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On a Functional Equation Related to Information Theory

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Abstract

The main aim of this paper is to obtain the general solutions of the functional equation (1.3) without imposing any regularity condition on the mappings appearing in it. To do so, the general solutions of the functional equation (1.5), without imposing any regularity condition on the mappings appearing in it are needed. To meet this need, the general solutions of the functional equation (1.6) without imposing any regularity condition on a mapping appearing have to be investigated. One solution of (1.3) is useful in information theory. Thus, indeed, is the reason to consider (1.3).
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1 Introduction

For $n = 1, 2, \ldots$, let $\Gamma_n = \{(p_1, \ldots, p_n) : 0 \leqslant p_i \leqslant 1, i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1\}$, denote the set of all discrete n-component complete probability distributions with non-negative elements. Let $I = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\} = [0, 1]$, $\mathbb{R}$ denoting the set of all real numbers.

The axiomatic characterization of the non-additive entropy of degree α (see [2]) defined as

$$H_n^{\alpha}(p_1, \ldots, p_n) = (2^{1-\alpha} - 1)^{-1} \left(\sum_{i=1}^{n} p_i^{\alpha} - 1 \right), \quad \alpha \neq 1$$

leads to the study of the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + \lambda \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \qquad (1.1)$$

in which $f : I \to \mathbb{R}$ is an unknown mapping, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $\lambda \neq 0$, $\lambda \in \mathbb{R}$ and n, m being positive integers.

By a general solution of a functional equation, we mean a solution obtained without imposing any condition such as differentiability, continuity, continuity at a point, measurability, boundedness, monotonicity etc on a(the) mapping(s) appearing in the functional equation under consideration.

The general solutions of (1.1), for fixed integers $n \geqslant 3$, $m \geqslant 3$ and $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ have been obtained in [5].

Losonczi [4] considered the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(p_i q_j) = \sum_{i=1}^{n} h_i(p_i) + \sum_{j=1}^{m} k_j(q_j) + \lambda \sum_{i=1}^{n} h_i(p_i) \sum_{j=1}^{m} k_j(q_j) \qquad (1.2)$$

with $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $\lambda \neq 0$, $\lambda \in \mathbb{R}$, $f_{ij} : I \to \mathbb{R}$, $h_i : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $i = 1, \ldots, n$; $j = 1, \ldots, m$, as unknown mappings. He found the measurable (in the sense of Lebesgue) solutions of (1.2) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ by taking $n \geqslant 3$, $m \geqslant 3$ as fixed integers, in Theorem 6 on p-69 in [4]. For the last more than two decades, the general solutions of (1.2) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $n \geqslant 3$, $m \geqslant 3$ being fixed integers, **are still not known so far**.

The main aim of this paper is to obtain the general solutions of the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i q_j) = \sum_{i=1}^{n} h(p_i) + \sum_{j=1}^{m} k_j(q_j) + \lambda \sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} k_j(q_j) \qquad (1.3)$$

which contains $m+1$ unknown real-valued mappings h and k_j ($j=1,\ldots,m$), each defined on $I=[0,1]$; $\lambda\in\mathbb{R}$, $\lambda\neq 0$ and $n\geqslant 3$, $m\geqslant 3$ being fixed integers. These general solutions have been obtained without making use of the difference operator D_i^r suggested on p-58 by Losonczi [4]. This paper is an improved version of the manuscript [9]. Nath and Singh [8] have also obtained the general solutions of

$$\sum_{i=1}^{n}\sum_{j=1}^{m}F(p_iq_j)=\sum_{i=1}^{n}G(p_i)+\sum_{j=1}^{m}H_j(q_j)+\lambda\sum_{i=1}^{n}G(p_i)\sum_{j=1}^{m}H_j(q_j)$$

with $F:I\to\mathbb{R}$, $G:I\to\mathbb{R}$, $H_j:I\to\mathbb{R}$, $j=1,\ldots,m$; $\lambda\neq 0$, $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$, $n\geqslant 3$, $m\geqslant 3$ being fixed integers.

