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Special Classes of H_b-Matrices

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Abstract

In the present paper we deal with constructions of 2×2 diagonal or upper-triangular or lower-triangular H_b-matrices with entries either of an H_b-field on $\mathbb{Z}_2$ or on $\mathbb{Z}_3$. We study the kind of the hyperstructures that arise, their unit and inverse elements. Also, we focus our study on the cyclicity of these hyperstructures, their generators and the respective periods.

Keywords: hope; H_v-structure; H_b-structure; H_v-matrix
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1 Introduction

F. Marty, in 1934 [13], introduced the hypergroup as a set H equipped with a hyperoperation $\cdot : H \times H \to \mathcal{P}(H) - \{\emptyset\}$ which satisfies the associative law: (xy)z = x(yz), for all $x, y, z \in H$ and the reproduction axiom: xH = Hx = H, for all $x \in H$. In that case, the reproduction axiom is not valid, the $(H, \cdot)$ is called semihypergroup.
In 1990, T. Vougiouklis [19] in the Fourth AHA Congress, introduced the H_v-structures, a larger class than the known hyperstructures, which satisfy the weak axioms where the non-empty intersection replaces the equality.

Definition 1.1. *[21], The $(\cdot)$ in H is called weak associative, we write WASS, if*

$$(xy)z \cap x(yz) \neq \emptyset, \forall x, y, z \in H.$$

The $(\cdot)$ is called weak commutative, we write COW, if

$$xy \cap yx \neq \emptyset, \forall x, y \in H.$$

The hyperstructure $(H, \cdot)$ is called H_v-semigroup if $(\cdot)$ is WASS. It is called H_v-group if it is H_v-semigroup and the reproduction axiom is valid.
Further more, it is called H_v-commutative group if it is an H_v-group and a COW.
If the commutativity is valid, then H is called commutative H_v-group.
Analogous definitions for other H_v-structures, as H_v-rings, H_v-module, H_v-vector spaces and so on can be given.

For more definitions and applications on hyperstructures one can see books [3], [4], [5], [6], [21] and papers as [2], [7], [9], [10], [12], [14], [20], [22], [23], [24], [26], [27].
An element $e \in H$ is called *left unit* if $x \in ex, \forall x \in H$ and it is called *right unit* if $x \in xe, \forall x \in H$. It is called *unit* if $x \in ex \cap xe, \forall x \in H$. The set of left units is denoted by E^ℓ [8]. The set of right units is denoted by E^r and by $E = E^\ell \cap E^r$ the set of units [8].
The element $a\prime \in H$ is called *left inverse* of the element $a \in H$ if $e \in a\prime a$, where e unit element (left or right) and it is called *right inverse* if $e \in aa\prime$. If $e \in a\prime a \cap aa\prime$ then it is called *inverse* element of $a \in H$. The set of the left inverses is denoted by $I^\ell(a, e)$ and the set of the right inverses is denoted by $I^r(a, e)$[8]. By $I(a, e) = I^\ell(a, e) \cap I^r(a, e)$, the set of inverses of the element $a \in H$, is denoted.
In an H_v-semigroup the *powers* are defined by: $h^1 = \{h\}, h^2 = h \cdot h, \cdots, h^n = h \circ h \circ \cdots \circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H, \cdot)$ is *cyclic of period s*, if there is an h, called *generator* and a natural s, the minimum: $H = h^1 \cup h^2 \cup \cdots \cup h^s$. Analogously the cyclicity for the infinite period

is defined [17],[21]. If there is an h and s, the minimum: $H = h^s$, then $(H, \cdot)$, is called *single-power cyclic of period s*.

Definition 1.2. *The fundamental relations β^*, γ^* and ϵ^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [18],[19],[21],[22], (see also [1],[3],[4]).*

More general structures can be defined by using the fundamental structures. An application in this direction is the general hyperfield. There was no general definition of a hyperfield, but from 1990 [19] there is the following [20], [21]:

Definition 1.3. *An H_v-ring $(R, +, \cdot)$ is called H_v-field if R/γ^* is a field.*

H_v-matrix is a matrix with entries of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m \times n$ and $n \times r$ respectively, is defined in the usual manner and it is a set of $m \times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H_v-matrices is not necessarily WASS. H_v-matrices is a very useful tool in Representation Theory of H_v-groups [15],[16], [25],[28] (see also [11], [29]).

