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PERIODIC SOLUTIONS OF A SECOND ORDER EVOLUTIVE VARIATIONAL INEQUALITY*

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ABSTRACT - Hereafter we shall analyse the existence and regularity properties of the solution of the periodic problem related to a second order evolutive variational inequality.

SUNTO - Si analizzano le questioni di esistenza e regolarità della soluzione per il problema periodico connesso ad una disequazione variazionale di evoluzione del secondo ordine.

INTRODUCTION

Let Ω_1 and Ω_2 be two open sets of R^N with $\Omega = \Omega_1 \cap \Omega_2 \neq \emptyset$, $V_l(l=1,2)$ a real separable Hilbert space with dense and continuous embedding in $L^2(\Omega_l)$.

Let us denote by:

$(\cdot,\cdot)$, $\vert\cdot\vert$	the inner product and the norm in $L^2(\Omega)$,
$(\cdot,\cdot)_l$, $\vert\cdot\vert_l$	the inner product and the norm in $L^2(\Omega_l)$,
$\Vert\cdot\Vert$	the norm in V_l,
$\langle\cdot,\cdot\rangle$	the pairing between $_l$ and its dual V_l'

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Furthermore, let K be the convex closed part of $V_1 \times V_2$:

$$\left\{(z_1, z_2) \in V_1 \times V_2 : z_1 \leq z_2 \ \ a.e. \text{ on } \Omega\right\}.$$

Given $f_l \in L^2\left(0, T; V_l'\right)$ $(0 < T < +\infty)$ and operators $A_l, B_l \in \mathcal{L}\left(V_l, V_l'\right)$ such that:

$$\langle A_l y, z\rangle_l = \langle A_l z, y\rangle_l, \quad \langle B_l y, z\rangle_l = \langle B_l z, y\rangle_l \quad \forall y, z \in V_l,$$

$$\langle A_l z, z\rangle_l \geq a_l \| z \|_l^2, \quad \langle B_l z, z\rangle_l \geq b_l \| z \|_l^2 \quad \forall z, \in V_l \quad (a_l, b_l = \text{const.} > 0),$$

we consider the following

PROBLEM *(P)*. Find $(u_1, u_2) \in \prod_{l=1}^{2} H^1(0, T; V_l)$ so that:

$$u_l'' \in L^2\left(0, T; V_l'\right),$$

$$u_l(0) = u_l(T), \ u_l'(0) = u_l'(T),$$

$$\left(u_1'(t), u_2'(t)\right) \in K \quad \text{a.e. on }]0, T[,$$

$$\sum_1^2 l \int_0^T \left\langle u_l''(t) + A_l u_l(t) + B_l u_l'(t) - f_l(t), \ v_l'(t) - u_l'(t)\right\rangle_l \ dt \geq 0$$

$$\forall (v_1, v_2) \in \prod_{l=1}^{2} H^1(0, T; V_l) \text{ with } v_l(0) = v_l(T) \text{ and } \left(v_1'(t), v_2'(t)\right) \in K \text{ a.e.on}]0, T[.$$

Of course our problem *(P)* will not have a unique solution. As a matter of fact:
if (u_1, u_2) is one solution of problem *(P)*, then any other solution will only be of the type $(u_1 + z_1, u_2 + z_2)$, where z_l is an arbitrary element of V_l.

Obviously $(u_1 + z_1, u_2 + z_2)$ satisfies our problem *(P)*. On the other hand, assuming that $(\bar{u}_1, \bar{u}_2)$ is an other solution, we immediately find that:

$$\left\|\bar{u}_l' - u_l'\right\|_{L^2(0,T;V_l)} = 0,$$

and therefore $\bar{u}_l = u_l + z_l$ with $z_l \in V_l$.

We shall now develop two existence theorems for our problem, theorems 4 and 5 (n. 3), with two different hypothesis on f_l: in the first we assume $f_l \in L^2\left(0, T; L^2(\Omega_l)\right)$, in the second $f_l \in H^1\left(0, T; V_l'\right)$, $f_l(0) = f_l(T)$.

For the proof we shall essentially adopt a penalty method [1], [3], [4] based on an existence theorem concerning periodic solutions of an abstract non linear second order differential equation. Moreover, in proving this theorem [th. 1, n. 1] we follow a technique inspired to the "elliptic regularization" [3], [5]. The results given by theorems 2 and 3 [n. 2] for the penalized problem will then be used to prove the above mentioned existence theorems for our problem *(P)*.

