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On Hyper Hoop-algebras

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Abstract

In this paper, we apply the hyper structure theory to hoop-algebras and introduce the notion of (quasi) hyper hoop-algebra which is a generalization of hoop-algebra and investigate some related properties. We also introduce the notion of (weak)filters on hyper hoop-algebras, and give several properties of them. Finally, we characterize the (weak) filter generated by a non-empty subset of a hyper hoop-algebra.

Keywords: Hoop-algebra, (quasi) hyper hoop-algebra, (weak) filter.

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1 Introduction

Hoop-algebras or Hoops are naturally ordered commutative residuated integral monoids were originally introduced by Bosbach in [7] under the name of complementary semigroups. It was proved that a hoop is a meet-semilattice. Hoop-algbras then investigated by Büchi and Owens in an unpublished manuscript [8] of 1975, and they have been studied by Blok and Ferreirim [2],[3], and Aglianò et.al. [1], among others. The study of hoops is motivated by their occurrence both in universal algebra and algebraic logic. Typical examples of hoops include both Brouwerian semilattices and the positive cones of lattice ordered abelian groups,

while hoops structurally enriched with normal multiplicative operators naturally generalize the normal Boolean algebras with operators. In recent years, hoop theory was enriched with deep structure theorems. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic introduced by Hájek in [12]. The algebraic structures corresponding to Hájek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops and MV-algebras, product algebras and Gödel algebras are the most known classes of BL-algebras. Hypersructure theory was introduced in 1934[13], when Marty at the 8th congress of scandinavian mathematicians, gave the definition of hypergroup and illustrated some applications and showed its utility in the study of groups, algebraic functions, and rational fraction. Till now, the hyperstructures have been studied from the theoretical point of view for their applications to many subject of pure and applied mathematics. Some fields of applications of the mentioned structures are lattices, graphs, coding, ordered sets, median algebra, automata, and cryptography[9]. Many researchers have worked on this area. R.A.Borzooei et al. introduced and studied hyper residuated lattices and hyper K-algebras in [4],[6] and S.Ghorbani et al.[11], applied the hyper structures to MV-algebras and introduced the concept of hyper MV-algebra, which is a generalization of MV-algebra.
In this paper we construct and introduce the notion of (quasi) hyper hoop-algebra which is a generalization of hoop-algebra. Then we study some properties of this structure. We also introduce the notion of (weak)filters on hyper hoop-algebras, and give several properties of them. Finally, we characterize the (weak) filter generated by a non-empty subset of a hyper hoop-algebra.

2 Preliminaries

In this section, we recall some definitions and theorems in hoop algebras which will be needed in this paper.

Definition 2.1. [1] A *hoop-algebra* or a *hoop* is an algebra $(A, *, \to, 1)$ of the type (2, 2, 0) such that, for all $x, y, z \in A$:
(H1) $(A, *, 1)$ is a commutative monoid,
(H2) $x \to x = 1$,
(H3) $(x \to y) * x = (y \to x) * y$,
(H4) $x \to (y \to z) = (x * y) \to z$.

On the hoop A, if we define $x \leq y$ iff $x \to y = 1$, for any $x, y \in A$, it is proved that $\leq$ is a partial order on A. A hoop A is bounded if there is an element

$0 \in A$ such that $0 \leq x$ for all $x \in A$.

Proposition 2.2. [1] Let A be a hoop-algebra. Then for every $a, b, c \in A$ the following hold:
(i) $(A, \leq)$ is a $\wedge$-semilattice and $a \wedge b = a * (a \to b)$,
(ii) $a \leq b \to c$ iff $a * b \leq c$,
(iii) $1 \to a = a$,
(iv) $a \to 1 = 1$, i.e. $a \leq 1$,
(v) $a \to b \leq (c \to a) \to (c \to b)$,
(vi) $a \leq b \to a$,
(vii) $a \leq (a \to b) \to b$,
(viii) $a \to (b \to c) = b \to (a \to c)$,
(ix) $a \to b \leq (b \to c) \to (a \to c)$,
(x) $a \leq b$ implies $b \to c \leq a \to c$ and $c \to a \leq c \to b$.