2 Constructions of 2×2 H_b-matrices with entries of an H_v-field on $\mathbb{Z}_2$

Consider the field $(\mathbb{Z}_2, +, \cdot)$. On the set $\mathbb{Z}_2$ also consider the hyperoperation $(\odot)$ defined by setting:

$$1 \odot 1 = \{0, 1\} \text{ and } x \odot y = x \cdot y \text{ for all } (x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_2 - \{(0, 1)\}.$$

Then $(\mathbb{Z}_2, +, \odot)$ becomes an H_b-field.
All the 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, are $2^4 = 16$. Let us denote them by:

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$a_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, a_5 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, a_6 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, a_7 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$a_8 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, a_9 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, a_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, a_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$a_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a_{13} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a_{14} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, a_{15} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

By taking a_i^2, $i = 1, \cdots, 15$ there exist 15 closed sets, let us say H_i , $i = 1, \cdots, 15$. Two of them are singletons, $H_2 = H_3 = \{\mathbf{0}\}$. Also, $H_7 = H_8$ and $H_{11} = H_{14} = H_{15}$.
So, we shall study, according to the hyperproduct $(\cdot)$ of two H_b-matrices, the following sets:

$$H_1 = \{\mathbf{0}, a_1\}, H_4 = \{\mathbf{0}, a_4\}, H_5 = \{\mathbf{0}, a_1, a_2, a_5\}, H_6 = \{\mathbf{0}, a_1, a_3, a_6\},$$

$$H_7 = \{\mathbf{0}, a_1, a_4, a_7\}, H_9 = \{\mathbf{0}, a_2, a_4, a_9\}, H_{10} = \{\mathbf{0}, a_3, a_4, a_{10}\},$$

$$H_{12} = \{\mathbf{0}, a_1, a_2, a_4, a_5, a_7, a_9, a_{12}\}, H_{13} = \{\mathbf{0}, a_1, a_3, a_4, a_6, a_7, a_{10}, a_{13}\},$$

$$H_{15} = \{\mathbf{0}, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}a_{11}, a_{12}, a_{13}, a_{14}, a_{15}\}.$$

2.1 The case of diagonal 2×2 H_b-matrices

Every set of H_1, H_4, H_7 consists of diagonal 2×2 H_b-matrices. Then, the multiplicative tables of the hyperproduct, are the following:

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$
$\mathbf{0}$	0	0
$\mathbf{a_1}$	0	H_1

,

$\cdot$	$\mathbf{0}$	$\mathbf{a_4}$
$\mathbf{0}$	0	0
$\mathbf{a_4}$	0	H_4

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_4}$	$\mathbf{a_7}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	0	$0, a_1$
$\mathbf{a_4}$	0	0	$0, a_4$	$0, a_4$
$\mathbf{a_7}$	0	$0, a_1$	$0, a_4$	H_7

In all cases:

$$x \cdot y = y \cdot x, \ \forall x, y \in H_i, \ i = 1, 4, 7$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \ \forall x, y, z \in H_i, \ i = 1, 4, 7$$

So, we get the next propositions:

Proposition 2.1. *Every set H, consisting of diagonal* 2×2 H_b*-matrices with entries of the* H_b*-field* $(\mathbb{Z}_2, +, \odot)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is a commutative semihypergroup.*

Notice that $H_1, H_4 \subset H_7$ and since $H_1 \cdot H_1 \subseteq H_1$, $H_4 \cdot H_4 \subseteq H_4$ then H_1, H_4 are sub-semihypergroups of $(H_7, \cdot)$.

