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ON IRREDUCIBLE BLOCKING SETS IN PROJECTIVE PLANES (*)

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Summary. - In a paper of Bruen and Silverman [7], it is proved that in a Desarguesian projective plane of square order q, q>4, in the interval of the admissible cardinalities of irreducible blocking sets there are integers k such that there is no irreducible blocking set with k points. In this paper we prove that in a finite projective plane there is a sub-interval in which for any integer k there is at least one irreducible blocking set with k points.

1. INTRODUCTION

Throughout this note, we denote by $\pi = \pi_q$ a finite projective plane of order q, where q is not necessarily a power of a prime.

A blocking set of π is a set K of points which contains no line but intersects every line. A blocking set is said to be irreducible if it contains no blocking set properly, otherwise it is said to be reducible. The index of a blocking set K is the minimum number of lines whose union contains K.

The following results are well-known (see [5], [6], [11]).

1.1 RESULT .- *Let* K *be an irreducible blocking set in* π. *Then*

(a) $q + \sqrt{q} + 1 \leq |K| \leq q\sqrt{q} + 1$.

(b) $|K| = q + \sqrt{q} + 1$ iff q *is a square and* K *is a Baer subplane.*

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(c) $|K| = q\sqrt{q} + 1$ iff q *is a square and* K *is a Hermitian arc.*

Examples and other results concerning the blocking sets in finite projective planes can be found in [1], [2], [4], [8], [9].

In particular, in [1] it is proved that:

1.2 RESULT .- *Let* π *be a Desarguesian projective plane of order* q, *where* $q=p^h$, p *a prime,* $q>2$. *Let* m(q) *be the function defined as follows:*

$$m(q) = \begin{cases} \sqrt{q} & \text{if } q \text{ is a square;} \\ (q+1)/2 & \text{if } q \text{ is a prime;} \\ p^{h-d} & \text{otherwise, where } d \text{ denotes the greatest divisor of } h \text{ different from } h. \end{cases}$$

Then for any integer k, with $q+m(q)+1 \le k \le q^2-m(q)$, *there exists a blocking set with k points.*

An obvious question is, whether there exist irreducible blocking sets in π for each cardinality belonging to the interval $(q+\sqrt{q}+1,\ q\sqrt{q}+1)$. In [7] is proved the following:

1.3 RESULT .- *If* π *is a Desarguesian projective plane,* q *square,* $q>4$, *then in the interval* $(q+\sqrt{q}+1,\ q+\sqrt{2q}+1-1/(2q))$ *there is no irreducible blocking set.*

In order to give an answer to this question we shall prove the following assertions:

(I) *For* $q>4$, *a finite projective plane has at least one irreducible blocking set of index* 4 *of cardinality* k *for any integer* k *with* $2q-1 \le k \le 3q-5$.

(II) *For* $q>4$, *a Desarguesian projective plane has at least one irreducible blocking set of index 4 of cardinality* k *for any integer* k *with* $2q-1 \le k \le 3q-3$.

We note that a complete characterization of irreducible blocking sets in the case q=3 can be found in [11], in the case q=4 in [2], in the case q=5 in [3]. Moreover we recall the following two results, see [8], [10], on the lower bound of the interval of (II).

1.4 RESULT. - *In a Desarguesian projective plane of order* q, *for every proper divisor* d *of* q *or of* q-1, *there exists an irreducible blocking set of index* 3 *having* 2q+1-d *points.*

1.5 RESULT. - *If in a Desarguesian projective plane of order* q *there is an irreducible blocking set* B *of index* 3 *whose cardinality is less than or equal to* 2q-1, *then* B *has* 2q+1-d *points exactly, where* d *is a proper divisor of* q *or of* q-1.