Now, we recall some basic notions of the hypergroup theory from [9]:
Let H be a non-empty set. A hypergroupoid is a pair $(H, \odot)$, where $\odot : H \times H \longrightarrow P(H) \setminus \emptyset$ is a binary hyperoperation on H. If $a \odot (b \odot c) = (a \odot b) \odot c$ holds, for all $a, b, c \in H$ then $(H, \odot)$ is called a semihypergroup, and it is said to be commutative if $\odot$ is commutative. An element $1 \in H$ is called a unit, if $a \in 1 \odot a \cap a \odot 1$, for all $a \in H$ and is called a scaler unit, if $\{a\} = 1 \odot a = a \odot 1$, for all $a \in A$. If the reproduction axiom $a \odot H = H = H \odot a$, for any element $a \in H$ is satisfied, then the pair $(H, \odot)$ is called a hypergroup. Note that if $A, B \subseteq H$, then $A \odot B = \bigcup_{a \in A, b \in B} (a \odot b)$.

3 Hyper hoop-algebras

Definition 3.1. A*quasi hyper hoop-algebra* or briefly, a *quasi hyper hoop* is a non-empty set A endowed with two binary hyperoperations $\odot, \to$ and a constant 1 such that, for all $x, y, z \in A$ satisfying the following conditions:
(HHA1) $(A, \odot, 1)$ is a commutative semihypergroup with 1 as the unit,
(HHA2) $1 \in x \to x$,
(HHA3) $(x \to y) \odot x = (y \to x) \odot y$,
(HHA4) $x \to (y \to z) = (x \odot y) \to z$,
A quasi hyper hoop $(A, \odot, \to, 1)$ is called a *hyper hoop* if the following hold;
(HHA5) $1 \in x \to 1$,
(HHA6) if $1 \in x \to y$ and $1 \in y \to x$ then $x = y$,
(HHA7) if $1 \in x \to y$ and $1 \in y \to z$ then $1 \in x \to z$.

In the sequel we will refer to the (quasi) hyper hoop $(A, \odot, \to, 1)$ by its universe A. On (quasi) hyper hoop A, for any $x, y \in A$, we define $x \leq y$ if and only if $1 \in x \to y$. If A is a hyper hoop, it is easy to see that $\leq$ is a partial order relation on A. Moreover, for all $B, C \subseteq A$ we define $B \ll C$ iff there exist $b \in B$ and $c \in C$ such that $b \leq c$ and define $B \leq C$ iff for any $b \in B$ there exists $c \in C$ such that $b \leq c$. A (quasi) hyper hoop A is bounded if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$.

In the following examples, we will show that the conditions (HHA5), (HHA6), and (HHA7) are independent from the other conditions.

Example 3.2. (i) Let $A = \{1, a, b\}$. Define the hyperoperations $\odot$, and $\to$ on A as follows:

$\odot$	1	a	b
1	$\{1\}$	$\{a\}$	$\{a, b\}$
a	$\{a\}$	$\{a\}$	$\{a, b\}$
b	$\{a, b\}$	$\{a, b\}$	$\{b\}$

$\to$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{b\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, a, b\}$	$\{1, a, b\}$	$\{1, a, b\}$

Then $(A, \odot, \to, 1)$ is a quasi hyper hoop, but doesn't satisfy the condition (HHA5). Since $1 \notin a \to 1$.

(ii) Let $A = \{1, a, b\}$. Define the hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{a\}$	$\{1\}$

$\to$	1	a	b
1	$\{1, b\}$	$\{a\}$	$\{1, b\}$
a	$\{1, b\}$	$\{1, b\}$	$\{1, b\}$
b	$\{1, b\}$	$\{a\}$	$\{1, b\}$

Then $(A, \odot, \to, 1)$ is a quasi hyper hoop, but doesn't satisfy the condition (HHA6). Since $1 \in b \to 1$ and $1 \in 1 \to b$, but $1 \neq b$.

