A LOCAL PROPERTY OF HAMILTONIAN MOON TOURNAMENTS (*)

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We prove that a hamiltonian tournament H_n, $n \geq 5$, is of Moon-type (i.e. its subtournaments are either hamiltonian or transitive) if and only if each hamiltonian 5-subtournament is too.

1. INTRODUCTION

Several properties of tournaments are known that are verified globally iff they are fulfilled by each subtournament of a given order.

It is well known, for example, that a tournament with at least 3 vertices is transitive iff each 3-subtournament T_3 is too.

Burzio and Demaria [3] proved that a hamiltonian tournament with at least 5 vertices is bineutral iff each hamiltonian k-subtournament is too, for some $5 \leq k \leq n$.

Other local properties of hamiltonian tournaments that are global properties too were studied by Demaria and Gianella [4], who proved the following:

- a hamiltonian tournament with at least 5 vertices has the minimum number, n-2, of 3-cycles iff every hamiltonian 5-subtournament has 3 3-cycles.
- a hamiltonian tournament with at least 6 vertices has only one spanning cycle iff every hamiltonian 6-subtournament has the same property.

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Burzio and Demaria [2] proved that a tournament with at least 4 vertices is of Moon-type (see next section for definitions) iff all of its 4-subtournaments are of the same type.

In this paper we prove that a hamiltonian tournament T_n, $n \geq 5$, is of Moon-type iff each hamiltonian 5-subtournament is too. (We remark that every Moon tournament is either hamiltonian or transitive).

2. PRELIMINARIES

In this section we give some definitions and well known results. See [1] and [5] for the definitions that we shall not refer to.

We denote by T_m a *tournament* of *order* m and by H_m a *hamiltonian* tournament of order m.

We often denote by T_m the set of labeled vertices of the tournament T_m.

C_r usually denotes a *cycle* of r vertices in a tournament, as well as the subtournament with the same vertices.

If (u,v) is an *arc* from the vertex u (called *predecessor*) to the vertex v (called *successor*) in T_m, then we write $u \longrightarrow v$.

$A \longrightarrow B$ means that each vertex of the subtournament A precedes all the vertices of the subtournament B in T_m.

We denote by Tr_m the *transitive* tournament of order m.

We say that a vertex v *cones* a subtournament R in T_m (or R is *coned* by v) iff either $v \longrightarrow R$ or $R \longrightarrow v$ in T_m. A subtournament R is *non-coned* if no vertex exists which cones R in T_m (see [2]).

We say that a subtournament S of T is an *e-component* of T, and its vertices are called *equivalent*, if S is coned by each vertex of T-S. Single vertices and T are trivial e-components.

Every tournament T_n can be partitioned (in a non-unique way) into disjoint e-components $S^1, S^2, \dots, S^m$. In such a case the e-components $S^1, S^2, \dots, S^m$ can be considered as the vertices ($v_1, v_2, \dots, v_m$ respectively) of a tournament Q_m, so that T_n can be obtained as the *composition* $Q_m(S^1, S^2, \dots, S^m)$ of the *quotient* Q_m with the e-components

$S^1, S^2, \ldots, S^m$.

In other words $T_n = S^1 \cup S^2 \cup \ldots \cup S^m$ and $a \to b$ in T_n iff either $a \to b$ in some S^j or $a \in S^h$, $b \in S^k$ and $v_h \to v_k$ (i.e. $S^h \to S^k$).

T_n is *simple* if it has no non-trivial e-component:

A subtournament R is called *shrinkable* in T_n if it is included in a non-trivial e-component of T_n (see [2]).

The *dual* (or *converse*) T* of a tournament T has the same vertex-set as T, but every arc is reversed.

Moon [6] considered tournaments whose subtournaments are either hamiltonian or transitive; we call them *Moon tournaments* or tournaments of *Moon-type*.

Burzio and Demaria (see [2] theorem 8) proved the following.

