A H_v-semigroup or a semihypergroup H is called cyclic if there exists $s \in H$, called generator, such that: $H = s^1 \cup s^2 \cup ... \cup s^n \cup ..., n \in N, n > 0$.*

For more definitions and applications on H_v-structures, see also the papers [4], [9], [10], [14], [15], [18], [20], [28].

3 The two approaches of fundamental relations

Searching for the quotient set, the definition of the relation between the elements of the hyperstructure plays an important role. The observation of a hyperoperation leads to the conclusion that elements belonging to the same hyperproduct act in a similar way with respect to the hyperoperation. This observation is the basis for defining the relation β. This definition is common to both approaches. Another common finding of the two approaches is the fact that the relation β is reflective, symmetric but not always transitive. The need for an equivalence relation that produces a quotient set such that it is a classical algebraic structure, makes it necessary to consider the β^* - relation that is the transitive closure of the β - relation and is, obviously, an equivalence relation. The last common point of the two approaches is the search for the smallest (with respect to the inclusion) equivalence relation having as quotient set the corresponding algebraic structure.

3.1 Koskas approach

Koskas, in his approach, introduces the equivalence relation that obtains as quotient set the corresponding algebraic structure by using the *strongly regular equivalence relation*. It is then shown that the transitive closure of the β - relation is the smallest strongly regular equivalence relation, i.e. β^* is the targeted relation. The proof is completed by the obvious finding that β^* is the desired equivalence relation such that the H/β^* is the corresponding algebraic structure. One can say that Koskas approach is a *deductive way* of defining fundamental relation on hyperstructures, since he starts considering a general definition and then specifying the β^*-relation as a subcase.

Taking into consideration the approach, the necessary definitions and theorems are mentioned, having as main sources the books [5], [7], [8].

Definition 3.1. *Let $(H, \circ)$ be a hypergroupoid, a, b elements of H and ρ be an equivalence relation on H. Then ρ is strongly regular on the left if the following implication holds:*

$$a\rho b \Rightarrow \forall u \in H, \forall x \in u \circ a, \forall y \in u \circ b : x\rho y.$$

Similarly, the strong regularity on the right can be defined. We call ρ strongly regular if it is strongly regular on both the left and the right.

Definition 3.2. *Let $(H, \cdot)$ be a semihypergroup and $n > 1$ be a natural number. We define the β_n relation as follows:*

$$x\beta_n y \quad \text{if there exist} \quad a_1, a_2, ..., a_n \quad \text{elements of} \quad H, \quad \text{so subsets} \quad \{x, y\} \subseteq \prod_{i=1}^{n} a_i.$$

and let

$$\beta = \bigcup_{n\geq 1} \beta_n, \quad \text{where} \quad \beta_1 = \{(x,x)/x \in H\} \quad \text{is the diagonal relation on } H.$$

Notice that relation β is reflexive and symmetric but, generally, not a transitive one.

Definition 3.3. *We denote β^* the transitive closure of relation β.*

Theorem 3.1. *β^* is the smallest strongly regular equivalence relation on H with respect to the inclusion.*

Theorem 3.2. *Let $(H,\cdot)$ be a semihypergroup (hypergroup), then the transitive closure of relation β is the smallest equivalence relation such that the quotient $H/\beta^*$is a semigroup (group).*

Definition 3.4. *β^* is called the fundamental relation on H and H/β^* is called the fundamental semigroup (group).*

Notice that [22] the term fundamental, given by Vougiouklis, is subsequent of Koskas definitions but totally used nowadays.

3.2 Vougiouklis approach

Unlike the previous ones, Vougiouklis approaches the issue in a straightforward way starting with the acquired question about the appropriate equivalence relations. He defines the relation that has as quotient set the corresponding algebraic structure. He then defines the relation β in a more general manner than previously defined. The approach has been completed by proving that the fundamental relation is no other than the transitive closure of the relation β. We can assume that Vougiouklis approach is an *inductive* way of defining the fundamental relation in hyperstructures because it starts with the partial and ends in a more general result.
It is important to note that Vougiouklis definitions were given for H_v-groups which are a wider class than the one of hypergroups. Also, the proof of theorem about β^* relation (see below) follows a remarkable strategy [3].

Taking into consideration the approach, the necessary definitions and theorems are mentioned, having as main source the book [25] and the papers [24], [26].