The functional equation (1.3) is a special case of (1.2). A particular case of (1.3) is the following

$$\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)=\sum_{i=1}^{n}h(p_i)+\sum_{j=1}^{m}k(q_j)+\lambda\sum_{i=1}^{n}h(p_i)\sum_{j=1}^{m}k(q_j)$$

in which $h:I\to\mathbb{R}$, $k:I\to\mathbb{R}$ and $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$. Nath and Singh [7] have obtained its general solution(s) for fixed integers $n\geqslant 3$, $m\geqslant 3$.

Let us define the mappings $f:I\to\mathbb{R}$ and $g_j:I\to\mathbb{R}$, $j=1,\ldots,m$ as

$$f(x)=x+\lambda\,h(x);\qquad g_j(x)=x+\lambda\,k_j(x) \tag{1.4}$$

for all $x\in I$. Then (1.3) reduces to the Pexider type functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m}f(p_iq_j)=\sum_{i=1}^{n}f(p_i)\sum_{j=1}^{m}g_j(q_j)\,. \tag{1.5}$$

We would like to mention that Kannappan and Sahoo [3] have obtained the general solutions of (1.3) and (1.5) on an open domain. In our case, the process of finding the general solutions of (1.5), for fixed integers $n\geqslant 3$, $m\geqslant 3$, needs determining the general solutions of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi(p_iq_j)=\sum_{i=1}^{n}\varphi(p_i)\sum_{j=1}^{m}\varphi(q_j)+m(n-1)\,\varphi(0)\sum_{i=1}^{n}\varphi(p_i) \tag{1.6}$$

where $\varphi:I\to\mathbb{R}$ and $n\geqslant 3$, $m\geqslant 3$ are fixed integers. This task has been accomplished in section 3. The corresponding general solutions of (1.5) and (1.3) have been investigated in sections 4 and 5 respectively. At the end of section 5, we have analysed the importance of the solutions of functional equation (1.3) from information-theoretic point of view. Section 2 contains some known definitions and results needed for the subsequent development of this paper.

2 Some preliminary results

In this section, we mention some known definitions and results.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle $\Delta = \{(x, y) : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant x + y \leqslant 1\}$ if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$. It is known [1] that if a mapping $a : I \to \mathbb{R}$ is additive on I, then it has a unique additive extension $A : \mathbb{R} \to \mathbb{R}$ in the sense that A is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative if $M(pq) = M(p)\,M(q)$ holds for all $p \in I$, $q \in I$.

Result 2.1. *[5] Let $n \geqslant 3$ be a fixed integer and c be a given constant. Suppose that a mapping $\psi : I \to \mathbb{R}$ satisfies the functional equation $\sum\limits_{i=1}^{n} \psi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. Then there exists an additive mapping $b : \mathbb{R} \to \mathbb{R}$ such that $\psi(p) = b(p) - \dfrac{1}{n}\, b(1) + \dfrac{c}{n}$ for all $p \in I$.*

Result 2.2. *[4] Let d be a given real constant and $\psi_j : I \to \mathbb{R}$, $j = 1, \ldots, m$, be mappings which satisfy the functional equation $\sum\limits_{j=1}^{m} \psi_j(q_j) = d$ for all $(q_1, \ldots, q_m) \in \Gamma_m$, $m \geqslant 3$ being a fixed integer. Then there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ and real constants c_j $(j = 1, \ldots, m)$ such that $\psi_j(p) = a(p) + c_j$ for all $p \in I$ with $a(1) + \sum\limits_{j=1}^{m} c_j = d$.*

3 The functional equation (1.6)

In this section, we prove:

Theorem 3.1. Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $\varphi : I \to \mathbb{R}$ be a mapping which satisfies the functional equation *(1.6)* for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Then φ is of the form

$$\varphi(p) = a(p) + \varphi(0) \tag{3.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$\begin{cases} \text{(i)} \quad a(1) = -\,nm\,\varphi(0) \quad \text{if } \varphi(1) + (n-1)\,\varphi(0) \neq 1 \\ \qquad\qquad \text{or} \\ \text{(ii)} \quad a(1) = 1 - n\,\varphi(0) \quad \text{if } \varphi(1) + (n-1)\,\varphi(0) = 1 \end{cases} \tag{3.2}$$

or

$$\varphi(p) = M(p) - B(p) \tag{3.3}$$

where $B : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $B(1) = 0$ and $M : I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$.