A regularity theorem "with respect to x" is also given in n. 3, theorem 6, when $V_1 = V_2 = H_0^1(\Omega)$ and operators A_l, B_l, within constant factors, are identical to a uniformly elliptic second order linear differential operator. The uniform ellipticity of this last operator allows us to obtain upper limitations by introducing a "special base" of $H_0^1(\Omega)$.

1. Let V and H be real Hilbert spaces: $V \subseteq H$, with dense and continuous embedding.

We identify H with its dual and denote with

$(\cdot\,,\cdot)\,,\,\lvert\cdot\rvert$	the inner product and the norm in H,
$\lVert\cdot\rVert$	the norm in V,
$\langle\cdot\,,\cdot\rangle$	the pairing between V and its dual V'.

Let also $f \in L^2(0,T;V')$ and A, B the operators from V into V': A linear and continuous, B strictly monotone and hemicontinuous. Let us suppose that:

$$\langle Ay,z\rangle = \langle Az,y\rangle \quad \forall y,\, z \in V,$$

$$\langle Az,z\rangle \geq a\lVert z\rVert^2 \quad \forall z \in V, \quad (a = \text{const.} > 0)$$

$$\langle Bz,z\rangle \geq b\lVert z\rVert^2 \quad \forall z \in V, \quad (b = \text{const.} > 0)$$

for each $v \in L^2(0,T;V)$ $Bv(\cdot) \in L^2(0,T;V')$,

operator $v \in L^2(0,T;V) \rightarrow Bv(\cdot)$ is bounded.

THEOREM 1. In the above stated assumptions, there exists only one solution $u \in H^1(0,T;V)$ of the following problem:

(1) $\quad u'' \in L^2(0,T;V')$,

(2) $\quad \langle u''(t),z\rangle + \langle Au(t),z\rangle + \langle Bu'(t),z\rangle = \langle f(t),z\rangle$ a.e. on $]0,T[$ $\forall z \in V$,

(3) $\quad u(0) = u(T),\ u'(0) = u'(T)$.

PROOF. Firstly, if $u_1, u_2 \in H^1(0,T;V)$ are solutions of the problem (1), (2), (3), then

$$\int_0^T \left\langle Bu_1'(t) - Bu_2'(t),\ u_1'(t) - u_2'(t) \right\rangle dt = 0$$

must hold and therefore

$$u_1'(t) = u_2'(t) \text{ a.e. on} \left]0, T\right[$$

owing to the strict monotonicity of B. Thus there exists a $z_0 \in V$ such that

$$u_1(t) = u_2(t) + z_0 \quad \forall t \in [0, T].$$

This condition in turn is fulfilled iff

$$\langle Az_0, z \rangle = 0 \quad \forall z \in V,$$

namely $z_0 = 0$. In order to prove the existence of the solution, we introduce the Hilbert space:

$$W = \left\{ v \in H^1(0,T;V) : v'' \in L^2(0,T;H),\ v(0) = v(T),\ v'(0) = v'(T) \right\}$$

equipped with norm

$$\| v \|_W = \left(\int_0^T \| v(t) \|^2 dt + \int_0^T \| v'(t) \|^2 dt + \int_0^T |v''(t)|^2 dt \right)^{\frac{1}{2}} \quad \forall v \in W$$

and then denote with $<< \cdot, \cdot >>$ the pairing between W and its dual W'.
Given $\varepsilon > 0$, we set, for any $u, v \in W$

$$<< C^\varepsilon u, v >> = \varepsilon \left[\int_0^T (u''(t), v''(t)) dt + \int_0^T \langle Au(t), v(t) \rangle dt \right] +$$
$$+ \int_0^T (u''(t), v'(t)) dt + \int_0^T \langle Au(t), v'(t) \rangle dt +$$
$$+ \int_0^T \langle Bu'(t), v'(t) \rangle dt - \int_0^T \langle f(t), v'(t) \rangle dt.$$

$C^{\varepsilon}: W \to W'$ is obviously a bounded, strictly monotone, hemicontinuous and coercive operator. Therefore ([3], theorem 2.1, pg. 171) there exists a unique $u_{\varepsilon} \in W$ solution of equation

$$(4) \qquad << C^{\varepsilon} u_{\varepsilon}, v >> = 0 \quad \forall v \in W.$$

