Ratio Mathematica **ISSN**: 1592-7415
Vol. 31, 2016, pp. 93--110 **eISSN**: 2282-8214

Groups of Transformations with a Finite Number of Isometries: the Cases of Tetrahedron and Cube

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Received on: 3-11-2016. **Accepted** on: 11-11-2017. **Published** on: 28-02-2017

doi: 10.23755/rm.v31i0.322

Abstract

This paper deals with groups of transformations with finite number of isometries and extends previous studies (Casolaro, F. L. Cirillo and R. Prosperi 2015) which are related to endless groups of transformations with isometrics. In particular, isometries of the tetrahedron and cube, which turn these figures in itself, are presented.

Keywords: Geometric transformations, isometries, symmetry.

2010 AMS subject classification: 97G50; 51N25.

1.Introduction

Compared with the operation of product of isometries, in previous studies, we presented some examples of infinite groups of transformations, whose we highlighted the following properties:

- *The isometries of the space form a group.*
- *The direct isometries of the space form a group, subgroup of the previous group.*
- *The translations of the space form a group, subgroup of the group of direct isometries.*
- *Rotations around a straight form a group, subgroup of direct isometries.*
- *The helical movements all having the same axis form a group, subgroup of the group of direct isometries. In this case, since the helical movements turn out to be products of rotations for translations having the direction of the axis of rotation, also translations (the rotation is reduced to the identity) and rotations (the translation is reduced to the identity) may be considered helical movements.*

It is also possible to obtain groups of transformation with a finite number of isometries.

In particular: about the tetrahedron, we show the *axial symmetry* μ having as an axis line *r*, *rotations* ρ of 120 ° and 240 ° around the height of the tetrahedron outgoing from a fixed vertex, *planar symmetry* σ relative to the plan π passing through two vertices of the tetrahedron and through the midpoint of the edge that joins the other two vertices; about the cube, *rotations* ρ around a line *r* connecting the centers of two opposite faces, *rotations* ρ around the line *r* joining the midpoints of two opposite edges, *planar symmetry* σ relative to the plan π passing through two vertices of the tetrahedron and through the midpoint of the edge that joins the other two vertices, *planar symmetry* σ relative to the pane π parallel to two faces passing through the midpoints of the four edges perpendicular to these two faces, *planar symmetries* σ relative to the pane π passing through two opposite edges that do not have face in common and a vertex in common.

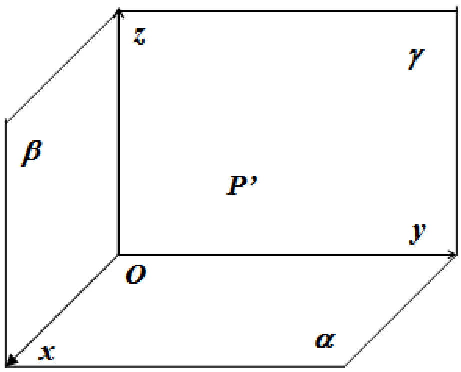

Figure 1

Consider three straight lines x, y, z, passing through the same point O and perpendicular to each other two by two. The three planes α, β, γ, respectively determined by the straight lines x and y, x and z, and y and z, are also perpendicular to each other two by two (Figure 1).

Let be:

I the identity,
s_x the axial symmetry having as an axis the line *x*,
s_y the axial symmetry having as an axis the line *y*,
s_z the axial symmetry having as an axis the line *z*,
s_α the planar symmetry relative to the plane α,
s_β the planar symmetry relative to the plane β,
s_γ the planar symmetry relative to the plane γ,
s_o the symmetry with center *O*,

It occurs that these eight isometries form a group. For this purpose, it is sufficient to prove that the product of any two of them is still one of the eight indicated isometries.

2. Tetrahedron's Isometries

Other examples of finite groups of isometries can be obtained considering all the isometries which leave fixed a given figure *F*, that is, such that in each of them F is united (*F* is transformed into itself). ABCD and A'B'C'D 'are two congruent tetrahedra. Then there exists one and only one isometry that transforms the vertices *A*, *B*, *C*, *D* neatly in the vertices *A '*, *B'*, *C '*, *D'* (Figure 2). This isometry is direct or reverse depending on whether or not the two tetrahedra are equally oriented.