Proposition 1. *A tournament T_n, $n \geq 4$, is a Moon tournament iff each 4-subtournament in T_n is a Moon tournament, i.e. iff no 3-cycle is coned in T_n.* ■

3. CHARACTERIZATION OF HAMILTONIAN MOON TOURNAMENTS

Proposition 1 allows us to determine easily the Moon tournaments in the set of the hamiltonian tournaments with 5 vertices, that are described, for example, in [5].

In fact the three hamiltonian tournaments $M_5^1 = C_3(v_1, v_2, Tr_3)$,

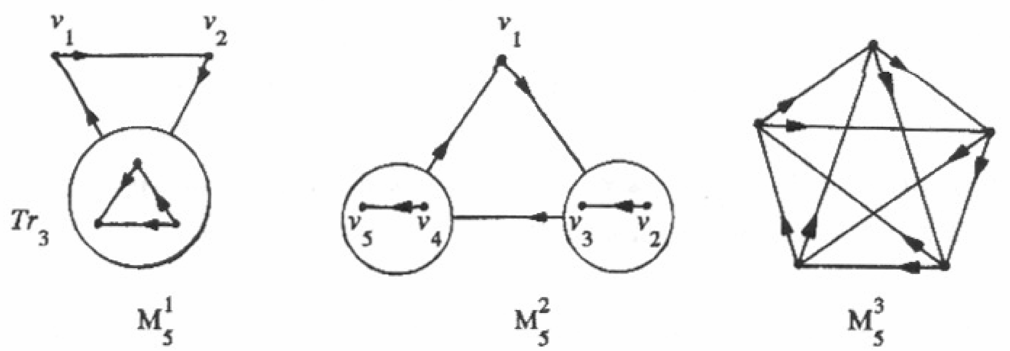

fig.1

$M_5^2 = C_3(v_1,\{v_2,v_3\},\{v_4,v_5\})$ and M_5^3 (i.e. the regular tournament of order 5) of fig.1 are of Moon-type.

On the other hand $N_5^1 = C_3(v_1,v_2,C_3)$, N_5^2 , N_5^3 described in fig.2 are the hamiltonian tournaments that are not of Moon-type, since each of them contains a coned 3-cycle.

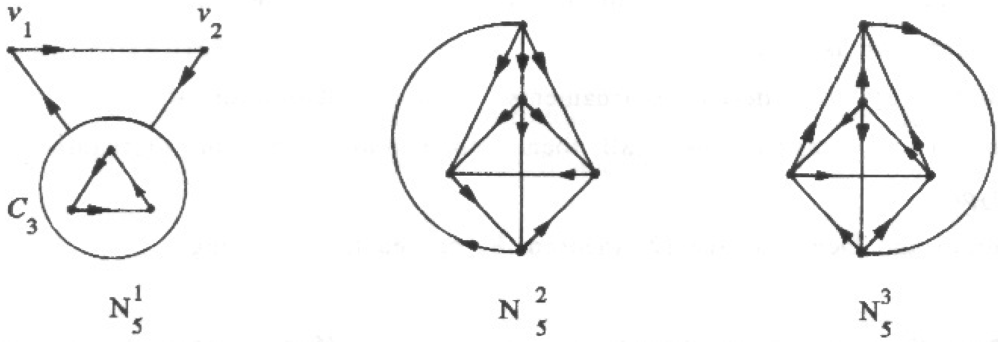

fig.2

Let us denote by $\mathcal{AH}$ the class of hamiltonian tournaments with at least 5 vertices whose hamiltonian 5-subtournaments are of Moon-type.

On the other hand let us denote by $\mathcal{MH}$ the class of hamiltonian Moon tournaments with at least 5 vertices.

We shall prove that $\mathcal{AH} = \mathcal{MH}$ (see proposition 4).

Remark 1. If T_n is of Moon-type then its dual tournament T_n^* is of Moon-type.

If $H_n \in \mathcal{AH}$, then its dual tournament $H_n^* \in \mathcal{AH}$.

Proposition 2. *If* $H_n \in \mathcal{AH}$ *and* S *is a non-trivial e-component of* H_n, *then* S *is a transitive subtournament of* H_n.

Proof. Since H_n is hamiltonian there exists a successor s of S and a predecessor p of S such as $s \longrightarrow p$.