Definition 3.5. *Let $(H,\cdot)$ be a H_v-group. The relation β^* on H is called fundamental equivalence relation if it is the smallest equivalence relation on H such that the quotient set H/β^* is a group, called fundamental group of H.*

Notice that the proof of the fundamental groups existence for any H_v-group is an obvious result of the following theorem's proof.

Let us denote by U the set of all finite hyperproducts of elements of H.

Definition 3.6. *Let $(H, \cdot)$ be a H_v-group. We define the relation β on H as follows:*

$$x\beta y \quad \textit{iff} \quad \{x, y\} \subseteq u, u \in U.$$

Theorem 3.3. *The fundamental equivalence relation β^* is the transitive closure of the relation β on H.*

Definition 3.7. *Let $(R, +, \cdot)$ be a H_v-ring. The relation γ^* on R is called fundamental equivalence relation on R if it is the smallest equivalence relation on R such that the quotient set R/γ^* is a ring, called fundamental ring of R.*

Let us denote by U the set of all finite polynomials of elements of R, over N.

Definition 3.8. *Let $(R, +, \cdot)$ be a H_v-ring. We define the relation γ on H as follows:*

$$x\gamma y \quad \textit{iff} \quad \{x, y\} \subseteq u, u \in U.$$

Theorem 3.4. *The fundamental equivalence relation γ^* is the transitive closure of the relation γ on R.*

Definition 3.9. [26] *A H_v-ring is called H_v-field if its fundamental ring is a field.*

Definition 3.10. *Let V be a H_v-vector space over a H_v-field R. The relation ε^* on V is called fundamental equivalence relation if it is the smallest equivalence relation on V such that the quotient set V/ε^* is a vector space over the field R/γ^*, called fundamental vector space of V over R.*

4 Fundamental classes

Searching for the classes of fundamental equivalence relations is a central question in studying the fundamental structures derived from hyperstructures. This quest is intertwined with the exploration of the conditions that must be accomplished so that the β relation is transitive, that is, $\beta = \beta^*$. It is clear that the two different approaches to the fundamental equivalence relation in hyperstructures settle on two different ways of searching or constructing the fundamental equivalence classes in a hyperstructure. We could also talk, in a similar way with 3.1 and 3.2, about the *deductive* and *inductive* way of finding equivalence classes.

4.1 Complete Parts

According to the deductive way that Koskas used and Corsini's school continued, the equivalence classes, that occur when the equivalence relation is strongly regular, are used. The notion of complete part of a hyperstructure's subset plays a key role in finding the β^* class of each element. The complete closure $C(A)$ of the part A is connected with an increasing chain of subsets of the hyperstructure which, in turn, are related to hyperproducts containing A. It then turns out that the introduction, in a natural way, of the equivalence relation K is essentially a consideration of the equivalence β^*. In this way the complete closure coincides to the fundamental equivalence class.
In particular, the definition of the complete part is used in the case of the singleton $\{x\}$, for each element x of the hyperstructure, so that we find ourselves in the environment of the fundamental equivalence relation. The increasing chain of sets created by the set $\{x\}$ constructs the fundamental class of the arbitrary element x. It is evident that, as in the introduction of the β^* - relation [16], a notion is used as a mediator, which comes in between the questions "how is the class" and "what is the class". This notion is K relation.
We now present the necessary propositions in order to describe the step by step approach of the fundamental classes notion. The main references we use are the books [5], [7], [8] and the papers [13], [19].

Definition 4.1. *Let $(H, \cdot)$ be a semihypergroup and A be a nonempty subset of H. We say that A is a complete part of H if for any nonzero natural number n and for all $a_1, a_2, , a_n$ elements of H, the following implication holds:*

$$A \cap \prod_{i=1}^{n} a_i \neq \emptyset \quad \Rightarrow \quad \prod_{i=1}^{n} a_i \subseteq A.$$

Notice that complete part A absorbs every hyperproduct containing one, at least, element of A.
According to theorem 3.1, $\beta^*(x)$ is a complete part of $H, \forall x \in H$. (Step 1)

Definition 4.2. *Let $(H, \cdot)$ be a semihypergroup and A be a nonempty subset of H. The intersection of the complete parts of H which contain A is called the complete closure of A in H; it will be denoted by $C(A)$.*

Denote $K_1(A) = A$ and for all $n > 0$

$$K_{n+1}(A) =$$

$$\left\{ x \in H | \quad \exists p \in N, \exists (h_1, h_2, ..., h_p) \in H^p : x \in \prod_{i=1}^{p} (h_i), \quad K_n(A) \cap \prod_{i=1}^{p} (h_i) \neq \emptyset \right\}.$$