Proposition 2.2. *For all commutative semihypergroups $(H, \cdot)$, consisting of diagonal 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$:*

$$E = \{a_i\},\ I(a_i, a_i) = \{a_i\},\ where\ a_i^2 = H.$$

Remark 2.1. *According to the above construction, the commutative semihypergroups $(H_1, \cdot), (H_4, \cdot)$ and $(H_7, \cdot)$, are single-power cyclic commutative semihypergroups with generators the elements a_1, a_4 and a_7, respectively, with single-power period 2.*

2.2 The case of upper- and lower- triangular 2×2 H_b-matrices

Every set of H_5, H_9, H_{12} consists of upper-triangular 2×2 H_b-matrices and every set of H_6, H_{10}, H_{13} consists of lower-triangular 2×2 H_b-matrices. Then, the multiplicative tables of the hyperproduct, are the following:

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_5}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	$0, a_2$	H_5
$\mathbf{a_2}$	0	0	0	0
$\mathbf{a_5}$	0	$0, a_1$	$0, a_2$	H_5

,

$\cdot$	$\mathbf{0}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_9}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_2}$	0	0	$0, a_2$	$0, a_2$
$\mathbf{a_4}$	0	0	$0, a_4$	$0, a_4$
$\mathbf{a_9}$	0	0	H_9	H_9

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_4}$	$\mathbf{a_5}$	$\mathbf{a_7}$	$\mathbf{a_9}$	$\mathbf{a_{12}}$
$\mathbf{0}$	0	0	0	0	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	$0, a_2$	0	$0, a_1, a_2, a_5$	$0, a_1$	$0, a_2$	$0, a_1, a_2, a_5$
$\mathbf{a_2}$	0	0	0	$0, a_2$	0	$0, a_2$	$0, a_2$	$0, a_2$
$\mathbf{a_4}$	0	0	0	$0, a_4$	0	$0, a_4$	$0, a_4$	$0, a_4$
$\mathbf{a_5}$	0	$0, a_1$	$0, a_2$	$0, a_2$	$0, a_1, a_2, a_5$	$0, a_1, a_2, a_5$	$0, a_2$	$0, a_1, a_2, a_5$
$\mathbf{a_7}$	0	$0, a_1$	$0, a_2$	$0, a_4$	$0, a_1, a_2, a_5$	$0, a_1, a_4, a_7$	$0, a_2, a_4, a_9$	H_{12}
$\mathbf{a_9}$	0	0	0	$0, a_2, a_4, a_9$	0	$0, a_2, a_4, a_9$	$0, a_2, a_4, a_9$	$0, a_2, a_4, a_9$
$\mathbf{a_{12}}$	0	$0, a_1$	$0, a_2$	$0, a_2, a_4, a_9$	$0, a_1, a_2, a_5$	H_{12}	$0, a_2, a_4, a_9$	H_{12}

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_3}$	$\mathbf{a_6}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	0	$0, a_1$
$\mathbf{a_3}$	0	$0, a_3$	0	$0, a_3$
$\mathbf{a_6}$	0	H_6	0	H_6

,

$\cdot$	$\mathbf{0}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{a_{10}}$
$\mathbf{0}$	0	0	0	0
$\mathbf{a_3}$	0	0	0	0
$\mathbf{a_4}$	0	$0, a_3$	$0, a_4$	H_{10}
$\mathbf{a_{10}}$	0	$0, a_3$	$0, a_4$	H_{10}

$\cdot$	$\mathbf{0}$	$\mathbf{a_1}$	$\mathbf{a_3}$	$\mathbf{a_4}$	$\mathbf{a_6}$	$\mathbf{a_7}$	$\mathbf{a_{10}}$	$\mathbf{a_{13}}$
$\mathbf{0}$	0	0	0	0	0	0	0	0
$\mathbf{a_1}$	0	$0, a_1$	0	0	$0, a_1$	$0, a_1$	0	$0, a_1$
$\mathbf{a_3}$	0	$0, a_3$	0	0	$0, a_3$	$0, a_3$	0	$0, a_3$
$\mathbf{a_4}$	0	0	$0, a_3$	$0, a_4$	$0, a_3$	$0, a_4$	$0, a_3, a_4, a_{10}$	$0, a_3, a_4, a_{10}$
$\mathbf{a_6}$	0	$0, a_1, a_3, a_6$	0	0	$0, a_1, a_3, a_6$	$0, a_1, a_3, a_6$	0	$0, a_1, a_3, a_6$
$\mathbf{a_7}$	0	$0, a_1$	$0, a_3$	$0, a_4$	$0, a_1, a_3, a_6$	$0, a_1, a_4, a_7$	$0, a_3, a_4, a_{10}$	H_{13}
$\mathbf{a_{10}}$	0	$0, a_3$	$0, a_3$	$0, a_4$	$0, a_3$	$0, a_3, a_4, a_{10}$	$0, a_3, a_4, a_{10}$	$0, a_3, a_4, a_{10}$
$\mathbf{a_{13}}$	0	$0, a_1, a_3, a_6$	$0, a_3$	$0, a_4$	$0, a_1, a_3, a_6$	H_{13}	$0, a_3, a_4, a_{10}$	H_{13}

