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# CONNECTEDNESS, ARCWISE-CONNECTEDNESS AND CONVEXITY FOR LEVEL-SETS OF MULTIDIMENSIONAL DISTRIBUTION FUNCTIONS

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**SUNTO** - Si individuano condizioni sufficienti per garantire le proprietà di connessione, arc-connessione e convessità degli insiemi di livello delle funzioni di ripartizione multidimensionali.

**ABSTRACT** - Sufficient conditions for guaranteeing connectedness, arcwise-connectedness and convexity for level-sets of multidimensional distribution functions are provided.

## 1. INTRODUCTION**

General concavity properties of $n$-dimensional $(n \geq 2)$ distribution functions (d.f.'s) have became of recent interest in the literature. See, for example, TONG [9], IYENGAR-TONG [3] for concavity of special distribution classes, TIBILETTI [7] for d.f. quasi-concavity, MARSHALL-OLKIN [4] for Schur-concavity.

In this note we confine our attention to connectedness, arcwise-connectedness and convexity of the d.f. level-sets. Sufficient conditions for guaranteeing above mentioned properties are stated. This issue can be relevant both for theoretical and applied statistical analysis. Recently,

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TIBILETTI [8] has introduced a new notion of $q$-th quantile (where $q \in [0,1]$) of the random vector $X$ as a point belonging to the $q$-th level-set of the d.f. of $X$. Previous properties characterise the quantile sets. The plan of the paper is as follows. Section 2 contains notations and preliminaries. Sufficient conditions for level-set connectedness, arcwise-connectedness and convexity are formulated in Section 3. Section 4 collects some final remarks.

## 2. NOTATION AND PRELIMINARIES

Let $X = (X_1, \ldots, X_n)$ be a random vector. Denote by

$$F(x) = \mathrm{P}\left\{ \bigcap_{i=1}^{n} (X_i \le x_i) \right\}$$

the distribution function of $X$ and by

$$F_i(x) = \mathrm{P}\{X_i \le x\}$$

the one-dimensional marginal associated to the random variable $X_i$. Our aim is to investigate the properties concerning the level-sets of $F$, *i.e.*,

$$I_q = \left\{ x \in \Re^n : F(x) = q \right\}, \text{ where } q \in [0,1].$$

For sake of completeness, we recall the definitions utilised throughout the work.

**Definition 1.** A set $D \in \Re^n$ is said to be *connected* if there exist nonempty disjoint sets $S^1 \subset \Re^n$, $S^2 \subset \Re^n$, such that neither contains cluser points of the other, satisfying $D = S^1 \cup S^2$.

**Definition 2.** A set $D \in \Re^n$ is said to be *arcwise-connected* if for every pair of points $x \in D, y \in D$ there exists a continuous vector valued function $H(x, y; \vartheta)$, called an arc, defined on the unit interval and with values in $D$ such that

$$H(x, y; 0) = y \qquad H(x, y; 1) = x.$$

We recall that every arcwise-connected set is connected , while the opposite does not necessarily hold.

**Definition 3.** A subset $D$ of the $n$-dimensional real Euclidean space $\Re^n$ is a *convex* set if for every $x, y \in D$ and $0 \le \lambda \le 1$ we have $\lambda x + (1-\lambda) y \in D$.

(See for example AVRIEL *et al.* [1] for further details).

## 3. CONNECTEDNESS, ARCWISE-CONNECTEDNESS AND CONVEXITY CONDITIONS

Below, sufficient conditions are provided in order to guarantee the connectedness, arcwise-connectedness and convexity of the d.f. level sets.

**Proposition 1.** *Assume F be continuous. Then,* $I_q$ *is a connected set, for all* $q \in [0,1]$.

*Proof.* Consider the set-valued map $g:[0,1] \to \Re^n$ such that

$$g(q) = I_q = \left\{ x \in \Re^n : F(x) = q \right\}.$$

Since $F$ is continuous $I_q = F^{-1}(q)$ is a closed set and the graph of $g$ is upper semi-continuous . It is immediate to prove that $g$ is also lower semi-continuous, then $g$ is continuous. Thus, $I_q$ turns out to be connected, because it is the image of a connected set.