Eq. (4), written with $v = u_{\varepsilon}$, gives the upper limitations

$$(5) \qquad \varepsilon\left[\int_0^T \left|u_{\varepsilon}''(t)\right|^2 dt + \int_0^T \left\|u_{\varepsilon}(t)\right\|^2 dt\right] \le c,$$

$$(6) \qquad \int_0^T \left\|u_{\varepsilon}'(t)\right\|^2 dt \le c, \qquad (c = \text{const.} \ > 0 \text{ indep. from } \varepsilon)$$

as well as

$$(7) \qquad \int_0^T \left\|\bar{u}_{\varepsilon}(t)\right\|^2 dt \le c$$

where $\bar{u}_{\varepsilon}(t) = u_{\varepsilon}(t) - u_{\varepsilon}(0)$. Inequalities (5), (6), (7) imply the existence of $\bar{u} \in H^1(0,T;V)$, with $\bar{u}(0) = \bar{u}(T) = 0$, of $g \in L^2(0,T;V')$ and of a positive numerical infinitesimal sequence $\{\varepsilon_n\}$ such that for $n \to +\infty$

$$(8) \qquad \begin{gathered} \bar{u}_{\varepsilon_n} \to \bar{u} \ \text{ weakly in } L^2(0,T;V), \\ u_{\varepsilon_n}' \to \bar{u}' \ \text{ weakly in } L^2(0,T;V), \\ Bu_{\varepsilon_n}'(\cdot) \to g \ \text{ weakly in } L^2(0,T;V'), \\ \varepsilon_n\left[\left\| u_{\varepsilon_n}'' \right\|_{L^2(0,T;H)} + \left\| u_{\varepsilon_n} \right\|_{L^2(0,T;V)}\right] \to 0. \end{gathered}$$

Starting from (4) and using (8), we come to relation

$$(9) \qquad \int_0^T \left(\bar{u}'(t), v''(t)\right) dt = \int_0^T \left\langle A\bar{u}(t) + g(t) - f(t), v'(t)\right\rangle dt \quad \forall v \in W.$$

Let now $\varphi_0 \in C_0^\infty(]0,T[)$ with $\int_0^T \varphi_0(t)dt = 1$, $\varphi \in C_0^\infty(]0,T[)$ and $z \in V$. From (9), setting

$$v(t) = \left(\int_0^t \left[\varphi(s) - \varphi_0(s)\int_0^T \varphi(t)dt\right]ds\right)z \quad \forall t \in [0,T]$$

we get:

$$\left(\int_0^T \bar{u}'(t)\varphi'(t)dt, z\right) =$$

$$\left\langle \int_0^T [A\bar{u}(t) + g(t) - f(t)]\varphi(t)dt,\, z\right\rangle +$$

$$+\left(\left[\int_0^T \bar{u}'(t)\varphi_0'(t)dt\right]\int_0^T \varphi(t)\,dt, z\right) +$$

$$-\left\langle\left[\int_0^T (A\bar{u}(t) + g(t) - f(t))\varphi_0(t)dt\right]\int_0^T \varphi(t)\,dt, z\right\rangle$$

or

$$\bar{u}''(t) = -[A\bar{u}(t) + g(t) - f(t)] - \theta \qquad \text{a.e. on }]0,T[\text{ in the } V' \text{ sense}$$

being θ the element of V'

$$\int_0^T \bar{u}'(t)\varphi_0'(t)\,dt - \int_0^T (A\bar{u}(t) + g(t) - f(t))\varphi_0(t)\,dt.$$

Let u_0 be the element of V for which $Au_0 = \theta$, and given $u = \bar{u} + u_0$, it is obvious that u satisfies (1), the first of (3) and that:

(10)
$$\langle u''(t), z\rangle + \langle Au(t), z\rangle + \langle g(t), z\rangle = \langle f(t), z\rangle \quad \text{a.e. on }]0,T[\ \forall z \in V.$$

We may also note that the second of (3) holds too. Indeed, chosen $\psi \in C^2([0,T])$ with $\psi(0) = \psi(T)$ and $\psi'(0) = \psi'(T) = 1$, recalling (9) and (10), we get $\forall z \in V$:

$$\begin{aligned}(u'(T) - u'(0), z) &= (u'(T)\psi'(T) - u'(0)\psi'(0), z) = \left\langle \int_0^T [u'(t)\psi'(t)]' dt, z \right\rangle = \\ &= \int_0^T \langle u''(t), \psi'(t) z \rangle dt + \int_0^T (u'(t), \psi''(t) z) dt = \\ &= \int_0^T \langle u''(t), \psi'(t) z \rangle dt + \int_0^T \langle Au(t) + g(t) - f(t), \psi'(t) z \rangle dt = \\ &= 0.\end{aligned}$$

