ADDITIONAL MONOTONICITY PROPERTIES AND INEQUALITIES FOR THE ZEROS OF BESSEL FUNCTIONS (*)

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1. INTRODUCTION.

For $\nu \geqslant 0$ we denote with $j_{\nu k}$ and $c_{\nu k}$ the k-th positive zeros of the Bessel functions $J_\nu(x)$ of the first kind and of the general cylinder function

$$C_\nu(x;\gamma) = C_\nu(x) =: \cos\gamma\, J_\nu(x) - \sin\gamma\, Y_\nu(x) \qquad 0 \leqslant \gamma < \pi$$

where $Y_\nu(x)$ is the Bessel function of the second kind.

In [2] A. Elbert and A. Laforgia introduced the notation $j_{\nu\kappa} = c_{\nu k}$ where $\kappa = k - \gamma/\pi$ and $k-1 < \kappa < k$. When $\kappa = k$ we get the zeros of the function $J_\nu(x)$. This notation has been used to prove several *monotonicity, concavity, convexity* properties of $j_{\nu\kappa}$ as a function of ν for κ fixed [2,3,4,5,9]. In particular we know that for $\nu \geqslant 0$ $j_{\nu\kappa}$ is concave for $\kappa \geqslant 0.344...$ and $j^2_{\nu\kappa}$ is convex for $\kappa \geqslant 0.7070...$ [5].

We observe that the study of the properties of $j_{\nu\kappa}$ was originated by the paper [19] of Putterman, Kac and Uhlenbeck. They have proposed a quantum mechanical explanation for the origin of the vortex lines produced in superfluid Helium when its container is rotated.

J.T. Lewis and M.E. Muldoon [16] have proved some monotonicity results using the Hellmann- Feynmann theorem of quantum chemistry [8,10,11] that here we recall

Hellmann-Feynmann Theorem. Let's be a pseudo inner product space with a pseudo-inner product $\langle\, .\, ,\, .\, \rangle$. Let (H_ν) be a family of symmetric operators on an inner product space and for $\nu \in (a,b)$ let ψ_ν be an eigenvector (eigenfunction) of H_ν corresponding to an eigenvalue λ_ν.

(*) Work sponsored by Ministero dell' Università e della Ricerca Scientifica e Tecnologica of Italy.

Suppose that for $\mu\in(a,b)$

$$\langle \psi_\mu , \psi_\nu \rangle \to \langle \psi_\mu , \psi_\mu \rangle \neq 0$$

as $\mu\to\nu$ and that

$$\lim_{\mu\to\nu} \langle \frac{H_\mu - H_\nu}{\mu - \nu} \psi_\mu , \psi_\nu \rangle$$

exists.

Moreover we define

$$\langle \frac{\partial H_\nu}{\partial \nu} \psi_\mu , \psi_\nu \rangle := \lim_{\mu\to\nu} \langle \frac{H_\mu - H_\nu}{\mu - \nu} \psi_\mu , \psi_\nu \rangle$$

Then

$$\frac{d\lambda_\nu}{d\nu} = \frac{\langle \frac{\partial H_\nu}{\partial \nu} \psi_\mu , \psi_\nu \rangle}{\langle \psi_\mu , \psi_\nu \rangle}$$

This version of the Theorem is taken from [11]. As a consequence of the Hellmann- Feynmann Theorem, Lewis and Muldoon [16] have proved that $j_{\nu 1}/\nu$ decreases with $\nu > 0$ and that $j^2_{\nu 1}/\nu$ increases with ν, $(3 \leqslant \nu < \infty)$. More recently C. Giordano and L.G. Rodonò proved that for $\nu>0$ and $\kappa \geqslant 0.7070...$ there exists a value ν_κ such that the function $j^2_{\nu\kappa}/\nu$ decreases on $(0,\nu_\kappa)$ and increases on (ν_κ,∞). Furthermore they showed that $j^2_{\nu\kappa}/\nu$ is convex on $(0,\nu_\kappa)$ [9]. We also recall that M.E.H. Ismail and M.E. Muldoon [12] proved the stronger monotonicity result that $j^2_{\nu 1}/(\nu+1)$ is an increasing function of ν on $(-1,\infty)$.

Further properties for $j_{\nu\kappa}$ and $c_{\nu k}$ can be established by the integral Watson formula [22, p. 508]

$$(1.1) \qquad \frac{d c_{\nu k}}{d\nu} = 2c_{\nu k}\int_0^\infty K_0\left(2c_{\nu k}\sinh t\right) e^{-2\nu t}\, dt , \qquad k=1,2 \ldots$$

where $K_0(x)$ is the standard modified Bessel function.

In section 2 we recall some montonicity properties recently established by A.Laforgia, M.E.Muldoon and L.Lorch and in section 3 we examine their application to the zeros of generalized Airy functions [6,15,17].