**Proposition 2.** *Assume F be continuous and partially strictly increasing*[1] *on* $C = [a_1, b_2] \times \ldots \times [a_n, b_n]$, $C \subseteq \Re^n$. *Then* $I_q$ *is arcwise-connected on C, for all* $q \in [0,1]$.

**Proof.** Introduce the function

$$\gamma(x) = F(x) - q.$$

Let $x \in I_q \cap C$. Clearly, $\gamma(x) = 0$. Suppose to fix $(n-1)$ components of $x$, let $x_1, \ldots, x_{n-1}$ be. Thus, the single-variable function $\gamma$ is continuous , strictly increasing and changes its sign on $[a_n, b_n]$. By the implicit function theorem

[1] $F$ is said to be partially strictly increasing on $C = [a_1, b_2] \times \ldots \times [a_n, b_n]$ if

$$t(x_i) = F\left(x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots x_n^*\right)$$

is a strictly increasing function on $[a_i, b_i]$ for all $x_j^* \in [a_j, b_j], j \ne i$.

(see, for example, NIKAIDO [5]) there exists one and only one continuous function $\varphi_n : \Re^{n-1} \to \Re$, defined for $a_s < x_s < b_s$, $(s = 1, \ldots, n-1)$, such that

$$\gamma(x_1, \ldots, x_{n-1}, \varphi_n(x_1, \ldots, x_{n-1})) = 0$$

Therefore, if $x \in I_q$ it follows that

$$F(x_1, \ldots, x_n) = F(x_1, \ldots, x_{n-1}, \varphi_n(x_1, \ldots, x_{n-1})) = G(x_1, \ldots, x_{n-1}),$$

where $G: \Re^{n-1} \to \Re$, is a continuous and partially strictly increasing function .
Also, the repeated use of the theorem shows that there exist continuous functions $\varphi_2, \ldots, \varphi_{n-1}$, such that

$$G(x_1, \ldots, x_{n-1}) = G(x_1, \ldots, x_{n-2}, \varphi_{n-1}(x_1, \ldots, x_{n-2})) = \ldots = K(x_1, x_2, x_3) =$$
$$= K(x_1, x_2, \varphi_3(x_1, x_2)) = W(x_1, x_2) = W(x_1, \varphi_2(x_1)) = T(x_1)$$

Note that: $\varphi_i, i = 2, .., n$ depends on the remaining $\varphi_j, i \neq j$; moreover, from the uniqueness of $\varphi_2, .., \varphi_n$ derives that fixed $x_1$ there exists a unique ($n$-1)-dimensional vector $(x_2, \ldots, x_n)$ such that $x = (x_1, \ldots, x_n) \in I_q$.

Consider $x, y \in I_q$, so that $F(x) = F(y) = q$. Let $H$ be defined on $[0,1] \subseteq \Re$ by

$$H(x, y; \vartheta) = (t_1, \ldots, t_n), \quad \text{where}$$

$t_1 = \vartheta x_1 + (1 - \vartheta) y_1$
$t_2 = \varphi_2(t_1),$
$t_3 = \varphi_3(t_1, t_2)$
..........................
$t_n = \varphi_n(t_1, t_2, \ldots, t_{n-1}).$

Since $H$ is a continuous arc with value in $I_q$ our claim comes out.

An example of arcwise-connected set is given below.