In all cases:

$$(x \cdot y) \cap (y \cdot x) \neq \emptyset,\ \forall x, y \in H_i,\ i = 5, 6, 9, 10, 12, 13$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z),\ \forall x, y, z \in H_i,\ i = 5, 6, 9, 10, 12, 13$$

So, we get the next proposition:

Proposition 2.3. *Every set H, consisting either of upper-triangular or lower-triangular 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a weak commutative semihypergroup.*

Notice that $H_5, H_9 \subset H_{12}$ and $H_6, H_{10} \subset H_{13}$. Since $H_5 \cdot H_5 \subseteq H_5$, $H_9 \cdot H_9 \subseteq H_9$, $H_6 \cdot H_6 \subseteq H_6$, $H_{10} \cdot H_{10} \subseteq H_{10}$, then H_5, H_9 are sub-semihypergroups of $(H_{12}, \cdot)$ and H_6, H_{10} are sub-semihypergroups of $(H_{13}, \cdot)$.

Proposition 2.4. *For all weak commutative semihypergroups $(H, \cdot)$, consisting either of upper-triangular or lower-triangular 2×2 H_b-matrices with entries of the H_b-field $(\mathbb{Z}_2, +, \odot)$, the following assertions hold*
i) If $a_i, a_j \in H : a_i \cdot a_j = H,\ a_i \in a_i^2,\ a_j^2 = H,\ a_i \in a_j \cdot a_i$, then

$$a)E^{\ell} = \{a_i, a_j\},\ b)I(a_i, a_i) = I(a_j, a_i) = \{a_i, a_j\}$$

$$c)I(a_j,a_j)=I^r(a_i,a_j)=\{a_j\},\ d)I^\ell(a_i,a_j)=\emptyset$$

ii) If $a_i,a_j\in H: a_j\cdot a_i=H,\ a_i\in a_i^2,\ a_j^2=H,\ a_i\in a_i\cdot a_j$, *then*

$$a)E^r=\{a_i,a_j\},\ b)I(a_i,a_i)=I(a_j,a_i)=\{a_i,a_j\}$$

$$c)I(a_j,a_j)=I^\ell(a_i,a_j)=\{a_j\},\ d)I^r(a_i,a_j)=\emptyset$$

iii) If $a_i,a_j\in H: a_i\cdot a_j=a_j\cdot a_i=H,\ a_i\in a_i^2,\ a_j^2=H$, *then*

$$a)E=\{a_i,a_j\},\ b)I(a_i,a_i)=I(a_j,a_i)=I(a_j,a_j)=\{a_i,a_j\},\ c)I(a_i,a_j)=\{a_j\}$$

Remark 2.2. *According to the above construction, the weak commutative semihypergroups* $(H_i,\cdot)$*, i=5,6,9,10,12,13 are single-power cyclic weak commutative semihypergroups with generators the elements* $a_5,a_6,a_9,a_{10},a_{12},a_{13}$ *respectively, with single-power period 2.*

3 Constructions of 2×2 H_b-matrices with entries of an H_b-field on $\mathbb{Z}_3$

Consider the field $(\mathbb{Z}_3,+,\cdot)$. On the set $\mathbb{Z}_3$, we consider four cases for the hyperoperation $(\odot_i), i=1,2,3,4$ defined, each time, by setting:

1) $1\odot_1 2=\{1,2\}$ and $x\odot_1 y=x\cdot y$ for all $(x,y)\in\mathbb{Z}_3\times\mathbb{Z}_3-\{(1,2)\}$.

2) $2\odot_2 1=\{1,2\}$ and $x\odot_2 y=x\cdot y$ for all $(x,y)\in\mathbb{Z}_3\times\mathbb{Z}_3-\{(1,2)\}$.