Proof. Let us put $p_1 = 1$, $p_2 = \ldots = p_n = 0$ in (1.6). We obtain

$$[\varphi(1) + (n-1)\,\varphi(0) - 1] \left[\sum_{j=1}^{m} \varphi(q_j) + m(n-1)\,\varphi(0) \right] = 0 \tag{3.4}$$

for all $(q_1, \ldots, q_m) \in \Gamma_m$. We divide our discussion into two cases.

Case 1. $\varphi(1) + (n-1)\,\varphi(0) \neq 1$.

In this case, (3.4) reduces to $\sum\limits_{j=1}^{m} \varphi(q_j) = -\,m(n-1)\,\varphi(0)$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that φ is of the form (3.1) with $a(1)$ as in (3.2)(i). Thus, we have obtained the solution (3.1) satisfying (i) in (3.2).

Case 2. $\varphi(1) + (n-1)\,\varphi(0) - 1 = 0$.

Let us write (1.6) in the form

$$\sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} \varphi(p_i q_j) - \varphi(q_j) \sum_{i=1}^{n} \varphi(p_i) - m(n-1)\varphi(0) q_j \sum_{i=1}^{n} \varphi(p_i) \right\} = 0\,.$$

By Result 2.1, there exists a mapping $A_1 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable, such that

$$\sum_{i=1}^{n} \varphi(p_i q) - \varphi(q) \sum_{i=1}^{n} \varphi(p_i) - m(n-1)\,\varphi(0)\,q \sum_{i=1}^{n} \varphi(p_i) \tag{3.5}$$
$$= \quad A_1(p_1, \ldots, p_n; q) - \varphi(0) \sum_{i=1}^{n} \varphi(p_i) + n\,\varphi(0)$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $q \in I$ with

$$A_1(p_1, \ldots, p_n; 1) = m\,\varphi(0) \left[\sum_{i=1}^{n} \varphi(p_i) - n \right]. \tag{3.6}$$

Let $x \in I$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Putting successively $q = xr_t$, $t = 1, \ldots, n$ in (3.5), adding the resulting n equations so obtained and then substituting the value of $\sum\limits_{t=1}^{n} \varphi(xr_t)$ calculated from (3.5), we get the equation

$$\sum_{i=1}^{n} \sum_{t=1}^{n} \varphi(xp_i r_t) - [\varphi(x) + m(n-1)\,\varphi(0)\,x - \varphi(0)] \tag{3.7}$$
$$\times \sum_{i=1}^{n} \varphi(p_i) \sum_{t=1}^{n} \varphi(r_t) - n^2\,\varphi(0)$$
$$= A_1(p_1, \ldots, p_n; x) + m(n-1)\,\varphi(0)\,x \sum_{i=1}^{n} \varphi(p_i)$$
$$+ A_1(r_1, \ldots, r_n; x) \sum_{i=1}^{n} \varphi(p_i)\,.$$

The symmetry of the left hand side of (3.7), in p_i and r_t, $i = 1, \ldots, n$; $t = 1, \ldots, n$ gives rise to the equation

$$[A_1(p_1, \ldots, p_n; x) + m(n-1)\,\varphi(0)\,x] \left[\sum_{t=1}^{n} \varphi(r_t) - 1 \right] \tag{3.8}$$
$$= \quad [A_1(r_1, \ldots, r_n; x) + m(n-1)\,\varphi(0)\,x] \left[\sum_{i=1}^{n} \varphi(p_i) - 1 \right].$$

Case 2.1. $\sum\limits_{t=1}^{n} \varphi(r_t) - 1$ vanishes identically on Γ_n.