(iii) Let $A = \{1, a, b, c\}$. Define hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{b\}$	$\{b\}$
c	$\{c\}$	$\{a, b\}$	$\{b\}$	$\{c\}$

$\to$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{a, 1\}$	$\{1, b, c\}$	$\{c\}$
b	$\{1\}$	$\{a\}$	$\{1, b, c\}$	$\{1, b, c\}$
c	$\{1\}$	$\{a\}$	$\{b\}$	$\{1, b, c\}$

Then $(A, \odot, \to, 1)$ is a quasi hyper hoop, but doesn't satisfy the condition (HHA7). Because $1 \in a \to b$ and $1 \in b \to c$ but $1 \notin a \to c$.

In the following, we give some examples of (quasi) hyper hoop algebras.

Example 3.3. (i) In any (quasi) hyper hoop $(A, \odot, \rightarrow, 1)$, if $x \odot y$ and $x \rightarrow y$ are singletons, for any $x, y \in A$, then $(A, \odot, \rightarrow, 1)$ is a hoop. Then (quasi) hyper hoops are generalizations of hoops.

(ii) Let $A = \{1\}$. If we consider $1 \rightarrow 1 = \{1\}$, $1 \odot 1 = \{1\}$, then it is clear that $A = (A, \odot, \rightarrow, 1)$ is a (quasi) hyper hoop.

(iii) Let $A = \{1, a\}$. Define the hyperoperations $\odot$ and $\rightarrow$ on A as follows:

$\odot$	1	a
1	$\{1\}$	$\{1, a\}$
a	$\{1, a\}$	{a}

$\rightarrow$	1	a
1	$\{1, a\}$	$\{a\}$
a	$\{1\}$	$\{1, a\}$

Then $(A, \odot, \rightarrow, 1)$ is a bounded (quasi) hyper hoop.

(iv) Let $A = \{1, a, b\}$. Define the hyperoperations $\odot$ and $\rightarrow$ on A as follows,

$\odot$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{a, b\}$	$\{a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{b\}$

$\rightarrow$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{1, b\}$
b	$\{1\}$	$\{a\}$	$\{1, b\}$

Then $(A, \odot, \rightarrow, 1)$ is a bounded (quasi) hyper hoop.

(v) Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot$ and $\rightarrow$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{a, b, c\}$	$\{a, c\}$
b	$\{b\}$	$\{a, b, c\}$	$\{b, c\}$	$\{b, c\}$
c	$\{c\}$	$\{a, c\}$	$\{b, c\}$	$\{c\}$

$\rightarrow$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b, c\}$	$\{1, c\}$
b	$\{1\}$	$\{a\}$	$\{1, b, c\}$	$\{b, c\}$
c	$\{1\}$	$\{a\}$	$\{b\}$	$\{1, b, c\}$

Then $(A, \odot, \to, 1)$ is a bounded (quasi) hyper hoop.

(vi) Let $A = \{1, a, b, c\}$. Define the hyperoperations $\odot$ and $\to$ on A as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{a,b,c\}$	$\{a,c\}$
b	$\{b\}$	$\{a,b,c\}$	$\{b,c\}$	$\{b,c\}$
c	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{c\}$

$\to$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1,a\}$	$\{b\}$	$\{1,c\}$
b	$\{1\}$	$\{a\}$	$\{1,b,c\}$	$\{b,c\}$
c	$\{1\}$	$\{a\}$	$\{b\}$	$\{1,b,c\}$

Then $(A, \odot, \to, 1)$ is an unbounded (quasi) hyper hoop. Hence, (quasi) hyper hoops may not be bounded, in general.

(vii) Let $A = [0, 1]$. Define the hyperoperations $\odot$ and $\to$ on A as follows:

$$x \odot y = \{1, x, y\} \qquad x \to y = \begin{cases} \{1, y\} & \text{, if } x \le y, \\ \{y\} & \text{, otherwise.} \end{cases}$$

Then $(A, \odot, \to, 1)$ is an infinite (quasi) hyper hoop.