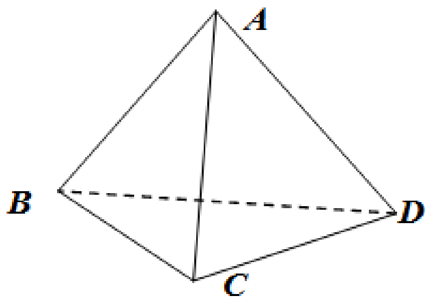

Figure 2

Isometries that turn a tetrahedron T into itself are 24 (twenty-four). They form a group S_T , obviously isomorphic to the group S_4 of the 24 permutations on four letters A, B, C, D.

Among the isometries ϕ that transform the tetrahedron T into itself, we present the following:

a) The *axial symmetry* μ having as an axis the straight line r, joining the midpoints of two opposite sides (*bimedian*), is a rotation of 180 ° around the straight line r.
The symmetries of this type present in the group are 3 (as many as the pairs of opposite sides of the tetrahedron); they have evidently period 2. Therefore there are 3 axial symmetries that leave T globally invariant, as many as the pairs of opposite sides.
A substitution is associated with each of these symmetries (M. Impedovo 1998).
- With symmetry μ_1 about the line r_1 joining the midpoints of the sides AB and CD, the following substitution is associated:

$$\mu_1 : \begin{pmatrix} A\ B\ C\ D \\ B\ A\ D\ C \end{pmatrix}$$

- With symmetry μ_2 about the line r_2 joining the midpoints of the sides AC e BD the following substitution is associated:

$$\mu_2 : \begin{pmatrix} A\ B\ C\ D \\ C\ D\ A\ B \end{pmatrix}$$

- With symmetry μ_3 about the line r_3 joining the midpoints of the sides AD e BC the following substitution is associated:

$$\mu_3 : \begin{pmatrix} A\ B\ C\ D \\ D\ C\ B\ A \end{pmatrix}$$

b) The *rotations* ρ of 120 ° and 240 ° around the height of the tetrahedron outgoing from a fixed vertex. For each height of the tetrahedron, you have two rotations of period 3 which hold the summit fixed. Since the tetrahedron heights are 4, these rotations are 8; therefore, there are 8 rotations of this type which transform T into itself, two for each height of the tetrahedron.
A substitution is associated with each of these rotations.

- With rotation ρ_1 about the height outgoing from A the following substitution is associated:

$$\rho_1 : \begin{pmatrix} A\,B\,C\,D \\ A\,C\,D\,B \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_2 : \begin{pmatrix} A\,B\,C\,D \\ A\,D\,B\,C \end{pmatrix}$$ relative to the amplitude of 240°

- With rotation ρ_3 about the height outgoing from B the following substitution is associated:

$$\rho_3 : \begin{pmatrix} A\,B\,C\,D \\ C\,B\,D\,A \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_4 : \begin{pmatrix} A\,B\,C\,D \\ D\,B\,A\,C \end{pmatrix}$$ relative to the amplitude of 240°

- With rotation ρ_5 about the height outgoing from C the following substitution is associated:

$$\rho_5 : \begin{pmatrix} A\,B\,C\,D \\ B\,D\,C\,A \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_6 : \begin{pmatrix} A\,B\,C\,D \\ D\,A\,C\,B \end{pmatrix}$$ relative to the amplitude of 240°

- With rotation ρ_7 about the height outgoing from D the following substitution is associated:

$$\rho_7 : \begin{pmatrix} A\,B\,C\,D \\ B\,C\,A\,D \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_8 : \begin{pmatrix} A\,B\,C\,D \\ C\,A\,B\,D \end{pmatrix}$$ relative to the amplitude of 240°

c) The *planar symmetry σ* relative to the plan π passing through the two vertices of the tetrahedron and the midpoint of the edge that joins the other two vertices. The σ symmetry *σ* is uniquely determined by the initial vertex. The symmetries of this type are 6 (as many as the pairs of vertices of the tetrahedron), and have period 2.

A substitution is associated with each of these symmetries

- With symmetry about the plane ABM_1, with M_1 medium point of *CD*, the following substitution is associated:

$$\sigma_1 : \begin{pmatrix} A\,B\,C\,D \\ A\,B\,D\,C \end{pmatrix}$$

- With symmetry about the plane ACM_2, with M_2 medium point of *BD*, the following substitution is associated:

$$\sigma_2 : \begin{pmatrix} A\,B\,C\,D \\ A\,D\,C\,B \end{pmatrix}$$

- With symmetry about the plane ADM_3, with M_3 medium point of *BC*, the following substitution is associated:

$$\sigma_3 : \begin{pmatrix} A\,B\,C\,D \\ A\,C\,B\,D \end{pmatrix}$$

- With symmetry about the plane BCM_4, with M_4 medium point of *AD*, the following substitution is associated:

$$\sigma_4 : \begin{pmatrix} A\,B\,C\,D \\ D\,B\,C\,A \end{pmatrix}$$

- With symmetry about the plane BDM_5, with M_5 medium point of AC, the following substitution is associated:

$$\sigma_5 : \begin{pmatrix} A\,B\,C\,D \\ C\,B\,A\,D \end{pmatrix}$$

- With symmetry about the plane CDM_6, with M_6 medium point of AB, the following substitution is associated:

$$\sigma_6 : \begin{pmatrix} A\,B\,C\,D \\ B\,A\,C\,D \end{pmatrix}$$

It is observed that the two sets of isometries described in points a) and b) each supplemented with the identity

$$I : \begin{pmatrix} A\,B\,C\,D \\ A\,B\,C\,D \end{pmatrix}$$

are closed about to the product.

The first set is a G_1 group of order 4 of involutorie transformations. The second set is a G_2 group of order 9 of periodic transformations of order 3.

The union of the two groups is a G_3 group of order 12, which is the group of *direct isometries* of T.

We will now examine the product of three symmetries, or we will fix an isometry σ_k of type c) (planar symmetry), and we will consider an isometry α_t ($t = 1, 2, \dots, 12$) variable in the G3 group. The product $\sigma_k \circ \alpha_t$ is still an isometry that changes the tetrahedron T into itself.

They are in number of 12; in fact, if we fix, for example, the isometry

$$\sigma_1 : \begin{pmatrix} A\,B\,C\,D \\ A\,B\,D\,C \end{pmatrix}$$

multiplying each isometry of the G_3 Group for σ_1, we will get 12 *reverse isometries* reverse, which can be summarized as:

$$\sigma_1 \circ \mu_1 = \varphi_1 : \begin{pmatrix} A & B & C & D \\ B & A & C & D \end{pmatrix}, \qquad \sigma_1 \circ \mu_2 = \varphi_2 : \begin{pmatrix} A & B & C & D \\ C & D & B & A \end{pmatrix},$$

$$\sigma_1 \circ \mu_3 = \varphi_3 : \begin{pmatrix} A & B & C & D \\ D & C & A & B \end{pmatrix}, \qquad \sigma_1 \circ \rho_1 = \varphi_4 : \begin{pmatrix} A & B & C & D \\ A & C & B & D \end{pmatrix},$$

$$\sigma \circ \rho_2 = \varphi_5 : \begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix}, \qquad \sigma_1 \circ \rho_3 = \varphi_6 : \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix},$$

$$\sigma \circ \rho_4 = \varphi_7 : \begin{pmatrix} A & B & C & D \\ D & B & C & A \end{pmatrix}, \qquad \sigma_1 \circ \rho_5 = \varphi_8 : \begin{pmatrix} A & B & C & D \\ B & D & A & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_6 = \varphi_9 : \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix}, \qquad \sigma_1 \circ \rho_7 = \varphi_{10} : \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_8 = \varphi_{11} : \begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}, \qquad \sigma_1 \circ \rho_9 = \varphi_{12} : \begin{pmatrix} A & B & C & D \\ A & B & D & C \end{pmatrix}.$$

It is easily seen that it results:

$$\phi_{12} = \sigma_1 , \quad \phi_5 = \sigma_2 , \quad \phi_4 = \sigma_3 , \quad \phi_7 = \sigma_4 , \quad \phi_6 = \sigma_5 , \quad \phi_1 = \sigma_6$$

That is the 12 isometries $\sigma_k \circ \alpha_t$ are given by the 6 planar symmetries σ_k of the type c) and by the 6 antirotations ϕ_k, with period 4. The isometries ϕ_k do not take firm no vertex and no edge of the tetrahedron.

In summary, we can say that the three axial symmetries of the G_1 group, the 8 rotations of the G_2 group, the 6 planar symmetries, the 6 latest found isometries, along with the identity, are the 24 isometries that leave the tetrahedron T globally invariant; their set is the S_T group of isometries of T.

S_T is the group of isometries that change the tetrahedron T in itself.

3.Isometries of Cube

Some examples of finite groups of isometries can be had considering all isometries leaving globally invariant a cube (A. Morelli, 1989).
ABCDEFGH e *A'B'C'D'E'F'G'H'* are two equal cubes. Then there exists one and only one isometry that transforms the vertices *A, B, C, D, E, F, G, H,* neatly in the vertices *A', B', C', D', E', F', G', H'* (Figure 3). This isometry is direct or reverse depending on whether or not the two cubes are equally oriented.