Now if a 3-cycle C_3 were contained in S, then H_n would contain the hamiltonian subtournament $N^1_5 = C_3(s,p,C_3)$ which contradicts the assumption.■

Definition 1. *We say that a cycle C_r in a tournament T_n can be extended to a cycle C_s , $s>r$, in T_n if there exist cycles C_{r+1} , $\dots,C_{s-1}$ such that C_{i+1} can be obtained from C_i by inserting a new vertex between two consecutive vertices, for every $r \leq i \leq s-1$ (or equivalently C_i can be obtained from C_{i+1} by deleting a vertex, for every $r \leq i \leq s-1$).*

Proposition 3. *Let C_m be a cycle in T_n, $m<n$. C_m can be extended to a spanning cycle in T_n if and only if C_m is unshrinkable.*

Proof. If S were a non-trivial e-component of T_n and $C_m \subseteq S$, then each vertex $v \in T_n$-S would cone S hence no cycle contained in S could be extended outside S.

For the "if" part of the proof, the assumption allows us to consider a vertex $v_{m+1} \in T_n - C_m$ that does not cone C_m, so that C_m can be extended to a cycle C_{m+1} containing v_{m+1} and, of course, C_{m+1} cannot be contained in any non-trivial e-component of T_n.

Therefore we can extend by induction C_m to a spanning cycle C_n in T_n.■

A trivial consequence of the preceding result is the following, that generalizes proposition 3 of [2].

Corollary 1. *A tournament is hamiltonian iff it contains an unshrinkable cycle.*■

Proposition 4. $\mathcal{AH} = \mathcal{MH}$.

Proof. The inclusion $\mathcal{MH} \subseteq \mathcal{AH}$ is immediate since every subtournament of a

Moon tournament is of Moon-type too.

Now let $H_n \in \mathcal{AH}$ and assume H_n is not of Moon-type.

If C_3 is any coned 3-cycle in H_n , then we deduce from proposition 2 that no non-trivial e-component of H_n can contain C_3 and consequently, by proposition 3, we can extend C_3 to some spanning cycle C_n in H_n.

If C_3 is coned by v in H_n we denote by $\mathcal{C}(C_3,v)$ the set of all minimal cycles in H_n that can be obtained by extending C_3 and contain v.

Let $\mathcal{C}$ denote the union of the family $\{\mathcal{C}(C_3,v) : C_3 \subseteq H_n,\ C_3$ coned by v, $v \in H_n\}$.

Eventually let us consider a hamiltonian tournament $C_r \in \mathcal{C}$ having minimum lenght in $\mathcal{C}$, and let C_r be an extension of a 3-cycle C_3 coned by $v \in C_r$. Of course $r \geq 5$.

We may assume, by remark 1, that $v \longrightarrow C_3$.

If $C_r = (...(C_3 \cup v_4) \cup ... \cup v_r)$, we must have $v_r = v$.

Otherwise, if $v_j = v$ then $3 < j < r$, $C_j = (...(C_3 \cup v_4) \cup ... \cup v_j)$ would be a smaller cycle than C_r extending C_3 and containing v, so C_r could not belong to $\mathcal{C}(C_3,v)$.

Consequently C_r - v is a cycle which extends C_3 and, since its lenght is smaller than the minimum r, C_r - v cannot contain any vertex which cones C_3.

Now let x be a predecessor of v in $<C_r>$.

If x precedes exactly two vertices (resp. one vertex) of C_3, then $<x,v,C_3>$ is isomorphic to N_5^2 (resp. N_5^3).

In any case H_n cannot belong to $\mathcal{AH}$, which is absurd.■

A trivial consequence of proposition 4 is the following.

Corollary 2. *Let H_n be a hamiltonian tournament. If every hamiltonian 5-subtournament of H_n is isomorphic to M_5^i for some $i=1,2,3$, then every non-hamiltonian subtournament of H_n is transitive.*■

Remark 2. We note that there exist tournaments with at least 6 vertices that are not of Moon-type although their hamiltonian 5-subtournaments are of Moon-type.

Easy examples are provided by $T_6 = T_2(v, M_5^1)$.

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