Proposition 3.4. Let A be a quasi hyper hoop. Then the following hold, for all $x, y, z \in A$ and $B, C, D \subseteq A$:
(HHA8) $B \ll C \Leftrightarrow 1 \in B \to C$,
(HHA9) $(B \odot C) \to D = B \to (C \to D)$,
(HHA10) $x \odot y \ll \{z\} \Leftrightarrow \{x\} \le y \to z$,
(HHA11) $B \odot C \ll D \Leftrightarrow B \ll C \to D$,
(HHA12) $x \to (y \to z) = y \to (x \to z)$,
(HHA13) $\{x\} \le y \to z \Leftrightarrow \{y\} \le x \to z$,
(HHA14) $\{x\} \le (x \to y) \to y$,
(HHA15) $x \odot (x \to y) \ll \{y\}$.

Proof. Let $x, y, z \in A$ and $B, C, D \subseteq A$. Then,
(HHA8): $B \ll C \Leftrightarrow$ there exist $b \in B$ and $c \in C$ such that $b \le c$ i.e. $1 \in b \to c \Leftrightarrow 1 \in B \to C$.

(HHA9): By (HHA4), the proof is clear.
(HHA10): $x \odot y \ll \{z\} \Leftrightarrow$ by (HHA8), $1 \in (x \odot y) \to z \Leftrightarrow$ by (HHA4), $1 \in x \to (y \to z) \Leftrightarrow$ by (HHA8), $\{x\} \leq y \to z$.
(HHA11): The proof is similar to the proof of (HHA10).
(HHA12): By (HHA4) and (HHA1),

$$x \to (y \to z) = (x \odot y) \to z = (y \odot x) \to z = y \to (x \to z).$$

(HHA13): $\{x\} \leq y \to z \Leftrightarrow$ by (HHA10), $x \odot y \ll \{z\} \Leftrightarrow$ by (HHA1), $y \odot x \ll \{z\} \Leftrightarrow$ by (HHA10), $\{y\} \leq x \to z$.
(HHA14): Since $x \to y \ll x \to y$, by (HHA1)and (HHA11), $x \odot (x \to y) \ll \{y\}$ and so by (HHA11), $\{x\} \leq (x \to y) \to y$.
(HHA15): By (HHA10) and (HHA14), the proof is clear. □

Proposition 3.5. Let A be a hyper hoop. Then the following hold, for all $x, y, z, t \in A$ and $B, C, D \subseteq A$,
(HHA16) $x \odot y \ll \{x\}, \{y\}$,
(HHA17) $\{y\} \leq x \to y$,
(HHA18) if $1 \in 1 \to x$, then $x = 1$,
(HHA19) $x \in 1 \to x$, and x is the maximum element of $1 \to x$,
(HHA20) $1 \odot 1 = \{1\}$,
(HHA21) if A is bounded, then $0 \in x \odot 0$,
(HHA22) if $B \ll C \leq D$, then $B \ll D$, and $\{x\} \leq B \leq \{y\}$ implies $x \leq y$,
(HHA23) if $B \leq C \leq D$, then $B \leq D$, and $\{x\} \leq \{y\} \leq B$ implies $\{x\} \leq B$,
(HHA24) if $B \ll \{x\} \ll C$, then $B \ll C$, and $B \ll \{x\} \leq C$ implies $B \ll C$,
(HHA25) if $x \leq y$, then $z \to x \leq z \to y$,
(HHA26) if $x \leq y$, then $y \to z \leq x \to z$,
(HHA27) $z \to y \leq (y \to x) \to (z \to x)$,
(HHA28) $z \to y \ll (x \to z) \to (x \to y)$,
(HHA29) if $x \leq y$, then $x \odot z \ll y \odot z$,
(HHA30) if $x \leq y$ and $z \leq t$, then $x \odot z \ll y \odot t$,
(HHA31) $(x \to y) \odot z \ll x \to (y \odot z)$.