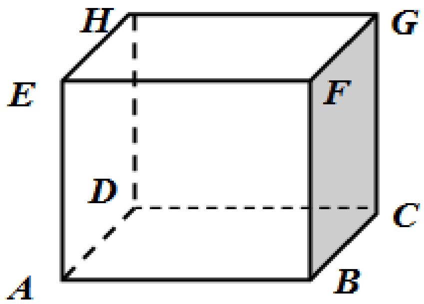

Figure 3

Isometries that transform a C cube to itself are forty eight. They forming a S_c group evidently isomorphic to S_8 group of forty eight permutations on eight letters *A, B, C, D, E, F, G, H.*

Among the isometries that transform the C Cube itself there are obviously the following:

a) The rotations ρ around a straight line r which joins the centers of two opposite faces.
   Since the faces of the cube are six, these lines are three; for each of these straight lines the cube is transformed into itself by the amplitude rotations, respectively, 90°, 180°, 270°.
   Therefore you have nine rotations of this type which transform C itself.
   For each of these rotations it is associated a substitution.
   - To ρ_1 rotation around the straight through M_1M_2, with M_1 the center of the *ABCD* face and M_2 the center of the *EFGH* face, is associated the substitution:

$$\rho_1 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ D\,A\,B\,C\,F\,G\,H\,E \end{pmatrix} \qquad \text{relative to the amplitude of } 90°$$

$\rho_2 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ C\,D\,A\,B\,G\,H\,E\,F \end{pmatrix}$ relative to the amplitude of 180°

$\rho_3 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ B\,C\,D\,A\,H\,E\,F\,G \end{pmatrix}$ relative to the amplitude of 270°

- To ρ_4 rotation around the straight through M_3M_4, with M_3 the center of the *ABFE* face and M_4 the center of the *DCGH* face, is associated the substitution:

$\rho_4 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ B\,G\,F\,C\,D\,E\,H\,A \end{pmatrix}$ relative to the amplitude of 90°

$\rho_5 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ G\,H\,E\,F\,C\,D\,A\,B \end{pmatrix}$ relative to the amplitude of 180°

$\rho_6 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ H\,A\,D\,E\,F\,C\,B\,G \end{pmatrix}$ relative to the amplitude of 270°

- To ρ_7 rotation around the straight through M_5M_6, with M_5 the center of the *AEHD* face and M_6 the center of the *BFGC* face, is associated the substitution:

$\rho_7 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ H\,G\,B\,A\,D\,C\,F\,E \end{pmatrix}$ relative to the amplitude of 90°

$\rho_8 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ E\,F\,G\,H\,A\,B\,C\,D \end{pmatrix}$ relative to the amplitude of 180°

$\rho_9 : \begin{pmatrix} A\,B\,C\,D\,E\,F\,G\,H \\ D\,C\,F\,E\,H\,G\,B\,A \end{pmatrix}$ relative to the amplitude of 270°

b) The rotations ρ around the straight line r that connects the midpoints of two opposite edges. Since the edges of the cube are twelve, these lines are six; for

each of these straight lines the cube is transformed into itself by rotations of 180 ° amplitude.
For each of these rotations it is associated a substitution.

- To rotation ρ_{10} around the straight line joining the midpoints of AB and EF edges, is associated the substitution:

$$\rho_{10}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix}$$

- To rotation ρ_{11} around the straight line joining the midpoints of CD and HG edges, is associated the substitution:

$$\rho_{11}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix}$$

- To rotation ρ_{12} around the straight line joining the midpoints of BC and HE edges, is associated the substitution:

$$\rho_{12}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & H \end{pmatrix}$$

- To rotation ρ_{13} around the straight line joining the midpoints of AD and FG edges, is associated the substitution:

$$\rho_{13}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & H & E & D & C & F & G & B \end{pmatrix}$$

- To rotation ρ_{14} around the straight line joining the midpoints of BC and HE edges, is associated the substitution:

$$\rho_{14}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & D \end{pmatrix}$$

- To rotation ρ_{15} around the straight line joining the midpoints of AD and FG edges, is associated the substitution:

$$\rho_{15}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & H & E & D & C & F & G & B \end{pmatrix}$$

c) The rotations ρ around the straight line r that contains a diagonal. The number of se lines is four; for each of these straight lines the cube is transformed into itself by the amplitude rotations respectively 120° and 240°. Therefore there are eight rotations of this type which transform C to itself.
For each of these rotations it is associated a substitution.