Because of (10), (2) is acquired as soon as we are able to prove that

$$Bu'(\cdot) = g. \tag{11}$$

From (4), with $v = u_{\varepsilon_n}$, we obtain:

$$\int_0^T \left\langle Bu'_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) \right\rangle dt \leq \int_0^T \left\langle f(t), u'_{\varepsilon_n}(t) \right\rangle dt$$

from which, because of the second of (8),

$$\lim'' \int_0^T \left\langle Bu'_{\varepsilon_n}(t), u'_{\varepsilon_n}(t) \right\rangle dt \leq \int_0^T \left\langle f(t), \bar{u}'(t) \right\rangle dt . \tag{12}$$

Assuming $z = u'(t)$, (10) produces the following equality:

$$\int_0^T \left\langle g(t), u'(t) \right\rangle dt = \int_0^T \left\langle f(t), u'(t) \right\rangle dt. \tag{13}$$

The second and third of (8), together with (12) and (13), imply (11), since operator

$$v \in L^2(0,T;V) \to Bv(\cdot)$$

is bounded, monotone and hemicontinuous ([3], proposition 2.5, pg. 179).

REMARK. Proof of the existence is essentially the same when assuming "B monotone" instead of "B strictly monotone".

2. Let us suppose

$$V = V_1 \times V_2, \quad H = L^2(\Omega_1) \times L^2(\Omega_2)$$

and write for each $z = (z_1, z_2)$, $y = (y_1, y_2) \in V$

$$\langle Az, y \rangle = \langle A_1 z_1, y_1 \rangle_1 + \langle A_2 z_2, y_2 \rangle_2,$$

$$\langle Lz, y \rangle = \frac{1}{\varepsilon}\left([z_1 - z_2]^+, y_1 - y_2 \right) \text{ with } \varepsilon > 0,$$

$$\langle Bz, y \rangle = \langle B_1 z_1, y_1 \rangle_1 + \langle B_2 z_2, y_2 \rangle_2 + \langle Lz, y \rangle.$$

Of course Hilbertian spaces V, H and operators A, B, from V into V', meet the assumptions stated at the beginning of n. 1. Therefore, from theorem 1, there exists a unique $(u_{1\varepsilon}, u_{2\varepsilon}) \in \prod_{l=1}^{2} H^1(0,T;V_l)$ solution of the problem

(14) $$u''_{l\varepsilon} \in L^2\left(0,T;V'_l\right),$$

(15) $$\sum_1^2 l\left\langle u''_{l\varepsilon}(t) + A_l u_{l\varepsilon}(t) + B_l u'_{l\varepsilon}(t) - f_l(t), z_l \right\rangle_l + \frac{1}{\varepsilon}\left(\left[u'_{1\varepsilon}(t) - u'_{2\varepsilon}(t) \right]^+, z_1 - z_2 \right) = 0$$

a.e. on $]0,T[\quad \forall (z_1, z_2) \in V_1 \times V_2$,

(16) $$u_{l\varepsilon}(0) = u_{l\varepsilon}(T), \quad u'_{l\varepsilon}(0) = u'_{l\varepsilon}(T).$$

THEOREM 2. If for $l = 1,2$ $f_l \in L^2\left(0,T;L^2(\Omega_l)\right)$ we then have:

(17) $$u''_{l\varepsilon} \in L^2\left(0,T;L^2(\Omega_l)\right),$$
$$\left\| u''_{l\varepsilon} \right\|_{L^2(0,T;L^2(\Omega_l))} \le c. \qquad (c = \text{const.} > 0 \text{ indep. from } \varepsilon)$$

PROOF. Let $\{z_{lj}\}$ be a base of V_l and, for each $n \in N$, V_{ln} be the space spanned by $\{z_{l1},\ldots,z_{ln}\}$. Theorem 1 and the finite dimensions of V_{ln} assure the existence of a unique $(w_{1n}, w_{2n}) \in \prod_{l=1}^{2} H^2(0,T;V_{ln})$ such that

$$
(18)\qquad
\begin{aligned}
&\sum_1^2 l\left\{\left(w_{ln}''(t), z_l\right)_l + \left\langle A_l w_{ln}(t) + B_l w_{ln}'(t), z_l\right\rangle_l - \left(f_l(t), z_l\right)_l\right\} + \\
&\frac{1}{\varepsilon}\left(\left[w_{1n}'(t) - w_{2n}'(t)\right]^+, z_1 - z_2\right) = 0 \\
&\qquad\qquad\qquad\qquad \text{a.e. on }]0,T[\quad \forall (z_1,z_2) \in V_{1n} \times V_{2n}, \\
&w_{ln}(0) = w_{ln}(T),\quad w_{ln}'(0) = w_{ln}'(T).
\end{aligned}
$$