In this case, by Result 2.1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that φ is of the form (3.1) but now $a(1)$ is as in (3.2)(ii).

Case 2.2. $\sum_{t=1}^{n} \varphi(r_t) - 1$ does not vanish identically on Γ_n.

Then, there exists a probability distribution $(r_1^*, \ldots, r_n^*) \in \Gamma_n$ such that

$$\left[\sum_{t=1}^{n} \varphi(r_t^*) - 1\right] \neq 0\,. \tag{3.9}$$

Setting $r_t = r_t^*$, $t = 1, \ldots, n$ in (3.8) and using (3.9), we obtain the equation

$$A_1(p_1, \ldots, p_n; x) = B(x) \left[\sum_{i=1}^{n} \varphi(p_i) - 1\right] - m(n-1)\,\varphi(0)\,x \tag{3.10}$$

where $B : \mathbb{R} \to \mathbb{R}$ is defined as $B(x) = \left[\sum_{t=1}^{n} \varphi(r_t^*) - 1\right]^{-1} [A_1(r_1^*, \ldots, r_n^*; x) + m(n-1)\,\varphi(0)\,x]$ for all $x \in \mathbb{R}$. It can be easily verified that $B : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $B(1) = m\,\varphi(0)$. From (3.5), (3.10), $B(1) = m\,\varphi(0)$ and the additivity of $B : \mathbb{R} \to \mathbb{R}$, it follows that

$$\sum_{i=1}^{n} [M(p_i q) - M(q)M(p_i) + n(m-1)\,\varphi(0)\,M(q)\,p_i] = 0 \tag{3.11}$$

where $M : I \to \mathbb{R}$ is defined as

$$M(x) = \varphi(x) + B(x) + m(n-1)\varphi(0)x - \varphi(0) \tag{3.12}$$

for all $x \in I$. From (3.12), it is easy to see that $M(0) = 0$ as $B(0) = 0$. Applying Result 2.1 on (3.11), there exists a mapping $E : \mathbb{R} \times I \to \mathbb{R}$, additive in the first variable such that

$$M(pq) - M(p)M(q) + n(m-1)\,\varphi(0)\,M(q)\,p = E(p, q) - \frac{1}{n}\,E(1, q) \tag{3.13}$$

for all $p \in I$, $q \in I$. The substitution $p = 0$ in (3.13) and the use of $M(0) = 0$ gives $E(1, q) = 0$ for all $q \in I$. Consequently,

$$M(pq) - M(p)M(q) + n(m-1)\,\varphi(0)\,M(q)\,p = E(p, q) \tag{3.14}$$

for all $p \in I$, $q \in I$. Since $E(1,q) = 0$, therefore $E(1,1) = 0$. Now, putting $p = q = 1$ in equation (3.14), we obtain

$$M(1)[1 - M(1) + n(m-1)\varphi(0)] = 0\,. \tag{3.14a}$$

We prove that $M(1) \neq 0$. To the contrary, suppose that $M(1) = 0$. Putting $q = 1$ in (3.14) and using $M(1) = 0$, we get $M(p) = E(p,1)$ for all $p \in I$. So, M is additive on I. Also, if we put $x = 1$ in (3.12), use $M(1) = 0$ and $\varphi(1) + (n-1)\varphi(0) = 1$, we obtain $n(m-1)\varphi(0) = -1$. Now from (3.9), (3.12) and the additivity of M on I, we have $1 \neq \sum\limits_{t=1}^{n} \varphi(r_t^*) = 1$ a contradiction. Hence $M(1) \neq 0$. Now, from (3.14a), it follows that

$$M(1) - 1 = n(m-1)\varphi(0)\,. \tag{3.15}$$

Our next task is to prove that $M : I \to \mathbb{R}$, defined by (3.12), is not additive. To the contrary, suppose that M is additive. Now from (3.9), (3.12), the additivity of M on I and (3.15), we have

$$1 \neq \sum_{t=1}^{n} \varphi(r_t^*) = M(1) - n(m-1)\varphi(0) = 1$$

a contradiction. Hence $M : I \to \mathbb{R}$ is not additive.