- To rotation ρ_{16} around the diagonal AF, it is associated the substitution:

$$\rho_{18}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & A & H & E & D & C & F \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{19}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & G & F & E & H & A & D \end{pmatrix}$$ relative to the amplitude of 240°

- To rotation ρ_{18} around the diagonal BE, it is associated the substitution:

$$\rho_{18}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & A & H & E & D & C & F \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{19}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & G & F & E & H & A & D \end{pmatrix}$$ relative to the amplitude of 240°

- To rotation ρ_{20} around the diagonal CH, it is associated the substitution:

$$\rho_{20}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ G & F & C & B & A & D & E & H \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{21}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & D & C & F & G & B & A & H \end{pmatrix}$$ relative to the amplitude of 240°

- To rotation ρ_{21} around the diagonal DG, it is associated the substitution:

$$\rho_{22}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & F & E & D & A & H & G & B \end{pmatrix}$$ relative to the amplitude of 120°

$$\rho_{23}:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & H & A & D & C & B & G & F \end{pmatrix}$$ relative to the amplitude of 240°

d) The planar symmetry σ with respect to π plane parallel to two faces through the midpoints of the four edges perpendicular to these two faces. The symmetries of the type indicated are three.
For each of these symmetries it is associated a substitution.

- At the planar symmetry σ_1 with respect to the plane π_1 parallel to *ABGH* and *EFCD* faces, is associated the substitution:

$$\sigma_1:\begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix}$$

- At the planar symmetry σ_2 with respect to the plane π_2 parallel to *ABDC* and *HGEF* faces, is associated the substitution:

$$\sigma_2:\begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix}$$

- At the planar symmetry σ_3 with respect to the plane π_3 parallel to *BCGH* and *ADHE* faces, is associated the substitution:

$$\sigma_3:\begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix}$$

e) The symmetries σ with respect to the π plan through two opposite edges that do not have common face and vertex. The symmetries of the type indicated are six.
For each of these symmetries it is associated a substitution.

- At the planar symmetry σ_4 respect to the π_4 plan through the edges *AD* and *GF* is associated with the substitution:

$$\sigma_4 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & H & E & D & C & F & G & B \end{pmatrix}$$

- At the planar symmetry σ_5 respect to the π_5 plan through the edges *BC* and *HE* is associated with the substitution:

$$\sigma_5 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ G & B & C & F & E & D & A & H \end{pmatrix}$$

-

- At the planar symmetry σ_6 respect to the π_6 plan through the edges *AB* and *EF* is associated with the substitution:

$$\sigma_6 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix}$$

- At the planar symmetry σ_7 with respect to the π_7 plan through the edges *CD* and *HG* is associated with the substitution:

$$\sigma_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix}$$

- At the planar symmetry σ_8 with respect to the π_8 plan through the edges *AH* and *CF* is associated with the substitution:

$$\sigma_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & D & C & B & G & F & E & H \end{pmatrix}$$

- At the planar symmetry σ_9 with respect to the π_9 plan through the edges *BG* and *DE* is associated with the substitution:

$$\sigma_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & A & D & E & H & G & F \end{pmatrix}$$

Note that the two sets of isometry described in points a), b) and c), each supplemented with the identity:

$$I:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & C & D & E & F & G & H \end{pmatrix},$$

are closed with respect to the product.

The first set G_1 is a group of order ten, the second set is a group G_2 of order seven, the third set is a group G_3 of order nine. The union of these three groups is a G_4 group of order twenty four which constitutes the group of direct isometries of C.

Let us now examine the product of three symmetries, that is fix an type d) isometry σ_k (planar symmetry), and consider an isometry α_t (t = 1, 2, ..., 24) variable in the G_4 group. The product $\sigma_k \circ \alpha_t$ is still an isometry which changes the C Cube to itself.