Immediate consequence of (18) are the upper limitations:

$$
(19)\qquad
\begin{aligned}
&\left\| w_{ln}' \right\|_{L^2(0,T;V_l)} \le c, \\
&\qquad\qquad (c = \text{const.} > 0 \text{ indep. from } \varepsilon \text{ and } n) \\
&\left\| w_{ln}'' \right\|_{L^2(0,T;L^2(\Omega_l))} \le c,
\end{aligned}
$$

and also

$$
\left\| \overline{w}_{ln} \right\|_{L^2(0,T;V_l)} \le c,
$$

where $\overline{w}_{ln}(t) = w_{ln}(t) - w_{ln}(0)$ Therefore $\overline{w}_l \in H^1(0,T;V_l)$, with

$$
(20)\qquad \overline{w}_l'' \in L^2\left(0,T;L^2(\Omega_l)\right),\quad \overline{w}_l(0) = \overline{w}_l(T) = 0,\quad \overline{w}_l'(0) = \overline{w}_l'(T),
$$

and $h \in L^2\left(0,T;L^2(\Omega)\right)$ exist so that, to within a subsequence, for $n \to +\infty$:

$$
(21)\qquad
\begin{aligned}
&\overline{w}_{ln} \to \overline{w}_l && \text{weakly in } L^2(0,T;V_l), \\
&w_{ln}' \to w_l' && \text{weakly in } L^2(0,T;V_l), \\
&w_{ln}'' \to w'' && \text{weakly in } L^2\left(0,T;L^2(\Omega_l)\right), \\
&\left[w_{1n}' - w_{2n}'\right]^+ \to h && \text{weakly in } L^2\left(0,T;L^2(\Omega)\right)
\end{aligned}
$$

Using (21) and equality

$$\overline{\bigcup_{n\in N} V_{ln}} = V_l,$$

we easily derive from the first of (18) this relation:

$$\sum_1^2 l \int_0^T \left\{ \left(\overline{w}_l''(t), \varphi'(t) z_l \right)_l + \left\langle A_l \overline{w}_l(t) + B_l \overline{w}_l'(t), \varphi'(t) z_l \right\rangle_l - \left(f_l(t), \varphi'(t) z_l \right)_l \right\} dt +$$

$$+ \frac{1}{\varepsilon} \int_0^T \left(h(t), \varphi'(t)(z_1 - z_2) \right) dt = 0 \quad \forall \varphi \in C_0^\infty(]0,T[) \text{ and } \forall (z_1, z_2) \in V_1 \times V_2,$$

the latter being equivalent, a.e. on $]0,T[$, to:

$$\frac{d}{dt}\left[\overline{w}_1''(t) + A_1 \overline{w}_1(t) + B_1 \overline{w}_1'(t) + \frac{1}{\varepsilon} h(t) - f_1(t) \right] = 0 \text{ in the sense of } V_1',$$

$$\frac{d}{dt}\left[\overline{w}''(t) + A_2 \overline{w}_2(t) + B_2 \overline{w}_2'(t) - \frac{1}{\varepsilon} h(t) - f_2(t) \right] = 0 \text{ in the sense of } V_2',$$

which in turn lead to existence of $F_l \in V_l'$ such that, a.e. on $]0,T[$:

$$\text{(22)} \qquad \begin{aligned} &\overline{w}_1''(t) + A_1 \overline{w}_1(t) + B_1 \overline{w}_1'(t) + \frac{1}{\varepsilon} h(t) - f_1(t) = F_1, \\ &\overline{w}_2''(t) + A_2 \overline{w}_2(t) + B_2 \overline{w}_2'(t) - \frac{1}{\varepsilon} h(t) - f_2(t) = F_2. \end{aligned}$$

Given $w_{l0} = A_l^{-1} F_l$ and $w_l = \overline{w}_l - w_{l0} \in H^1(0,T;V_l)$, from (20), (22) we get respectively:

$$\text{(23)} \qquad w_l'' \in L^2\left(0,T; L^2(\Omega_l)\right), \quad w_l(0) = w_l(T), \quad w_l'(0) = w_l'(T)$$

$$\sum_1^2 l\left\{\left(w_l''(t),z_l\right)_l+\left\langle A_l w_l(t)+B_l w_l'(t),z_l\right\rangle_l-\left(f_l(t),z_l\right)_l\right\}+ \tag{24}$$
$$+\frac{1}{\varepsilon}\left(h(t),z_1-z_2\right)=0 \quad \text{a.e. on} \,]0,T[\quad \forall\left(z_1,z_2\right)\in V_1\times V_2.$$