**Example** Let $F$ be a d.f. defined via

$$F(x_1, x_2) = (1 - e^{-x_1})(1 - e^{-x_2}).$$

Observe that

$$I_q = \left\{ (x_1, x_2) \in \Re^2 : x_2 = \log \frac{\left(1 - e^{-x_1}\right)}{\left(1 - e^{-x_1} - q\right)} \right\}, \quad q \in [0,1].$$

It is clear that $I_q$ is arcwise-connected. In fact, we can connect each pair of points $x = (x_1, x_2), y = (y_1, y_2) \in I_q$ by the arc

$$H(x, y; \vartheta) = \left( t, \log \frac{\left(1 - e^{-t}\right)}{\left(1 - e^{-t} - q\right)} \right).$$

where $t = \vartheta x_1 + (1 - \vartheta) y_1$ and $\vartheta \in [0,1]$.

**Proposition 3.** *Assume F be continuous and partially strictly increasing on* $C = [a_1, b_2] \times \ldots \times [a_n, b_n]$. *Let* $x = (x_1, \ldots, x_n) \in I_q \cap C$, *with* $q \in (0,1)$. *For each* $x_i, i = 1,..,n$ *there exists an unique (n-1)-dimensional vector such that*

$$x = (x_1, \ldots, x_i, \ldots, x_n) \in I_q \cap C.$$

In one dimension, the $q$-th level set $I_q$ is connected, then also convex for all $q \in [0,1]$. Nevertheless, this property is not preserved in higher dimensions. In fact, all level sets of $F$ are convex iff $F$ is *quasi-monotone*, *i.e.*, $F$ is both quasi-concave and quasi-convex, and this property holds only for a class of generalised uniform distributions.

## 4 SOME FURTHER FINAL REMARKS

Now, some further remarks are collected.

**Remark 1.** To check whether $F$ is partially strictly increasing or not, it could be useful to calculate $\nabla F$. If $\nabla F$ exists (obviously, a sufficient condition is that the density function is continuous) and $\nabla F > 0$ the property holds.

**Remark 2.** Straightforward calculations show that for the d.f. $F$ results

$$\frac{\partial F}{\partial x_i} = f_i(x_i) F(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n|x_i) \text{ for all } i = 1,..,n$$

where

$f(x_i)$ is the one-dimensional marginal density function of $X_i$,
$F(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n|x_i)$ is the conditional distribution function of $X$ given $X_i = x_i$.
Thus, the condition $\nabla F > 0$ requires that all one-dimensional densities and all conditional d.f.'s be not null.
Note that under these assumptions, the implicit function theorem guarantees that functions $\varphi_i, i = 2,..,n$ are also continuously differentiable.

**Remark 3.** $F$ can be only partially increasing, even if each one-dimensional marginal $F_i(i = 1,..,n)$ is strictly increasing[2]. Additional conditions have to be added. Clearly, the assumption that the copula has to be partially strictly increasing yields our claim (for further details on the copula and a historical overview, the reader can refer to SCHWEIZER [6]).

**Remark 4.** Above discussion deals with the d.f. (also called cumulative function) $F$ of the random vector $X = (X_1,\ldots,X_n)$, nevertheless, analogous results can be expanded upon for every partially decumulative-cumulative function

$$F_{i_1,\ldots,i_n}(x) = \mathrm{P}\left\{\left(\bigcap_{i=i_1,\ldots,i_k}\{X_i \le x_i\}\right) \cap \left(\bigcap_{i=i_{k+1},\ldots,i_n}\{X_i > x_i\}\right)\right\},$$

of $X$, where $i_1,..,i_n$ are integers from 1 to $n$, such that $i_j \ne i_t$ if $j \ne t$ (for further details on the partially cumulative-decumulative function level-sets see TIBILETTI [8]).

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[2] For example, consider the Fréchet cumulative function

$$F(x,y) = Min(F_1(x_1), F_2(x_2)),$$

where $F_1$ and $F_2$ are strictly increasing (see FRÉCHET [2]). Let $x,y,z \in \Re$ such that

$$F_1(x) < F_2(y) < F_2(z),$$

where $y < z$. Since $F(x,y) = F(x,z) = F_1(x_1)$, then $F$ is not partially strictly increasing.

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