Proof. (HHA16): By (HHA2)and(HHA5), $\{y\} \leq x \to x$ and so by (HHA10), $x \odot y \ll \{x\}$. Moreover by (HHA5), $\{x\} \leq y \to y$ and so by (HHA10), $x \odot y \ll \{y\}$.
(HHA17): By (HHA16) and (HHA10), the proof is clear.
(HHA18): Let $1 \in 1 \to x$. Since by (HHA5), $1 \in x \to 1$, by (HHA6), $1 = x$.
(HHA19): For all $u \in 1 \to x$ by (HHA2), $1 \in u \to (1 \to x)$. Then by (HHA12), $1 \in 1 \to (u \to x)$ and so there exists $v \in u \to x$ such that $1 \in 1 \to v$. Then by (HHA18), $v = 1$. Hence $1 \in u \to x$ and so $u \leq x$. On the other hand, by (HHA17), $\{x\} \ll 1 \to x$. Then there exists a $t \in 1 \to x$ such that $x \leq t$. Since for all $u \in 1 \to x$ we have $u \leq x$, by considering $u = t$, we have $t \leq x \leq t$ and

so by (HHA6), $x = t$. Hence $x \in 1 \to x$ and so x is the maximum element of $1 \to x$.
(HHA20): By (HHA1), 1 is the unit and so $1 \in 1 \odot 1$. Let $1 \neq a \in 1 \odot 1$. Then $1 \odot 1 \ll a$ and so by (HHA10), $1 \leq 1 \to a$. Hence $1 \in 1 \to a$ and by (HHA18), $a = 1$. Then $1 \odot 1 = \{1\}$.
(HHA21): Let A be bounded. Since by (HHA2), $1 \in 0 \to 0$, we get $\{x\} \leq 0 \to 0$, for all $x \in A$. Then by (HHA10), $x \odot 0 \ll \{0\}$. Hence since A is bounded, we get $0 \in x \odot 0$.
(HHA22): Straightforward, by (HHA7).
(HHA23): Straightforward, by (HHA7).
(HHA24): Straightforward, by (HHA7).
(HHA25): Let $x \leq y$. For all $u \in z \to x$ we have $\{u\} \leq (z \to x)$ and so by (HHA10), $u \odot z \ll \{x\}$. Since $x \leq y$, by (HHA24), $u \odot z \ll \{y\}$ and so by (HHA10), $\{u\} \leq z \to y$. Hence $z \to x \leq z \to y$.
(HHA26): Let $x \leq y$. For all $u \in y \to z$ we have $\{u\} \ll (y \to z)$ and so by (HHA13), $\{y\} \ll u \to z$. Since $x \leq y$, by (HHA23), $\{x\} \ll (u \to z)$. Hence by (HHA13), $\{u\} \ll (x \to z)$ and so $y \to z \leq x \to z$.
(HHA27): For all $u \in z \to y$ we have $\{u\} \ll z \to y$ and so by (HHA10) and (HHA14), $u \odot z \ll \{y\} \ll (y \to x) \to x$. Hence by (HHA24) and (HHA10), $\{u\} \ll z \to ((y \to x) \to x)$ and so by (HHA12), $\{u\} \ll (y \to x) \to (z \to x)$. Therefore, $z \to y \leq (y \to x) \to (z \to x)$.
(HHA28): By (HHA27), $(x \to z) \ll (z \to y) \to (x \to y)$. Hence by (HHA13), $(z \to y) \ll (x \to z) \to (x \to y)$.
(HHA29): Let $x \leq y$. Since $y \odot z \ll y \odot z$, by (HHA10), $\{y\} \leq z \to (y \odot z)$. Hence by (HHA23), $\{x\} \ll z \to (y \odot z)$ and so by (HHA10), $(x \odot z) \ll (y \odot z)$.
(HHA30): Let $x \leq y$ and $z \leq t$. Since $z \leq t$, by (HHA29), $y \odot z \ll y \odot t$. Then by (HHA10), $\{y\} \leq z \to (y \odot t)$. Hence by (HHA23), $\{x\} \leq z \to (y \odot t)$ and so by (HHA10), $x \odot z \ll y \odot t$.
(HHA31): Since $x \to y \ll x \to y$, by (HHA10), $(x \to y) \odot x \ll \{y\}$. Hence by (HHA29), $(x \to y) \odot x \odot z \ll y \odot z$. Therefore, by (HHA10), $(x \to y) \odot z \ll x \to (y \odot z)$. □