Since from (18) and the second of (21)

$$\lim_n{}'' \frac{1}{\varepsilon}\int_0^T\left(\left[w_{1n}'(t)-w_{2n}'(t)\right]^+,\ w_{1n}'(t)-w_{2n}'(t)\right)dt\le$$
$$\le\sum_1^2 l\int_0^T\left(f_l(t),\ w_l'(t)\right)_l dt-\sum_1^2 l\int_0^T\left\langle B_l w_l'(t),\ w_l'(t)\right\rangle_l dt$$

and from (23), (24)

$$\frac{1}{\varepsilon}\int_0^T\left(h(t),w_1'(t)-w_2'(t)\right)dt=\sum_1^2 l\int_0^T\left(f_l(t),w_l'(t)\right)_l dt+$$
$$-\sum_1^2 l\int_0^T\left\langle B_l w_l'(t),w_l'(t)\right\rangle_l dt,$$

we may write

$$h(t)=\left[w_1'(t)-w_2'(t)\right]^+ \quad \forall t\in[0,T] \tag{25}$$

since

$$v\in L^2(0,T;V)\to Lv(\cdot)$$

is a bounded, monotone and hemicontinuous operator. From (23), (24) and (25) we see that $w_l=u_l\varepsilon$. Consequently (17) hold: the first of (17) is true because of the first of (20), the second because of (19) and the third of (21).

THEOREM 3. If for $l=1,2$ $f_l \in H^1\left(0,T;V_l'\right)$ and $f_l(0)=f_l(T)$, then we have:

$$u_{l\varepsilon}'' \in L^2(0,T;V_l),$$
$$(c = \text{const.} > 0 \text{ indep. from } \varepsilon)$$
$$\left\| u_{l\varepsilon}'' \right\|_{L^2(0,T;V_l)} \le c.$$

PROOF. Proceeding as with the beginning of the proof of theorem 2 and considering the present hypothesis on f_l, we see that there exists a unique $(w_{1n}, w_{2n}) \in \prod_{l=1}^{2} H^3(0,T;V_{ln})$ which fulfills the following conditions:

(26)

$$\sum_1^2 l\left\{\left(w_{ln}''(t), z_l\right)_l + \left\langle A_l w_{ln}(t) + B_l w_{ln}'(t) - f_l(t), z_l\right\rangle_l\right\} +$$
$$+\frac{1}{\varepsilon}\left(\left[w_{1n}'(t) - w_{2n}'(t)\right]^+, z_1 - z_2\right) = 0 \ \forall t \in [0,T], \text{ and } \forall (z_1,z_2) \in V_1 \times V_2,$$

(27) $$w_{ln}(0) = w_{ln}(T), \quad w_{ln}'(0) = w_{ln}'(T), \quad w_{ln}''(0) = w_{ln}''(T).$$

By differentiating the left member of (26), accounting for both the second and third of (27) and inequality:

$$\left(\frac{d}{dt}\left[w_{1n}'(t) - w_{2n}'(t)\right]^+, \quad w_{1n}''(t) - w_{2n}''(t)\right) \ge 0 \quad \forall t \in [0,T],$$

we obtain

$$\left\| w_{ln}'' \right\|_{L^2(0,T;V_l)} \le c. \qquad (c = \text{const.} > 0 \text{ indep. from } \varepsilon \text{ and } n)$$

This proof is completed by proceeding similarly to what reasoned with theorem 2.

3. Results obtained in n. 2 will now allow us to produce some existence theorems for problem *(P)*.

THEOREM 4. If for $l=1,2$ $f_l \in L^2\left(0,T;L^2(\Omega_l)\right)$ then there exists a $(u_1,u_2) \in \prod_{l=1}^{2} H^1(0,T;V_l)$, with $u_l'' \in L^2\left(0;T;L^2(\Omega_l)\right)$, which is solution of problem *(P)*.

