A SIMPLE CONSTRUCTIVE PROOF OF VON NEUMANN EQUILIBRIUM

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A proof of existence of general equilibrium in the Von Neumann model of an expanding economy is given based on elementary results of Linear Programming. General equilibrium is computable by solving a sequence of Linear Programs.

1. Introduction

Over the years, existence of the equilibrium in the Von Neumann model has been proved with recourse to analytical tools differing widely in nature as well as in complexity. The original Von Neumann proof made extensive use of game theory. Other proofs rely on fixed point theorems such as Brouwer or Kakutani (for a review, see Burmeister Dobell 1970, p. 207).

In 1960, Charles W. Howe exhibited a proof that did not require game theory or fixed point theorems but used a result on linear inequalities due to A.W. Tucker. Unfortunately though, this result requires independent derivation. To the non-initiated, this derivation can appear lengthy. Murata for instance, derives Tucker' theorem from Farka's Lemma through a sequence of about 10 intermediate theorems (Murata, 1977, pp. 283-288).

Unaware, to my knowledge, of more elementary proofs, I wondered if all this mathematical machinery is really necessary. Here I provide a proof based solely on Linear Programming (henceforth LP). Namely, I employ the only two notions that 1. if a primal-dual pair of LP problems have feasible solutions, they have optimal solutions with a common value, and 2. these satisfy the so-called complementarity slackness conditions: positive primal variables are associated to active dual constraints and loose primal constraints are associated to zero dual variables. The rest of the proof is elementary algebra.

Besides greater simplicity, the advantage is to characterise solutions through a constructive procedure. This feature is not usually obtained using fixed point theorems or linear inequality results.

2. Model Description

Von Neumann economy comprises m goods and n processes. Process j is characterised by an activity level y_j. When it is operated at unit level ($y_j =$ 1) its input demands and output supplies are described by two non-negative vectors

$$\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \quad \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{bmatrix}.$$

Thus technology is completely specified by an input matrix A and an output matrix B, both non-negative with m rows (goods) and n columns (processes).

Each good must be demanded by at least one process so there is at least one positive entry in each row of A. Each process must supply at least one good, so there is at least one positive entry in each column of B. We refer to these two assumptions as KMT (from Kemeny Morgenstern Thompson (1956) who formulated them in order to relax those originally made by Von Neumann).

Labour is treated as one amongst the m goods. It is supplied to other processes by a "domestic" sector that produces it using other goods (like food, housing, good TV, etc). So consumption is one of the rows of A and there is no exogenous demand. Labor being paid in nature-goods, there is no wage rate and net income is made up by profits only. These are entirely re-invested in the form of input purchases for next period. Under these conditions an equilibrium is sought so as to satsfy

1. Output at some period By must cover input of next period AY where $Y = (1+g)y$, and g is a (balanced) growth rate. Letting $\alpha = 1+g$

$$\alpha Ay \leq By. \tag{1}$$

2. Revenues $p'B$ cannot exceed production costs $p'A$ plus competitive profits $rp'A$ where r is a (uniform) interest rate. Letting $\beta = 1+r$

$$\beta p'A \geq p'B. \tag{2}$$

3. Goods in excess of next period demand (components of (1) with strict inequality) are worthless

$$p'(B - \alpha A)y = 0. \tag{3}$$

4. Processes generating less than competitive profits (components of (2) with strict inequality) remain idle

$$p'(B - \beta A)y = 0. \tag{4}$$

5. Something of value is produced in the economy $p'By > 0$.

3. Von Neumann Theorem

Theorem. *(Von Neumann, 1937)*

A solution $y \geq 0$ $p \geq 0$ satisfying (1-5) exists if and only if $\alpha = \beta = \gamma$ where

$$\gamma = \max\{\alpha : By - \alpha Ay \geq 0, \quad y \geq 0\}$$

$$\gamma = \min\{\beta : p'B - \beta p'A \leq 0, \quad p \geq 0\}.$$

Proof. *Necessity.* Assume that $y, p \geq 0$ satisfying 1-5 exist. From (3) and (4) it follows $p'By = \alpha p'Ay$ and $p'By = \beta p'Ay$. Since $p'By > 0$, if a solution exists the r.h.s. are positive and $p'Ay > 0$. Hence $\alpha = \beta$.

Furthermore, from (1) and (2)

$$p'By \geq \alpha p'Ay \quad \text{and} \quad -p'By \geq -\beta p'Ay.$$

Adding each side and recalling that $p'Ay > 0$ it follows $\alpha \leq \beta$. Thus there exists only one common value to α and β, call it γ. This necessarily satisfies

$$\max\{\alpha : By - \alpha Ay \geq 0, \quad y \geq 0\} = \gamma = \min\{\beta : p'B - \beta p'A \leq 0, \quad p \geq 0\}.$$

Sufficiency. Define $\gamma = \max\{\delta : By - \delta Ay \geq 0, \quad y \geq 0\}$. The maximum certainly exists, since the set of $y \geq 0$ satisfying (1) is either empty or closed and bounded. Furthermore, $\gamma \geq 0$ due to the non-negativity of A, B, y. Now we have to show that there exist a $y \geq 0$ and a $p \geq 0$ satisfying 1-5, for some α, β. Choose $\alpha = \beta = \gamma$. Letting

$$C = B - \gamma A$$

and introducing scalars $Y_1, Y_2 \geq 0$ and $P_1, P_2 \geq 0$, consider the LP problem

$$\min \quad \left[0'|1|-1\right] \begin{bmatrix} y \\ Y_1 \\ Y_2 \end{bmatrix}$$

$$\begin{bmatrix} C & \mathbf{1} & -\mathbf{1} \\ \mathbf{1}' & 0 & 0 \\ -\mathbf{1}' & 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ Y_1 \\ Y_2 \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix}. \tag{6}$$