Notation: Let A be a bounded (quasi) hyper hoop. Then for any $x \in A$, we consider $x' = x \to 0$.

Proposition 3.6. Let A be a bounded quasi hyper hoop. Then $1 \in 0'$ and for any $x \in A$, $\{x\} \leq x''$.

Proof. By (HHA2), $1 \in 0 \to 0$. Then $1 \in 0'$. Since by (HHA12),

$$(x \to 0) \to (x \to 0) = x \to ((x \to 0) \to 0) = x \to x''$$

and by (HHA2), $1 \in (x \to 0) \to (x \to 0)$. Then $1 \in x \to x''$ and so, $\{x\} \leq x''$. □

Proposition 3.7. Let A be a bounded hyper hoop. Then the following hold, for any $x, y \in A$,
(i) $x \leq y$, implies that $y' \leq x'$,
(ii) $x' \leq x \to y$,
(iii) $x \to y \leq y' \to x'$.

Proof. (i) If $x \leq y$, then by (HHA26), $y \to 0 \leq x \to 0$. Hence $y' \leq x'$.
(ii) Since $0 \leq y$, by (HHA25), $x \to 0 \leq x \to y$. Hence $x' \leq x \to y$.
(iii) By Proposition 3.6 , $y \leq y''$. Then by (HHA25) and (HHA12),

$$x \to y \leq x \to y'' = x \to ((y \to 0) \to 0) = (y \to 0) \to (x \to 0) = y' \to x'.$$

□

Theorem 3.8. *Any (quasi) hyper hoop of order n, can be extend to a (quasi) hyper hoop of order $n + 1$, for any $n \in \mathbb{N}$.*

Proof. Let A be a (quasi) hyper hoop of order $n \in \mathbb{N}$, e be an element such that $e \notin A$ and $A_1 = A \cup \{e\}$. Then we define two hyperoperations $\odot'$ and $\to'$ on A_1 by:

$$a \odot' b = \begin{cases} a \odot b & \text{if } a, b \in A, \\ \{a\} & \text{if } a \in A, b = e, \\ \{b\} & \text{if } b \in A, a = e \end{cases} \qquad a \to' b =$$

$$\begin{cases} a \to b \cup \{e\} & \text{if } a, b \in A, 1 \in a \to b, \\ a \to b & \text{if } a, b \in A, 1 \notin a \to b, \\ \{e\} & \text{if } b = e, \\ \{b\} & \text{if } a = e \end{cases}$$

By some modification we can prove that $(A_1, \odot', e)$ is a commutative semihypergroup with e as the unit and satisfies the conditions (HHA2), (HHA3), (HHA4), (HHA5), (HHA6), and (HHA7). Therefore, $(A_1, \odot', \to', e)$ is a (quasi) hyper hoop and e is the unit element of it.

□

Corollary 3.9. There exist at least one (quasi) hyper hoop of order n, for any $n \in \mathbb{N}$

Proof. By Theorem 3.8 and Example 3.3 (ii), the proof is clear.

□

Note: From now on, we let A be a hyper hoop, unless otherwise is stated.

4 Some filters on hyper hoop-algebras

In this section we define the concepts of some filters on hyper hoops and we get some properties.

Definition 4.1. Let F be a non-empty subset of A. Then F is called an upset of A, if $x \in F$ and $x \leq y$ imply $y \in F$, for all $x, y \in A$,

Definition 4.2. Let F be a non-empty subset of A. Then:
(i) F is called a weak filter of A, if F is an upset and for all $x, y \in F$, $x \odot y \cap F \neq \emptyset$.
(ii) F is called a filter of A, if F is an upset and for all $x, y \in F$, $x \odot y \subseteq F$.