PROOF. For any $\varepsilon>0$ let $(u_{1\varepsilon},u_{2\varepsilon}) \in \prod_{l=1}^{2} H^1(0,T;V_l)$ be a solution of problem (14), (15), (16). Because of theorem 2 $u_{l\varepsilon}'' \in L^2\left(0,T;L^2(\Omega_l)\right)$ and we have:

(28) $$\left\| u_{l\varepsilon}'' \right\|_{L^2\left(0,T;L^2(\Omega_l)\right)} \le c. \qquad (c = \text{const.} > 0 \text{ indep. from } \varepsilon)$$

Consequence of (15), (16) are the upper limitations:

(29) $$\left\| u_{l\varepsilon}' \right\|_{L^2(0,T;V_l)} \le c,$$

$$(c = \text{const.} > 0 \text{ indep. from } \varepsilon)$$

(30) $$\left\| \left[u_{1\varepsilon}' - u_{2\varepsilon}' \right]^+ \right\|^2_{L^2\left(0,T;L^2(\Omega)\right)} \le c\varepsilon .$$

The existence of $(u_1,u_2) \in \prod_{l=1}^{2} H^1(0,T;V_l)$ with

$$u_l'' \in L^2\left(0,T;L^2(\Omega_l)\right), \quad u_l(0)=u_l(T), \quad u_l'(0)=u_l'(T),$$

and of a positive numerical infinitesimal sequence $\{\varepsilon_n\}$ such that for $n \to +\infty$:

(31) $$\begin{aligned} u_{l\varepsilon_n} - u_{l\varepsilon_n}(0) \to u_l &\quad \text{weakly in } L^2(0,T;V_l), \\ u_{l\varepsilon_n}' \to u_l' &\quad \text{weakly in } L^2(0,T;V_l), \\ u_{l\varepsilon_n}'' \to u_l'' &\quad \text{weakly in } L^2\left(0,T;L^2(\Omega_l)\right). \end{aligned}$$

is guaranteed by (28) and (29).

Thus, solution of problem *(P)* is (u_1, u_2). Indeed, on one hand

$$\left(u_1'(t), u_2'(t)\right) \in \mathrm{K} \qquad \text{a.e. on }]0,T[$$

since, holding (30) and the second of (31), we have:

$$\int_0^T \left|\left[u_1'(t) - u_2'(t)\right]^+\right|^2 dt \le \lim \int_0^T \left|\left[u_{1\varepsilon_n}'(t) - u_{2\varepsilon_n}'(t)\right]^+\right|^2 dt = 0 .$$

On the other hand, by virtue of inequality

$$\int_0^T \left(\left[u_{1\varepsilon_n}'(t) - u_{2\varepsilon_n}'(t)\right]^+, \left[v_1'(t) - u_{1\varepsilon_n}'(t)\right] - \left[v_2'(t) - u_{2\varepsilon_n}'(t)\right]\right) dt \le 0,$$

where (v_1, v_2) is an arbitrary element of $\prod_{l=1}^{2} H^1(0,T;V_l)$, satisfying conditions:

$$\left(v_1'(t), v_2'(t)\right) \in \mathrm{K} \qquad \text{a.e. on }]0,\mathrm{T}[,\ v_l(0) = v_1(T),$$

from (15), (16) we get:

$$\sum_1^2 l \int_0^T \left\{\left(u_{l\varepsilon_n}''(t), v_l'(t)\right)_l + \left\langle A_l\left[u_{l\varepsilon_n}(t) - u_{l\varepsilon_n}(0)\right], v_l'(t)\right\rangle_l + \left\langle B_l\, u_{l\varepsilon_n}'(t), v_l'(t)\right\rangle_l + \right.$$

$$\left. -\left(f_l(t), v_l'(t) - u_{l\varepsilon_n}'(t)\right)_l\right\} dt \ge \sum_1^2 l \int_0^T \left\langle B_l\, u_{l\varepsilon_n}'(t), u_{l\varepsilon_n}'(t)\right\rangle_l dt$$

and from here, taking the limit as $n \to +\infty$, we obtain because of (31):

$$\sum_1^2 l \int_0^T \left\{ \left(u_l''(t), v_l'(t) \right)_l + \left\langle A_l u_l(t), v_l'(t) \right\rangle_l + \left\langle B_l u_l'(t), v_l'(t) \right\rangle_l + \right.$$

$$\left. - \left(f_l(t), v_l'(t) - u_l'(t) \right)_l \right\} dt \geq \sum_1^2 l \int_0^T \left\langle B_l u_l'(t), u_l'(t) \right\rangle_l dt.$$

Using theorem 3, with the above procedure we prove the following

THEOREM 5. If for $l = 1,2$ $f_l \in H^1\left(0,T;V_l'\right)$ and $f_l(0) = f_l(T)$ then there exists a $(u_1, u_2) \in H^2(0,T;V_l)$ solution of problem *(P)*.

Let us complete the study of problem *(P)* by analsysing a particular case. Let: $\Omega_1 = \Omega_2 = \Omega$ be a $C^{1,1}$ open, bounded, connected set of R^n and $V_1 = V_2 = H_0^1(\Omega)$.