3) $1\odot_3 1=\{1,2\}$ and $x\odot_3 y=x\cdot y$ for all $(x,y)\in\mathbb{Z}_3\times\mathbb{Z}_3-\{(1,2)\}$.

4) $2\odot_4 2=\{1,2\}$ and $x\odot_4 y=x\cdot y$ for all $(x,y)\in\mathbb{Z}_3\times\mathbb{Z}_3-\{(1,2)\}$.

Then, each time, $(\mathbb{Z}_3,+,\odot_i), i=1,2,3,4$ becomes an H_b-field.

Now, consider the set H of the diag(b_{11},b_{22}), $b_{11},b_{22}\in\mathbb{Z}_3$ with $b_{11}b_{22}\neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3,+,\odot_i)$. Let us denote them by:

$$a_{11}=\begin{pmatrix}1&0\\0&1\end{pmatrix},a_{12}=\begin{pmatrix}1&0\\0&2\end{pmatrix},a_{21}=\begin{pmatrix}2&0\\0&1\end{pmatrix},a_{22}=\begin{pmatrix}2&0\\0&2\end{pmatrix}.$$

So, H = $\{a_{11},a_{12},a_{21},a_{22}\}$.

3.1 The case of $1 \odot_1 2 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{11}, a_{12}	a_{11}, a_{21}	H
$\mathbf{a_{12}}$	a_{12}	a_{11}	a_{12}, a_{22}	a_{11}, a_{21}
$\mathbf{a_{21}}$	a_{21}	a_{21}, a_{22}	a_{11}	a_{11}, a_{12}
$\mathbf{a_{22}}$	a_{22}	a_{21}	a_{12}	a_{11}

Notice that in the above multiplicative table:
i) $x \cdot H = H \cdot x = H, \forall x \in H$
ii) $(x \cdot y) \cap (y \cdot x) \neq \emptyset, \forall x, y \in H$
iii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset, \forall x, y, z \in H$

So, we get the next proposition:

Proposition 3.1. *The set H, consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_1)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is an* H_v*-commutative group.*

Proposition 3.2. *For the* H_v*-commutative group (H, ·), consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_1)$:

$$i)\ E = \{a_{11}\}\ ii)\ I^r(x, a_{11}) = \{a_{22}\}, \forall x \in H\ iii)\ I^\ell(x, a_{11}) = \{a_{11}\}, \forall x \in H$$

Proposition 3.3. *The* H_v*-commutative group (H, ·), consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_1)$*, is a single-power cyclic* H_v*-commutative group with generator the element* a_{22}*, with single-power period 3.*

3.2 The case of $2 \odot_2 1 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{12}	a_{21}	a_{22}
$\mathbf{a_{12}}$	a_{11}, a_{12}	a_{11}	a_{21}, a_{22}	a_{21}
$\mathbf{a_{21}}$	a_{11}, a_{21}	a_{12}, a_{22}	a_{11}	a_{12}
$\mathbf{a_{22}}$	H	a_{11}, a_{21}	a_{11}, a_{12}	a_{11}

As in the paragraph 3.1:

Proposition 3.4. *The set H, consisting of the diag*(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ *with* $b_{11}b_{22} \neq 0$ H_b*-matrices, with entries of the* H_b*-field* $(\mathbb{Z}_3, +, \odot_2)$*, equipped with the usual hyperproduct* $(\cdot)$ *of matrices, is an* H_v*-commutative group.*

Now, take a map f onto and 1:1, $f : H \to H$, such that

$$f(a_{11}) = a_{22},\ f(a_{12}) = a_{21},\ f(a_{21}) = a_{12},\ f(a_{22}) = a_{11}$$

Then, the successive transformations of the above multiplicative table are:

$\cdot$	$\mathbf{a_{22}}$	$\mathbf{a_{21}}$	$\mathbf{a_{12}}$	$\mathbf{a_{11}}$
$\mathbf{a_{22}}$	a_{11}	a_{12}	a_{21}	a_{22}
$\mathbf{a_{21}}$	a_{11}, a_{12}	a_{11}	a_{21}, a_{22}	a_{21}
$\mathbf{a_{12}}$	a_{11}, a_{21}	a_{12}, a_{22}	a_{11}	a_{12}
$\mathbf{a_{11}}$	H	a_{11}, a_{21}	a_{11}, a_{12}	a_{11}