Its dual is

$$\max \quad [\mathbf{0}'|1|-1] \begin{bmatrix} p \\ P_1 \\ P_2 \end{bmatrix}$$

$$[p'|P_1|P_2] \begin{bmatrix} C & \mathbf{1} & -\mathbf{1} \\ \mathbf{1}' & 0 & 0 \\ -\mathbf{1}' & 0 & 0 \end{bmatrix} \leq [\mathbf{0}'|1|-1]. \tag{7}$$

Rewriting (6,7) as

$$\min \quad Y_1 - Y_2 : \quad Cy + (Y_1 - Y_2)\mathbf{1} \geq 0, \quad \mathbf{1}'y = 1$$

$$\max \quad P_1 - P_2 : \quad p'C + (P_1 - P_2)\mathbf{1}' \leq 0, \quad p'\mathbf{1} = 1$$

it is clear that, for any pair $y, p \geq 0$ with components adding to one, it is always possible to choose $Y_1 - Y_2$ and $P_2 - P_1$ large enough so as to satisfy the constraints. Since there exist feasible solutions there exist optimal solutions, which necessarily satisfy

$$Y_1 - Y_2 = P_1 - P_2.$$

We show now that $Y_1 - Y_2 = P_1 - P_2 = 0$. From the dual constraints we have $P_1 - P_2 \leq 0$. But, if it were $P_1 - P_2 < 0$ then $Y_1 - Y_2 < 0$ and from $Cy + (Y_1 - Y_2)\mathbf{1} \geq 0$ it would follow

$$Cy = [B - \gamma A]y > 0$$

against the assumption that γ is the largest value of α for which (1) has a solution $y \geq 0$. Thus $P_1 - P_2 = 0 = Y_1 - Y_2$ and this shows that there exist two non-zero vectors $y \geq 0$ and $p \geq 0$ satisfying (1) e (2). Conditions 3 and 4, for $\alpha = \beta = \gamma$ are the complementarity slackness conditions of LP and these are obviously satisfied given the optimality of the pair y, p.

We finally prove that 5 holds. Observe first that

$$p'By \geq \bar{p}'\bar{B}\bar{y}$$

where $\bar{p}$ and $\bar{y}$ are vectors obtained from p and y by deleting exactly those zero-components that are associated to active constraints in the respective duals (if there are any) and $\bar{B}$ is the matrix obtained from B by deleting the corresponding rows and columns. Therefore, it is enough to prove 5 under the assumption

$$p + Cy > 0 \qquad y - C'p > 0. \tag{8}$$

Rearranging components if necessary, the optimal solutions can be partitioned as

$$y = \{y_1|y_2\}' \quad p = \{p_1|p_2\}'$$

where the first components of each vector are positive and the remaining components (if there are any) are zero. Partitioning conformally A and B, we get from (3,4) (with $\alpha = \beta = \gamma$)

$$p_1'(B_{11} - \gamma A_{11})y_1 = 0.$$

Now, if $p'By = 0$, then $B_{11} = 0$ (being positive p_1, y_1) and thus also $A_{11} = 0$, and thus also

$$B_{11} - \gamma A_{11} = 0.$$

From (8), if there are components $p_2 = 0$, then the constraints associated to p_2 in the dual of $p'(B - \gamma A) \leq 0$ must be loose, that is

$$(B_{12} - \gamma A_{12})y_1 > 0.$$

But then it is false that γ is the greatest value of α for which (1) has a solution $y \geq 0$. Therefore $p'By > 0$.

If instead there are no components $p_2 = 0$, then there must be components $y_2 = 0$ and the proof is the same, with the roles of y and p interchanged. Q.E.D.

4. Remarks

The proof offered is constructive. Suppose one has an LP algorithm able to compute the optimum value $v = Y_1 - Y_2$ (as well as the solution y) of problem (6). Regard this algorithm as a function

$$f : \gamma \to v.$$

The problem is simply to find the zero of this function. Call y_0 the solution of (6) at $\gamma = 0$. Due to the feasibility of (1) at $\gamma = 0$, we have $v(0) \leq 0$.

Let now $\gamma^* > 0$ be a value of γ for which $B - \gamma A < 0$ (such a value certainly exists due to KMT assumption). Due to infeasibility of (1) at $\gamma = \gamma^*$, we have $v(\gamma^*) > 0$. Since f is a contiuous function, it must have a zero in $[0, \gamma^*]$. But this zero is unique since it is the optimal value of a primal-dual LP pair. So the problem is solvable by a simple line-search.

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Biografia

Paolo Caravani, associato di Modelli e Simulazione nell'Universita' dell'Aquila, si occupa di Economia Matematica, Teoria del Controllo e Teoria dei Giochi. E' autore di circa 40 pubblicazioni nel settore. E' socio della Society for Economic Dynamics and Control.

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4. Remarks

The proof offered is constructive. Suppose one has an LP computer code able to produce either an optimal solution or an infeasibility message. Consider now the primal LP problem formulated in the proof of the theorem. Starting at $\alpha = 0$ (where a solution certainly exists) choose a non-converging increasing sequence $\{\alpha_k, \quad k = 1, 2, \ldots\}$ and use at each k the LP code. Suppose at $k = m$ an infeasibility message is received for the first time (clearly, $m < \infty$). Then choose a sequence $\{\alpha_h, \quad h = 1, 2, \ldots\}$ converging to α_m with $\alpha_1 = \alpha_{m-1}$ and use again the code at each h. It is obvious that, proceeding in this manner, γ can be approximated to any desired accuracy - and likewise the solution.

A final comment to the KMT assumption. I retained it as a homage to the tradition but it seems that all is needed in the proof is non-negativty and non-nullity of A and B.

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