We now consider the uniformly elliptic second order linear differential operator

$$A = -\sum_1^n ij \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial}{\partial x_i} \right) \qquad \text{with } a_{ij} = a_{ji} \in C^{0,1}(\Omega),$$

and let

$$A_l = \alpha_l A, \qquad B_l = \beta_l A,$$

α_1 and β_1 being positive constants.

THEOREM 6. If for $l = 1,2$ $f_l \in L^2\left(0,T;L^2(\Omega)\right)$, then there exists a $(u_1, u_2) \in \left[H^1\left(0,T;H_0^1(\Omega) \cap H^2(\Omega)\right) \right]^2$ with $u_l'' \in L^2\left(0,T;L^2(\Omega)\right)$, solution to problem *(P)*.

PROOF. In the light of the proof given for theorem 4 and of statements made in the introduction concerning solutions of problem *(P)*, it is evidently enough to prove that for the solution $(u_{1\varepsilon}, u_{2\varepsilon})$ of the problem (14), (15), (16) we have:

$$u_{l\varepsilon}' \in L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right), \quad u_{l\varepsilon}'' \in L^2\left(0,T;L^2(\Omega)\right),$$

(32) $(c = \text{const.} > 0 \text{ indep. from } \varepsilon)$

$$\left\| u_{l\varepsilon}' \right\|_{L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right)} + \left\| u_{l\varepsilon}'' \right\|_{L^2\left(0,T;L^2(\Omega)\right)} \le c.$$

Assumptions made for Ω and operator A assure ([2], Remark 31, pg. 308; [7], theor. 2.1, pg. 201) the existence of a base $\{z_j\}$ of $H_0^1(\Omega)$ of functions of $H_2(\Omega)$ such that

(33) $$Az_j = \lambda_j z_j$$

where $\{\lambda_j\}$ is a positively diverging sequence of positive numbers. Let V_n be the space spanned by $\{z_1,\dots,z_n\}$, from theorem 1 there is a unique $(w_{1n}, w_{2n}) \in \left[H^2(0,T;V_n)\right]^2$ solution of the problem:

(34)

$$\sum_1^2 l\left(w_{ln}''(t)+\alpha_l A w_{ln}(t)+\beta_l A w_{ln}'(t)-f_l(t),\ y_l\right)+$$
$$+\frac{1}{\varepsilon}\left(\left[w_{1n}'(t)-w_{2n}'(t)\right]^+,\ \ y_1-y_2\right)=0 \quad \text{a.e. on}]0,T[\quad \forall(y_1,y_2)\in V_n^2,$$

(35) $$w_{ln}(0)=w_{ln}(T), \quad w_{ln}'(0)=w_{ln}'(T).$$

Recalling (33), (34), may also be written as:

$$\sum_1^2 l\left(w_{ln}''(t)+\alpha_l A w_{ln}(t)+\beta_l A w_{ln}'(t)-f_l(t), Ay_l\right)+$$
$$+\frac{1}{\varepsilon}\left(\left[w_{1n}'(t)-w_{2n}'(t)\right]^+,\ A[y_1-y_2]\right)=0$$
$$\text{a.e. on}]0,T[\quad \forall(y_1,y_2)\in V_n^2$$

and this, together with (35) and the following inequality

$$\left(\left[w_{1n}'(t)-w_{2n}'(t)\right]^{+},\ A\left[w_{1n}'(t)-w_{2n}'(t)\right]\right)\geq 0 \qquad \forall t\in[0,T],$$

allow us to acquire the upper limitation:

$$\left\| A w_{ln}' \right\|_{L^2(0,T;L^2(\Omega))} \leq c \quad (c = \text{const.} > 0 \text{ indep. from } \varepsilon \text{ and } n)$$

from which ([7], theor. 2.1, pg. 201):

$$(36) \qquad \left\| w_{ln}' \right\|_{L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))} \leq c.$$

The further upper limitation

$$(37) \quad \left\| w_{ln}'' \right\|_{L^2(0,T;L^2(\Omega))} \leq c \quad (c = \text{const.} > 0 \text{ indep. from } \varepsilon \text{ and } n)$$

follows because of (34), (35), (36).

Inequalities (36), (37) bring us to (32) with the same technique used for theorem 2 considering that now $F_l \in L^2(\Omega)$ and that $w_{l0} = A_l^{-1} F_l \in H_0^1(\Omega)\cap H^2(\Omega)$ ([7], theor. 2.1, pg. 201).

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