Obviously, $K_n(A), n > 0$ is an increasing chain of subsets of H as we mentioned. If $x \in H$, we denote $K_n(\{x\}) = K_n(x)$. This implies that

$$K_n(A) = \bigcup_{a \in A} K_n(a)$$

In particular, if P is the set of all finite hyperproducts of elements of H, and $x \in H$ we have:

$$K_1(x) = \{x\}, \quad K_2(x) = \bigcup_{u \in P} u : x \in u, \quad K_3(x) = \bigcup_{u \in P} u : u \cap K_2(x) \neq \emptyset,$$

$$..., K_{n+1}(x) = \bigcup_{u \in P} u : u \cap K_n(x) \neq \emptyset \quad \text{and} \quad C(A) = \bigcup_{a \in A} C(a), A \subseteq H.$$

Notice that $K_2(x) = \{z \in H | z\beta x\} = \beta(x)$. (Step 2)

Theorem 4.1. *Let $(H, \cdot)$ be a semihypergroup and K a binary relation defined as follows:*

$$xKy \Leftrightarrow x \in C(y), (x, y) \in H^2.$$

Then,K is an equivalence relation that coincides with β^*. (Step 3)

Thus, the relation K and the chain of sets K_n behave as a mediator, which comes in between $C(x)$ and β^*, connecting the construction of the class with the class itself.
Now we present some propositions mainly about the transitivity of β-relation.

Theorem 4.2. [12] *Let $(H, \cdot)$ be a hypergroup then, $\beta = \beta^*$.*

Theorem 4.3. *Let $(H, \cdot)$ be a hypergroupoid. Then,*

$$\beta = \beta^* \Leftrightarrow C(x) = K_2(x), \forall x \in H.$$

Theorem 4.4. [13] *Let $(H, \cdot)$ be a H_v-group having, at least, one identity. Then,*

$$\beta = \beta^*.$$

Theorem 4.5. [13] *Let $(H, \cdot)$ be a H_b-group then, $\beta = \beta^*$.*

4.2 Constructing Fundamental Classes

According to the inductive way that Vougiouklis introduced, the direct approach to the fundamental class is used. Vougiouklis, while studying the fundamental classes, follows the same philosophy and strategy as he does in his approach to the introduction of the fundamental relationship and the corresponding

structure. He does not attempt to create an environment of desirable definitions in which he will then place the fundamental class of each element, as Koskas has chosen to do.
On the contrary, he considers the fundamental class of each element as a set and tries to specify its elements through their common properties. He prefers, in short, a straightforward reference to the common behaviour of these elements in the generation of hyperproducts. Therefore, he develops propositions that relate each other the elements which behave in a similar way, thus achieving the accumulation of all the equivalent, with respect to β^*, elements which belong to the same fundamental class. It is clear that this straightforward dealing with the class study is an inductive type of answer to the question of the nature and form of the fundamental classes.
The fundamental class which is a singleton, whenever it exists, plays an essential role in the study of H_v-groups and $H_v$rings. The element of each such class is called single element. Its value lies in the fact that each hyperproduct having a single element as a factor, is an entire fundamental class. Thus, finding the classes is achieved by multiplying a single element with each element of the hyperstructure. In fact, it is not necessary to perform the hyperoperation with all the elements. Moreover, the existence of one, at least, single element is a sufficient condition such that $\beta = \beta^*$ holds in H_v-groups.
Finally, the above mentioned approach also includes the technique of "translation" of hyperproducts which allows us to find the fundamental structure of an H_v-group and its classes using isomorphism between the quotient sets. We now present the necessary propositions in order to describe the direct approach of fundamental classes. The main references we use are the book [25] and the papers [24], [26].