Note: Let F be a (weak) filter of A and $x \in F$. Since F is an upset and $x \leq 1$, we get $1 \in F$.

Example 4.3. (i) In Example 3.3(iv), $F = \{b, 1\}$ is a filter.
(ii) In Example 3.3(v), $F = \{b, 1\}$ is a weak filter.

Example 4.4. It is clear that A is a (weak) filter of A. By (HHA20), $\{1\} = 1 \odot 1$ and so $1 \odot 1 \subseteq \{1\}$. Then $\{1\}$ is a (weak)filter of A.

Proposition 4.5. Any filter of A is a weak filter.

Proof. Let F be a filter of A. Then F is an upset and $x \odot y \subseteq F$, for all $x, y \in F$. Hence $(x \odot y) \cap F \neq \emptyset$, for all $x, y \in F$. Then F is a weak filter. □

Note: Any weak filter is not a filter, in general. It can be verified by the following Example.

Example 4.6. In Example 3.3(vi), $F = \{b, 1\}$ is a weak filter, but it is not a filter.

Theorem 4.7. *Let F be a non-empty subset of A. Then F is a weak filter of A if and only if F is an upset and $F \ll x \odot y$, for all $x, y \in F$.*

Proof. $(\Rightarrow)$ Straightforward.
$(\Leftarrow)$ Let F be an upset and $F \ll x \odot y$, for all $x, y \in F$. Hence there exist $u \in F$ and $v \in x \odot y$ such that $u \leq v$. Since F is an upset and $u \in F$, then $v \in F$ and so $x \odot y \cap F \neq \emptyset$. Hence F is a weak filter of A. □

Theorem 4.8. *Let F be a filter of A. Then for all $x, y, z \in A$,*
(i) if $x \to y \subseteq F$ and $x \in F$, then $y \in F$,
(ii) If $x \to y \subseteq F$ and $x \odot z \subseteq F$, then $y \odot z \subseteq F$,
(iii) If $x, y \in F$ and $x \ll y \to z$, then $z \in F$.

Proof. (i) Let $x \in F$ and $x \to y \subseteq F$, for $x, y \in A$. Then $x \odot (x \to y) = \bigcup_{u \in x \to y} x \odot u \subseteq F$. On the other hand, since $x \to y \ll x \to y$, by (HHA11), $(x \to y) \odot x \ll y$. Therefore, there is $v \in (x \to y) \odot x$ such that $v \leq y$. Since $v \in F$, we get $y \in F$.

(ii) By (HHA16), $x \odot z \ll x, z$. Then there exists $u \in x \odot z \subseteq F$ such that $u \leq x, z$. Since $u \in F$ and F is a filter, we get $x, z \in F$. Now, since $x \in F$ and $x \to y \subseteq F$, by (i) $y \in F$. Finally, since $y, z \in F$ and F is a filter, $y \odot z \subseteq F$.

(iii) Let $x, y \in F$. Since F is a filter, $x \odot y \subseteq F$ and since $x \ll y \to z$, by (HHA10), $x \odot y \ll z$. Then there exists $u \in x \odot y \subseteq F$ such that $u \leq z$. Since F is a filter and $u \in F$, we get $z \in F$. □

Theorem 4.9. *Let F be a non-empty subset of A. Then F is a filter of A if and only if $1 \in F$ and $F \ll x \to y$ and $x \in F$ implies $y \in F$, for any $x, y \in A$.*