Now we prove that, indeed, $M(1) - 1 = 0$. If possible, suppose $[M(1) - 1] \neq 0$. Putting $q = 1$ in (3.14) and using (3.15), we obtain

$$[M(1)p - M(p)] = [M(1) - 1]^{-1} E(p,1)$$

for all $p \in I$. Since $p \longmapsto E(p,1)$ is additive on I, it follows that $p \longmapsto M(1)p - M(p)$ must also be additive on I. But $p \longmapsto M(1)p$ is additive on I. Hence M is additive on I contradicting the fact that M is not additive. Hence $M(1) - 1 = 0$, that is, $M(1) = 1$.

Now, from (3.15), it follows that $\varphi(0) = 0$. Consequently, equation (3.14) reduces to the equation

$$M(pq) - M(p)\,M(q) = E(p,q) \tag{3.16}$$

for all $p \in I$, $q \in I$ and (3.12) reduces to (3.3) for all $p \in I$ with $B(1) = 0$.

The left hand side of (3.16) is symmetric in p and q. Hence $E(p,q) = E(q,p)$ for all $p \in I$, $q \in I$. Consequently, E is also additive on I in the second variable. We may assume that $E(p,\cdot)$ has been extended additively to the whole of $\mathbb{R}$.

Let $p \in I$, $q \in I$, $r \in I$. From (3.16), we have

$$\begin{aligned} E(pq,r) + M(r)\,E(p,q) &= M(pqr) - M(p)\,M(q)\,M(r) \\ &= E(qr,p) + M(p)\,E(q,r)\,. \end{aligned} \tag{3.17}$$

We prove that $E(p,q) = 0$ for all $p \in I$, $q \in I$. If possible, suppose there exists a $p^* \in I$ and a $q^* \in I$ such that $E(p^*,q^*) \neq 0$. Then, (3.17) gives

$$M(r) = [E(p^*,q^*)]^{-1}\{E(q^*r,p^*) + M(p^*)E(q^*,r) - E(p^*q^*,r)\}$$

from which it follows that M is additive on I contradicting the fact that M is not additive. Hence $E(p,q) = 0$ for all $p \in I$, $q \in I$. Now, from (3.16), it follows that $M(pq) = M(p)\,M(q)$ for all $p \in I$, $q \in I$. Thus, $M : I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$. □

4 The functional equation (1.5)

In this section, we prove:

Theorem 4.1. Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $f : I \to \mathbb{R}$, $g_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ be mappings which satisfy the functional equation *(1.5)* for all

$(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Then, any general solution of *(1.5)*, for all $p \in I$, is of the form

$$\begin{cases} f(p) = b(p) \\ g_j \text{ any arbitrary real-valued mapping} \end{cases} \tag{4.1}$$

or

$$\begin{cases} f(p) = [f(1) + (n-1)\, f(0)]\, a(p) + f(0), \\ \qquad\qquad\qquad\qquad [f(1) + (n-1)\, f(0)] \neq 0 \\ g_j(p) = a(p) + A^*(p) + g_j(0) \end{cases} \tag{4.2}$$

or

$$\begin{cases} f(p) = f(1)[M(p) - B(p)]\,, & f(1) \neq 0 \\ g_j(p) = M(p) - B(p) + A^*(p) + g_j(0) \end{cases} \tag{4.3}$$

where $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$, $A^* : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}$ are additive mappings with

$$\begin{cases} \text{(i)} & b(1) = 0 \\ \text{(ii)} & B(1) = 0 \\ \text{(iii)} & a(1) = 1 - nf(0)[f(1) + (n-1)f(0)]^{-1} \\ \text{(iv)} & A^*(1) = -\sum\limits_{j=1}^{m} g_j(0) + nm\, f(0)[f(1) + (n-1)f(0)]^{-1} \end{cases} \tag{4.4}$$

and $M : I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$.