The number of these product is twenty four; in fact, it fixed eg. the isometry

$$\sigma_1:\begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix},$$

multiplying each isometry of the G_4 group σ_1, you get twentyfour reverse isometries, which can be summarized as:

$$\sigma_1 \circ \rho_1 = \varphi_1:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & B & A & D & E & H & G & F \end{pmatrix},$$

$$\sigma_1 \circ \rho_2 = \varphi_2:\begin{pmatrix} A & B & C & D & E & F & G & H \\ B & A & D & C & F & E & H & G \end{pmatrix},$$

$$\sigma_1 \circ \rho_3 = \varphi_3:\begin{pmatrix} A & B & C & D & E & F & G & H \\ A & D & C & B & G & F & E & H \end{pmatrix},$$

$$\sigma_1 \circ \rho_4 = \varphi_4:\begin{pmatrix} A & B & C & D & E & F & G & H \\ C & F & G & B & A & H & E & D \end{pmatrix},$$

$$\sigma_1 \circ \rho_5 = \varphi_5:\begin{pmatrix} A & B & C & D & E & F & G & H \\ F & E & H & G & B & A & D & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_6 = \varphi_6:\begin{pmatrix} A & B & C & D & E & F & G & H \\ E & D & A & H & G & B & C & F \end{pmatrix},$$

$$\sigma_1 \circ \rho_7 = \varphi_7 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ A & B & G & H & E & F & C & D \end{pmatrix},$$

$$\sigma_1 \circ \rho_8 = \varphi_8 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & F & E & D & C & B & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_9 = \varphi_9 : \begin{pmatrix} A & B & C & D & E & F & G & H \\ E & F & C & D & A & B & G & H \end{pmatrix},$$

$$\sigma_1 \circ \rho_{10} = \varphi_{10} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & G & B & A & D & C & F & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{11} = \varphi_{11} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & F & E & H & G & B & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{12} = \varphi_{12} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & B & G & H & A & D & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{13} = \varphi_{13} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & H & A & B & G & F & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_{14} = \varphi_{14} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & B & G & H & A & D & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{15} = \varphi_{15} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & H & A & B & G & F & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_{16} = \varphi_{16} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & E & D & A & B & C & F & G \end{pmatrix},$$

$$\sigma_1 \circ \rho_{17} = \varphi_{17} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & G & H & A & D & E & F & C \end{pmatrix},$$

$$\sigma_1 \circ \rho_{18} = \varphi_{18} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ H & A & B & G & F & C & D & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{19} = \varphi_{19} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & G & B & C & D & A & H & E \end{pmatrix},$$

$$\sigma_1 \circ \rho_{20} = \varphi_{20} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ B & C & F & G & H & E & D & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_{21} = \varphi_{21} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ F & C & D & E & H & A & B & G \end{pmatrix},$$

$$\sigma_1 \circ \rho_{22} = \varphi_{22} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & E & F & C & B & G & H & A \end{pmatrix},$$

$$\sigma_1 \circ \rho_{23} = \varphi_{23} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & A & H & E & F & G & B & C \end{pmatrix},$$

$$\sigma_1 \circ I = \varphi_{24} : \begin{pmatrix} A & B & C & D & E & F & G & H \\ D & C & B & A & H & G & F & E \end{pmatrix}.$$

It is easily seen that results:

$\varphi_{24} = \sigma_1$, $\varphi_8 = \sigma_2$, $\varphi_2 = \sigma_3$, $\varphi_7 = \sigma_6$, $\varphi_9 = \sigma_7$, $\varphi_3 = \sigma_8$, $\varphi_1 = \sigma_9$

that is, the twentyfour isometries $\sigma_k \circ \alpha_t$ are given from nine symmetries σ_k planar type d), e), and fifteen anti rotations φ_k.

In summary therefore it can be said that the twenty three rotations of the G_4 group, the nine planar symmetries and the latest isometries found, along with the identity, are the forty eight isometries which leave the cube C globally invariant; their set is the S_C group of isometries of the cube C.

S_C is the group of the isometries that change C cube to itself.

Conclusions

As already shown in a previous work (Casolaro, F., Cirillo, L. and Prosperi, R. 2015), the geometric Universe is three-dimensional, so the transformations taking place in it are generated in space. Then, we believe, for a correct analysis of the physical phenomena that occur in the universe, that it is essential to the knowledge of the real transformations that take place in it. Recent results of other branches of mathematics, in particular the modern algebra, have highlighted the interrelationships between movements in the plane and in space with some properties of the Theory of Groups (Casolaro, F. 1992), for which we consider essential to the deepening of these issues both in education and in the field of pure research (Casolaro, F. and Eugeni, F. 1996). Unfortunately, teaching (Casolaro F. 2014) in both the Secondary School that the University has been anchored to old programs that do not take into account the development of mathematics in the last 150 years, so we hope that this work will stimulate teachers and researchers to expand their views.

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