$\cdot$	$\mathbf{a_{22}}$	$\mathbf{a_{21}}$	$\mathbf{a_{12}}$	$\mathbf{a_{11}}$
$\mathbf{a_{11}}$	H	a_{11}, a_{21}	a_{11}, a_{12}	a_{11}
$\mathbf{a_{12}}$	a_{11}, a_{21}	a_{12}, a_{22}	a_{11}	a_{12}
$\mathbf{a_{21}}$	a_{11}, a_{12}	a_{11}	a_{21}, a_{22}	a_{21}
$\mathbf{a_{22}}$	a_{11}	a_{12}	a_{21}	a_{22}

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{11}, a_{12}	a_{11}, a_{21}	H
$\mathbf{a_{12}}$	a_{12}	a_{11}	a_{12}, a_{22}	a_{11}, a_{21}
$\mathbf{a_{21}}$	a_{21}	a_{21}, a_{22}	a_{11}	a_{11}, a_{12}
$\mathbf{a_{22}}$	a_{22}	a_{21}	a_{12}	a_{11}

Then, the last multiplicative table is the table of the paragraph 3.1. So, we get:

Proposition 3.5. *The H_v-commutative group $(H, \cdot)$ consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_2)$, is isomorphic to H_v-commutative group $(H, \cdot)$ consisting of the diag(b_{11}, b_{22}) , $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1)$.*

3.3 The case of $1 \odot_3 1 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	H	a_{12}, a_{22}	a_{21}, a_{22}	a_{22}
$\mathbf{a_{12}}$	a_{12}, a_{22}	a_{11}, a_{21}	a_{22}	a_{21}
$\mathbf{a_{21}}$	a_{21}, a_{22}	a_{22}	a_{11}, a_{12}	a_{12}
$\mathbf{a_{22}}$	a_{22}	a_{21}	a_{12}	a_{11}

Notice that in the above multiplicative table:
i) $x \cdot H = H \cdot x = H$, $\forall x \in H$

ii) $x \cdot y = y \cdot x,\ \forall x, y \in H$
iii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset,\ \forall x, y, z \in H$

So, we get the next proposition:

Proposition 3.6. *The set H, consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_3)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a commutative H_v- group.*

Proposition 3.7. *For the commutative H_v-group $(H, \cdot)$, consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_3)$:*

i) $E = E^r = E^\ell = \{a_{11}\}$ *ii)* $I(x, a_{11}) = I^r(x, a_{11}) = I^\ell(x, a_{11}) = \{x\}, \forall x \in H$

Proposition 3.8. *The commutative H_v-group $(H, \cdot)$, consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_3)$:*
i) is a single-power cyclic commutative H_v-group with generator the element a_{11}, with single-power period 2.
ii) is a single-power cyclic commutative H_v-group with generator the element a_{22}, with single-power period 4.
iii) is a cyclic commutative H_v-group of period 3 to each of the generators a_{12} and a_{21}.

3.4 The case of $2 \odot_4 2 = \{1, 2\}$

The multiplicative table of the hyperproduct, is the following:

$\cdot$	$\mathbf{a_{11}}$	$\mathbf{a_{12}}$	$\mathbf{a_{21}}$	$\mathbf{a_{22}}$
$\mathbf{a_{11}}$	a_{11}	a_{12}	a_{21}	a_{22}
$\mathbf{a_{12}}$	a_{12}	a_{11}, a_{12}	a_{22}	a_{21}, a_{22}
$\mathbf{a_{21}}$	a_{21}	a_{22}	a_{11}, a_{21}	a_{12}, a_{22}
$\mathbf{a_{22}}$	a_{22}	a_{21}, a_{22}	a_{12}, a_{22}	H

Notice that in the above multiplicative table:
i) $x \cdot H = H \cdot x = H,\ \forall x \in H$
ii) $x \cdot y = y \cdot x,\ \forall x, y \in H$
iii) $(x \cdot y) \cdot z = x \cdot (y \cdot z),\ \forall x, y, z \in H$

So, we get the next proposition:

Proposition 3.9. *The set H, consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_4)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a commutative hypergroup.*

Proposition 3.10. *For the commutative hypergroup $(H, \cdot)$, consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_4)$:*

$$i)\ E = \{a_{11}\}\ ii)\ I(x, a_{11}) = \{x\}, \forall x \in H$$

Proposition 3.11. *The commutative hypergroup $(H, \cdot)$, consisting of the diag(b_{11}, b_{22}), $b_{11}, b_{22} \in \mathbb{Z}_3$ with $b_{11}b_{22} \neq 0$ H_b-matrices, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_4)$ is a single-power cyclic commutative hypergroup with generator the element a_{22}, with single-power period 2.*

4 Construction of 2×2 upper-triangular H_b-matrices with entries of an H_b-field on $\mathbb{Z}_3$

On the set $\mathbb{Z}_3$, consider the hyperoperation $(\odot_1)$ defined, by setting:

$1 \odot_1 2 = \{1, 2\}$ and $x \odot_1 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$

Now, consider the set H of the 2×2 upper-triangular H_b-matrices with $b_{11}, b_{22} \in \mathbb{Z}_3$ and $b_{11}b_{22} \neq 0$, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1)$. Let us denote the elements of H by:

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, a_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, a_4 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix},$$

$$a_5 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, a_6 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, a_7 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, a_8 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

$$a_9 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, a_{10} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, a_{11} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, a_{12} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$$

So, H = $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}\}$.
Since the multiplicative table is long enough, it is omitted. From this table we get:
i) $x \cdot H = H \cdot x = H,\ \forall x \in H$
ii) $(\cdot)$ is non-commutative
iii) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset,\ \forall x, y, z \in H$

So, we get the next proposition:

Proposition 4.1. *The set H, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}, b_{22} \in \mathbb{Z}_3$ and $b_{11}b_{22} \neq 0$, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a non-commtative H_v-group.*

Proposition 4.2. *For the non-commtative H_v-group $(H,\cdot)$, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}, b_{22} \in \mathbb{Z}_3$ and $b_{11}b_{22} \neq 0$, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1) : E = E^\ell = E^r = \{a_1\}, \forall x \in H$.*

Proposition 4.3. *The non-commtative H_v-group $(H,\cdot)$, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}, b_{22} \in \mathbb{Z}_3$ and $b_{11}b_{22} \neq 0$, with entries of the H_b-field $(\mathbb{Z}_3, +, \odot_1)$:*
i) is a single-power cyclic non-commutative H_v-group with generator the element a_{12}, with single-power period 4.
ii) is a single-power cyclic non-commutative H_v-group with generator the element a_{10}, with single-power period 3.

Now, take any H_b-field $(\mathbb{Z}_p, +, \odot_1)$, $p = prime \neq 2$ and then consider a set H consisting of the 2×2 upper-triangular H_b-matrices with entries of this H_b-field, with $b_{11}b_{22} \neq 0$, $b_{11}, b_{22} \in \mathbb{Z}_p$.
Then, for any such a set $\mathbb{Z}_p$, take for example the elements $a_3, a_7 \in H$, then:

$$a_7 \cdot a_3 = a_{11} \; and \; a_3 \cdot a_7 = \{a_1, a_7\}$$

So, we get the next general proposition:

Proposition 4.4. *Any set H, consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}b_{22} \neq 0$, $b_{11}, b_{22} \in \mathbb{Z}_p$, $p = prime \neq 2$, with entries of the H_b-field $(\mathbb{Z}_p, +, \odot_1)$, equipped with the usual hyperproduct $(\cdot)$ of matrices, is a non-commutative hyperstructure.*

Remark 4.1. *The above proposition means that, the **minimum non-commutative** H_v- group, equipped with the usual hyperproduct $(\cdot)$ of matrices and consisting of the 2×2 upper-triangular H_b-matrices with $b_{11}b_{22} \neq 0$, is that with entries of the H_b-field $(\mathbb{Z}_p, +, \odot_1)$, where $1 \odot_1 2 = \{1, 2\}$ and $x \odot_1 y = x \cdot y$ for all $(x, y) \in \mathbb{Z}_3 \times \mathbb{Z}_3 - \{(1, 2)\}$.*

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