2. MONOTONICITY RESULTS.

In [17] L.Lorch has deduced the inequality

$$\frac{\nu}{c_{\nu k}} \cdot \frac{d c_{\nu k}}{d\nu} < 1 , \qquad \text{for} \quad c_{\nu k} \geqslant \nu > 0 ,$$

which implies that, for $k=1,2,\ldots,\ 0<\nu<\infty$

$$\frac{\nu}{c_{\nu k}} \qquad \text{increases for} \quad c_{\nu k} \geqslant \nu .$$

When $c_{\nu k} > \nu + \pi/4$, there follows the stronger result that [14,17]

$$\frac{\nu+\frac{1}{2}}{c_{\nu k}} \qquad \text{increases for} \quad 0 \leqslant \nu < \infty .$$

More generally A.Laforgia and M.E.Muldoon in [14] have showed the inequality

$$c_{\nu k} > \nu + c_{0k} , \qquad 0< \nu <\infty , \ k= 2,3,\ldots$$

and

$$(2.1) \qquad \left(\nu + a_k\right) \frac{d\,c_{\nu k}}{d\nu} < c_{\nu k} , \quad 0 < \nu < \infty , \quad k = 2,3, \ldots$$

where the positive number a_k is defined by

$$a_k^{-1} = c_{0k}^{-1} \left[\frac{d\,c_{\nu k}}{d\nu} \right]_{\nu = 0}$$

The previous results hold also in the case $k=1$, but $0 \leqslant \gamma \leqslant \pi/2$.
Inequality (2.1) can be employed to yield the stronger monotonicity result

$$\frac{\nu + a_k}{c_{\nu k}} \qquad \text{increases for } \nu > 0 \ \text{ when } k = 2,3,\ldots$$

If $0 \leqslant \gamma \leqslant \pi/2$ this monotonicity holds also for $k=1$.
Differentiating we get

$$c_{\nu k} \left(\frac{\nu + a_k}{c_{\nu k}} \right)' = 1 - \frac{\nu + a_k}{c_{\nu k}} \frac{dc_{\nu k}}{d\nu} > 0$$

The previous results can be employed to deduce other monotonicity results as the following ones recently found by A.Laforgia, M.E.Muldoon and L.Lorch [15,17].
We recall now some of them.
If $\nu>0$ the function

$$\left(\frac{c_{\nu k}}{\nu}\right)^{\nu} \qquad \text{increases for } c_{\nu k} \geqslant \nu + \frac{\pi}{4}$$

and

$$\left(\frac{c_{\nu k}}{2\nu}\right)^{2\nu} \qquad \text{decreases for } 0<\nu \leqslant c_{\nu k} \leqslant 2\nu$$

The proof is based on the properties [17] of the function

$$\delta_\nu = \frac{d}{d\nu} \left\{ \ln \left[\frac{c_{\nu k}}{\beta \nu} \right]^{\nu} \right\} , \qquad \nu > 0 , \ \beta > 0$$

in the case $\beta=1$, $\beta=2$, respectively.
The monotonicity of δ_ν implies that if $\delta_\nu \geq 0$, then $[c_{\nu k}/(2\nu)]^{2\nu}$ increases for $0<\nu\leq\mu$, while $\delta_\nu \leq 0$ implies that this function of ν decreases for all $\nu>\mu$, provided of course that $c_{\nu k} \geq \nu+\pi/4$.
The hypothesis that $c_{\nu k} > \nu+\pi/4$ is satisfied when k=2,3,... and for any $c_{\nu 1}$ for which $0\leq\gamma\leq\pi/2$; in particular for $c_{\nu 1}=j_{\nu 1}$.
In the case k=1, $c_{\nu k}=j_{\nu 1}$ there exists a value $\nu=\nu_1$ such that

$$(2.2)\qquad \left(\frac{j_{\nu 1}}{2\nu}\right)^{2\nu} \qquad \text{increases,}\quad 0<\nu\leq\nu_1$$

and

$$(2.3)\qquad \left(\frac{j_{\nu 1}}{2\nu}\right)^{2\nu} \qquad \text{decreases,}\quad \nu\geq\nu_1$$