Theorem 4.6. *Let $(H, \cdot)$ be a $H_v$group, then*

$$x\beta^* y \quad \textit{iff there exist} \quad A, A^{'} \subseteq \beta^*(a), \quad B, B^{'} \subseteq \beta^*(b), \quad (a, b) \in H^2,$$

$$\textit{such that} \quad xA \cap B \neq \emptyset \quad \textit{and} \quad yA^{'} \cap B^{'} \neq \emptyset.$$

Theorem 4.7. *Let $(H, \cdot)$ be a H_v-group, then $u \in \omega_H$ iff there exist $A \subseteq \beta^*(a)$, for some $a \in H$, such that $uA \cap A \neq \emptyset$.*

Definition 4.3. *Let H be a H_v-structure. An element $s \in H$ is called single if $\beta^*(s) = \{s\}$.*

We denote by S_H the set of singles elements of H

Theorem 4.8. *Let $(H, \cdot)$ be a H_v-group and $s \in S_H$. Let $(a, v) \in H^2$ such that $s \in av$, then*

$$\beta^*(a) = \{h \in H : hv = s\} \quad \textit{and the core of H is} \quad \omega_H = \{z \in H : zs = s\}.$$

Theorem 4.9. *Let $(H,\cdot)$ be a H_v-group and $S_H \neq \emptyset$, then $\beta^* = \beta$.*

Definition 4.4. *(Translations)* [25] *Let $(H,\cdot)$ be a H_v-group and $x \in H$, then $H/\beta^* \cong (H/l_x)/\beta^*$, where l_x is the translation equivalence relation.*

4.3 Using the fundamental equivalence relations

The fundamental equivalence relations on hyperstructures, on the one hand, connect the theory of hyperstructures to that of the corresponding classical structures, and on the other hand, are a tool for the introduction of new hyperstructure classes. The two approaches to the concept of the fundamental equivalence relations were initially referred to semihypergroups and hypergroups. However, they form the driving lever to apply similar definitions to other hyperstructures as hyperrrings and hyperfields or to study specific behaviour of some hyperstructures using the quotient set.
Freni [11] and others (P. Corsini, B. Davazz) introduced and studied equivalence relations that refer to individual properties or characteristics of elements of a hyperstructure. As an example, we mention the equivalence relations Δ_{h^*} and α^* which are related to the cyclicity and commutativity of a hyperstructure, respectively. Our main reference is [8].
Relation α was introduced by Freni who used the letter γ instead of α. Thus, there was a confusion on symbolism since Vougiouklis had already used the letter γ for the fundamental equivalence relation on hyperrings.

Vougiouklis focused on the study of hyperstructures which have a desirable quotient.(h/v)-structures are a typical example of generalization using the fundamental structures. They are a larger class than that of H_v-structures. Also, the use of the fundamental classes of equivalence as hyperproducts, led to constructions of new hyperstructure classes. As additional examples of hyperstructures with desirable quotient we mention the $s1, s2, ..., sn$-hyperstructures [3], the complete (in the sense of Corsini) H_v-groups and the constructions S_1 and S_2 [25].

Definition 4.5. [23] *The H_v-semigroup $(H,\cdot)$ is called h/v-group if H/β^* is a group.*

Notice that the reproductivity is not necessarily valid. However, the reproductivity of classes is valid.

Definition 4.6. [1], [2]. *We shall say that a hyperstructure is an sn-hyperstructure if all its fundamental classes are singleton except for n of them, $n \in N, n > 0$.*

5 Conclusions - Findings

The aim of this paper was to present the two different approaches to the concept of the fundamental equivalence relation through a comparison of the choices and methods applied by Koskas and Vougiouklis, but also through the results they have achieved. Summarizing our conclusions and findings, we can note that:
a) The approach of Koskas is a deductive way based on the introduction of general notions willing to draw conclusions to more specific ones. On the contrary, Vougiouklis makes the reverse. Beginning with the specific notion leads to propositions generalizing his conclusions.
b) The directness of Vougiouklis approach allows for the "transfer" of definitions and corresponding theorems from the H_v-groups to H_v-rings, H_v-modules and the other weak hyperstructures. In essence, it is a widely applied model that imposed, among other things, on the terminology of the "fundamental" relations.
c) The nature of the step-by-step approach of Koskas has led to frequent use of the *mathematical induction method* in proving many basic theorems. On the other hand, due to the direct reference of Vougiouklis to the equivalence classes and their view as sets of elements, the most frequent proving method used is the *proof by contradiction*.
d) The Koskas approach raises the question: *What is the quotient set of the hyperstructure that we are studying?* That is why increasing chains of relations are the main study tool. Vougiouklis approach reverses the question with the following one: *Which hyperstructure produces a particular desired fundamental quotient?* In this inversion, there is a trigger for the introduction of weak and h/v-structures.
Finally, we consider that the mathematical value of fundamental equivalence relations in hyperstructures is important and their asynchronized approaches by Koskas and Vougiouklis is an interesting moment of the short history of algebraic hyperstructures from their beginning at 1934 until today.

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