2. IRREDUCIBLE BLOCKING SETS

We begin with the following

2.1 THEOREM. - *In a finite projective plane of order* q, q>4, *for any integer* k *with* $2q-1 \leq k \leq 3q-5$ *there exists at least one irreducible blocking set of index* 4 *having* k *points.*

Proof.- Let r and s be two lines of a finite projective plane, let V be their intersection point, let R_i, i=0, 1, ..., q-1, be the points on r different from V and let S_j, j=0,1,...,q-1, be the points on s different from V. Denote by t the line R_0S_0. The set

$$K = r \cup s \cup t - \{V, R_0, S_0\}$$

is a well-known irreducible blocking set, with 3(q-1) points called a triangle without vertices. Let T_r, r=1, 2, ..., q-1, be the points of t distinct from R_0 and S_0. Consider an arbitrary point S on s, different from V and S_0, and an arbitrary point T_1 on t, different from R_0 and S_0. Denote by R the intersection point of the line ST_1 with r. Put

$$U_1 = R_0S \cap VT_1, \qquad U_2 = R_0S \cap RS_0.$$

There are two possible cases:

(a) $U_1 = U_2$;

(b) $U_1 \neq U_2$.

Let n be an integer such that $0 \leq n \leq q-3$ if case (a) holds and $0 \leq n \leq q-4$ if case (b) holds. Moreover let U_{2+i}, i = 1,..., n be n arbitrary points of R_0S, distinct two by two, and different from R_0, S, U_1, U_2. Denote by R_i and by T_i the intersection points of the line S_0U_i with r and t respectively, for any i=1, 2, ..., n+2. The set

$$K' = K - \{R_1, R_2, \dots R_{n+2}, T_1, T_2, \dots, T_{n+2}\} \cup \{U_1, U_2, \dots U_{n+2}\}$$

is an irreducible blocking set of index 4 with 3 (q-1)-(n+1) points if case (a) holds and with 3(q-1)-(n+2) if case (b) holds. If case (a) holds, since $0 \leq n \leq q-3$, we obtain irreducible blocking sets of cardinality 3(q-1)-r for any integer r such that $1 \leq r \leq q-2$. If case (b) holds, since $0 \leq n \leq q-4$, we have irreducible blocking sets of cardinality 3(q-1)-r for any integer r such that $2 \leq r \leq q-2$.

Now we deal with Desarguesian case.

2.2 COROLLARY .- *In a Desarguesian projective plane of even order greater than* 2 *for any integer k with* $2q-1 \leq k \leq 3q-3$ *there exists at least one irreducible blocking set of index* 4 *having* k *points.*

Proof.- The assertion follows by the proof of the previous theorem and by observing that in a Desarguesian projective plane the order is even if and only if case (a) holds.

Finally we prove the following:

2.3 PROPOSITION .- *In a Desarguesian projective plane of odd order greater than* 3 *for any integer k with* $2q-1 \leq k \leq 3q-3$ *there exists at least one irreducible blocking set of index* 4 *having k points.*

Proof.- By using the same notation as in the proof of the previous theorem, since in a Desarguesian projective plane the order is odd if and only if case (b) holds, we prove that if $U_1 \neq U_2$ it is possible to construct an irreducible blocking set of index 4 with 3(q-1)-1 points, so the assertion follows by the proof of the theorem. Let we denote by R_i, i=1, 2, the intersection points of the line S_0U_i with r and by T_i, i=1, 2, the intersection points of the line U_iV with t.
Put $Q = S_0U_2 \cap T_1V$ and $S'' = R_0Q \cap s$. The set

$$K'' = K - \{T_1, R_1, R_2\} \cup (U_1, Q\}.$$

is a blocking set of index 4 with 3(q-1)-1 points. In order to prove that K'' is irreducible, since the line ST_1 contains R_2, it is sufficient to prove that the line $S''T_1$ passes through the point R_1. Put

$$X = S''T_1 \cap r .$$

Denote by (ABCD) the cross-ratio of four collinear points A, B, C and D and by (P) the perspectivity of centre a point P. It results:

$$(VR_2R_0X) \overset{(T_1)}{=} (VSS_0S'') = (S''S_0SV).$$

Moreover we have:

$$(VR_2R_0R_1) \overset{(U_1)}{=} (QR_2U_2S_0) \overset{(S)}{=} (QT_1U_1V) \overset{(R_0)}{=} (S''S_0SV)$$

It follows that

$$(VR_2R_0R_1) = (VR_2R_0X)$$

and then

$$R_1 = X.$$

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