M.E.Muldoon has calculated some values of $[j_{\nu 1}/(2\nu)]^{2\nu}$ near 1 from which it derives that $1.003<\nu_1<1.006$.
If the order ν is kept constant, but k or γ varies in $c_{\nu k}(\gamma)$ it could be convenient to use the notation introduced by Á.Elbert and A.Laforgia in [2]. They show that $j_{\nu\kappa}$ increases with $\kappa>0$, for fixed ν.
L.Lorch in [17] extends (2.2), (2.3) by implying the existence of unique ν_κ such that

$$(2.4)\qquad \left(\frac{j_{\nu\kappa}}{2\nu}\right)^{2\nu} \qquad \text{increases for } 0<\nu\leq\nu_\kappa \text{ and decreases for } \nu\geq\nu_\kappa,$$

where ν_κ is an increasing function of $\kappa>0$.
The behaviour of $[c_{\nu k}/(2\nu)]^{2\nu}$ suggests to consider the character of $[c_{\nu k}/(2\nu)]^{2\nu}/\nu$ [15,17].
First of all A.Laforgia and M.E.Muldoon in [15] have obtained that for some $0<\nu_0\leq 1/2$ and some k= 1,2,..., under some hypothesis, the function $[c_{\nu k}/(2\nu)]^{2\nu}/\nu$ decreases as ν increases for $0<\nu\leq\nu_0$.
Recently L.Lorch has established in [17] that $[c_{\nu 1}/(2\nu)]^{2\nu}/\nu$ decreases for $0<\nu<\infty$ if $c_{\nu 1}\geq\nu$. The proof is divided into four parts. In each subinterval $\lambda\leq\nu\leq\mu$ he has proved that the expression

$$\frac{1}{2}\frac{d}{d\nu}\left\{\ln\left[\left(\frac{c_{\nu 1}}{2\nu}\right)^{2\nu}\cdot\frac{1}{\nu}\right]\right\}$$

is negative , using previous results in [15, 17] .

3. APPLICATION TO THE ZEROS OF GENERALIZED AIRY FUNCTIONS.

The generalized Airy functions are the solutions on $0 \leqslant x < \infty$ of the equation

(3.1) $$y'' + x^{\alpha} y = 0$$

where α is a positive number. The Airy functions correspond to the case $\alpha = 1$.

The case $\alpha = 1$ arises in a fundamental way in the asymptotic solution of certain kinds of differential equations [18,20,21] . A function of this kind also occurs in work concerned with weighted averages of a function at a jump discontinuity [1].

Equation (3.1) is connected to the Bessel equation

$$t^2 \frac{d^2u}{dt^2} + t \frac{du}{dt} + \left(t^2 - \nu^2\right) u = 0$$

by means of the transformations

(3.2) $$y(x) = x^{1/2} u(t) \qquad t = 2\nu x^{1/(2\nu)}$$

where $\nu = 1/(\alpha + 2)$.

So it is possible to use many known results about monotonicity with respect to order of zeros of Bessel functions. We denote $a_{\alpha k}$ the k-th positive zero of a solution of (3.1).

Because of (3.2), it is clear that

(3.3) $$a_{\alpha k} = \left(\frac{c_{\nu k}}{2\nu}\right)^{2\nu} \qquad \text{where } 0 < \nu = \frac{1}{\alpha + 2} .$$

Using the connection (3.3) between zeros of cylinder functions and generalized Airy functions, it is possible to obtain results for $a_{\alpha k}$. As Corollary to Theorem 2.1 in [14] there follows that for each k=2,3,... $a_{\alpha k}$ decreases to 1 as α increases , $0 < \alpha < \infty$. If $y(0) = 0$ this decrease holds also for $a_{\alpha 1}$.

Muldoon's bounds for ν_1 permit to assert (2.2) and (2.3) with $1.003 < \nu_1 < 1.006$.

Consequently

(3.4)
$$a_{\alpha 1} \quad \text{increases} \quad -1.00596 < -2 + 1/\nu_1 \leqslant \alpha < \infty$$
$$a_{\alpha 1} \quad \text{decreases} \quad -2 < \alpha \leqslant -2 + 1/\nu_1 < -1.00299$$

Taking into account (2.4), the result (3.4) can be extended to

(3.5)
$$a_{\alpha k} \quad \text{increases} \quad -2 + 1/\nu_k \leqslant \alpha < \infty$$
$$a_{\alpha k} \quad \text{decreases} \quad \alpha < -2 + 1/\nu_k$$

This implies that

$$(\alpha+2)\,a_{\alpha 1} \quad \text{increases}, \quad -1.0029 < \alpha < \infty$$

and more generally

$$(\alpha+2)\,a_{\alpha k} \quad \text{increases}, \quad -2+1/\nu_k < \alpha < \infty$$

for $k=1,2,\ldots$.

In [6] Á. Elbert and A. Laforgia continued the investigation about the behaviour of $a_{\alpha k}$ and they proved the following result

Theorem. For each fixed $k = 2,3,\ldots$, let $a_{\alpha k}$ be the k-th positive zero of (3.1). Then for $\alpha \geq 0$, the function $\log a_{\alpha k}$ is convex.

The first important consequence of this theorem is the convexity of $a_{\alpha k}$, since a positive log-convex function is also convex.

The proof is based on some known properties of $c_{\nu k}$ and the Watson's formula (1.1).

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