Proof. Put $p_1 = 1$, $p_2 = \ldots = p_n = 0$ in (1.5). We obtain

$$\sum_{j=1}^{m} [f(q_j) + (n-1)f(0)] = [f(1) + (n-1)\, f(0)] \sum_{j=1}^{m} g_j(q_j) \tag{4.5}$$

for all $(q_1, \ldots, q_m) \in \Gamma_m$.

Case 1. $f(1) + (n-1)\, f(0) = 0$.

Then, (4.5) reduces to the equation $\sum\limits_{j=1}^{m} f(q_j) = -\, m(n-1)\, f(0)$. Put $q_1 = 1$, $q_2 = \ldots = q_m = 0$ in this equation and using the fact $f(1) + (n-1)\, f(0) = 0$, we have

$f(0) = 0 = f(1)$. Hence $\sum_{j=1}^{m} f(q_j) = 0$. By Result 2.1, there exists an additive mapping $b : \mathbb{R} \to \mathbb{R}$ such that $f(p) = b(p)$ with $b(1) = 0$. Consequently, for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, it is easy to verify that $\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) = b(1) = 0$. Now, from (1.5), it follows that g_j ($j = 1, \ldots, m$) are, indeed, arbitrary real-valued mappings. Thus, we have obtained the solution (4.1) of (1.5) where $b(1)$ is given by (4.4)(i).

Case 2. $f(1) + (n-1)\, f(0) \neq 0$.

In this case, (4.5) can be written in the form

$$\sum_{j=1}^{m} \left\{ g_j(q_j) - [f(1) + (n-1)\, f(0)]^{-1} [f(q_j) + (n-1)\, f(0)] \right\} = 0\,. \tag{4.6}$$

By Result 2.2, there exists an additive mapping $A^* : \mathbb{R} \to \mathbb{R}$ such that

$$g_j(p) = [f(1) + (n-1)\, f(0)]^{-1} [f(p) - f(0)] + A^*(p) + g_j(0) \tag{4.7}$$

for $j = 1, \ldots, m$ with $A^*(1)$ given by (iv) in (4.4). The elimination of $\sum_{j=1}^{m} g_j(q_j)$ from equations (1.5) and (4.6) gives the equation

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) &= [f(1) + (n-1)\, f(0)]^{-1} \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \\ &\quad + [f(1) + (n-1)\, f(0)]^{-1} m(n-1)\, f(0) \sum_{i=1}^{n} f(p_i) \end{aligned} \tag{4.8}$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Define a mapping $\varphi : I \to \mathbb{R}$ as

$$\varphi(x) = [f(1) + (n-1)\, f(0)]^{-1}\, f(x) \tag{4.9}$$

for all $x \in I$. Then (4.8) reduces to the functional equation (1.6) with $\varphi(1) + (n-1)\, \varphi(0) = 1$. So, we need to consider only those solutions of (1.6) which satisfy the requirement $\varphi(1) + (n-1)\, \varphi(0) = 1$.

The solutions (3.1) (with (3.2)(ii)) and (3.3) of (1.6) satisfy the condition $\varphi(1) + (n-1)\,\varphi(0) = 1$. Making use of (4.9), (4.7), (3.1) (with (3.2)(ii)) and (3.3), the solutions (4.2) and (4.3) can be obtained in which $B(1)$, $a(1)$ and $A^*(1)$ are given respectively by (ii), (iii) and (iv) in (4.4). $\square$

5 The functional equation (1.3)

In this section, we prove:

Theorem 5.1. Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $h : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ be mappings which satisfy the functional equation *(1.3)* for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$ and $\lambda \neq 0$. Then, any general solution of *(1.3)*, for all $p \in I$, is of the form

$$\begin{cases} h(p) = \dfrac{1}{\lambda}\,[b(p) - p] \\ k_j \text{ any arbitrary real-valued mapping} \end{cases} \tag{5.1}$$

or

$$\begin{cases} h(p) = \dfrac{1}{\lambda}\big\{[\lambda(h(1) + (n-1)\,h(0)) + 1]\,a(p) - p\big\} + h(0), \\ \qquad\qquad\qquad [\lambda(h(1) + (n-1)\,h(0)) + 1] \neq 0 \\ k_j(p) = \dfrac{1}{\lambda}\big\{a(p) + A^*(p) - p\big\} + k_j(0) \end{cases} \tag{5.2}$$

or

$$\begin{cases} h(p) = \dfrac{1}{\lambda}\,\big\{[\lambda\,h(1) + 1][M(p) - B(p)] - p\big\}\,, \quad [\lambda\,h(1) + 1] \neq 0 \\ k_j(p) = \dfrac{1}{\lambda}\,\big\{M(p) - B(p) + A^*(p) - p\big\} + k_j(0) \end{cases} \tag{5.3}$$

where $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$, $A^* : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}$ are additive mappings with

$$\begin{cases} \text{(i)} \quad b(1) = 0 \\ \text{(ii)} \quad B(1) = 0 \\ \text{(iii)} \quad a(1) = 1 - \lambda n\, h(0)[\lambda(h(1) + (n-1)h(0)) + 1]^{-1} \\ \text{(iv)} \quad A^*(1) = -\lambda \sum_{j=1}^{m} k_j(0) + \lambda nmh(0)[\lambda(h(1) + (n-1)h(0)) + 1]^{-1} \end{cases} \tag{5.4}$$

and $M : I \to \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$.

Proof. From (1.4) and the solutions of the functional equation (1.5) i.e., (4.1), (4.2), (4.3) with (4.4); we obtain respectively the solutions (5.1), (5.2), (5.3) with (5.4); of the functional equation (1.3). The details are omitted. □

Remarks. The object of this remark is to point out the importance of various solutions of Theorem 5.1 from information-theoretic point of view.

1. The summand $\sum_{i=1}^{n} h(p_i)$ of the mapping h appearing in (5.1) is independent of the probabilities $p_1, \ldots, p_n$. The solution (5.1) may be of some importance in information theory provided k_j is chosen as a suitable mapping of probability p, $p \in I$.

2. In solution (5.2), the summands $\sum_{i=1}^{n} h(p_i)$ and $\sum_{j=1}^{m} k_j(q_j)$ are independent of the probabilities $p_1, \ldots, p_n$ and $q_1, \ldots, q_m$ respectively. So, this solution does not seem to be of any relevance in information theory.

3. In solution (5.3)

$$\sum_{i=1}^{n} h(p_i) = \frac{1}{\lambda}\{\beta_1 \sum_{i=1}^{n} M(p_i) - 1\}$$

and

$$\sum_{j=1}^{m} k_j(q_j) = \frac{1}{\lambda}\Big\{\sum_{j=1}^{m} M(q_j) - 1\Big\} + \beta_2$$

where

$$\beta_1 = \lambda h(1) + 1$$

$$\beta_2 = nm\, h(0)[\lambda(h(1) + (n-1)h(0)) + 1]^{-1}\,.$$

If $\beta_1 = 1$ and $\beta_2 = 0$, then $\sum\limits_{i=1}^{n} h(p_i) = L_n^{\lambda}(p_1, \dots, p_n)$ and $\sum\limits_{j=1}^{m} k_j(q_j) = L_m^{\lambda}(q_1, \dots, q_m)$ where (see Nath and Singh [6])

$$L_t^{\lambda}(x_1, \dots, x_t) = \frac{1}{\lambda}\Big[\sum_{i=1}^{t} M(x_i) - 1\Big]. \tag{5.5}$$

The non-additive measure of entropy $H_t^{\alpha}(x_1, \dots, x_t) = (2^{1-\alpha} - 1)^{-1}(\sum\limits_{i=1}^{t} x_i^{\alpha} - 1)$, $\alpha \neq 1$, is a particular case of (5.5) when $\lambda = 2^{1-\alpha} - 1$, $\alpha > 0$, $\alpha \neq 1$ and $M : I \to \mathbb{R}$ is of the form $M(p) = p^{\alpha}$, $p \in I$, $\alpha \neq 1$, $\alpha > 0$, $0^{\alpha} := 0$, $1^{\alpha} := 1$.

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