Proof. $(\Rightarrow)$ Let F be a filter, $F \ll x \to y$ and $x \in F$, for $x, y \in A$. Hence there exist $u \in F$ and $v \in x \to y$ such that $u \leq v$. Since $u \in F$ and F is an upset, we get $v \in F$ and since F is a filter, we get $x \odot v \subseteq F$. By $v \in x \to y$ we have $\{v\} \leq x \to y$. Then by (HHA10), $v \odot x \ll y$ and so there exists $t \in v \odot x \subseteq F$ such that $t \leq y$. Since F is an upset, we get $y \in F$.
$(\Leftarrow)$ Let $x \leq y$ and $x \in F$, for $x, y \in A$. Then $1 \in x \to y$ and since $1 \in F$, we get $F \ll x \to y$. Then, by hypothesis $y \in F$ and so F is an upset. Now, let $x, y \in F$ and $u \in x \odot y$. Then $x \odot y \ll u$ and so by (HHA10), $\{y\} \leq x \to u$. Since $y \in F$, we get $F \ll x \to u$ and so by hypothesis, $u \in F$. Hence $x \odot y \subseteq F$ and so F is a filter of A.

□

Definition 4.10. Let S be a non-empty subset of A. If S is a hyper hoop with respect to the hyperoperations $\odot$ and $\to$ on A, we say that S is a hyper hoop-subalgebra of A.

Theorem 4.11. *Let S be a non-empty subset of A. Then S is a hyper hoop-subalgebra of A iff $x \odot y \subseteq S$ and $x \to y \subseteq S$, for all $x, y \in S$.*

Proof. $(\Rightarrow)$ The proof is clear.

$(\Leftarrow)$ Let $x \in S$. By (HHA2), $1 \in x \to x$ and by assumption, $x \to x \subseteq S$. Hence $1 \in S$. It is easy to show that $(S, \odot, \to, 1)$ is a hyper hoop. Then S is a hyper hoop-subalgebra of A.

□

Example 4.12. (i) In Example 3.3(iv), $F = \{b, 1\}$ is a hyper hoop-subalgebra.
(ii) In Example 3.3(iii), $F = \{1\}$ is a (weak)filter, but it is not a hyper hoop-subalgebra.
(iii) In Example 3.3(vi), $F = \{a, 1\}$ is a hyper hoop-subalgebra, but it is not a

(weak)filter. Since $a \leq c$ and $a \in F$, but $c \notin F$ and so F is not an upset.

Theorem 4.13. *If $\{F_i\}$ is a finite family of filters of A, then $\cap\{F_i\}$ is a filter of A.*

Proof. The proof is easy. $\square$

Definition 4.14. Let D be a subset of A. The intersection of all (weak) filters of A containing D is called the (weak) filter generated by D. The filter generated by D denoted by $[D)$ and the weak filter generated by D denoted by $[D)_w$. It is trivial to verify that $[D)$ is the least filter containing D and $[D)_w$ is the least weak filter containing D.

Theorem 4.15. *If $\emptyset \neq D \subseteq A$, then*

$$[D)_w \subseteq \{x \in A | \exists\, a_1, ..., a_n \in D, \text{ s.t. } a_1 \odot .... \odot a_n \ll \{x\}\}$$

Proof. Let

$$F = \{x \in A | \exists\, a_1, ..., a_n \in D, \text{ s.t. } a_1 \odot a_2 \odot ...... \odot a_n \ll \{x\}\}$$

It is sufficient to show that F is a weak filter containing D. Let $x \leq y$ and $x \in F$, for $x, y \in A$. Then there exist $a_1, ..., a_n \in D$, such that, $a_1 \odot ...... \odot a_n \ll \{x\}$. Since $x \leq y$, by (HHA23), $a_1 \odot ...... \odot a_n \ll \{y\}$ and so $y \in F$. Hence F is an upset. Now, let $x, y \in F$. Then there exist $a_1, ..., a_n, b_1, ..., b_m \in D$, such that, $a_1 \odot ...... \odot a_n \ll \{x\}$ and $b_1 \odot ...... \odot b_m \ll \{y\}$. Hence there exist $u \in a_1 \odot ..... \odot a_n$ and $v \in b_1 \odot ..... \odot b_m$, such that $u \leq x$ and $v \leq y$. By (HHA30) $u \odot v \ll x \odot y$. Then $a_1 \odot ...... \odot a_n \odot b_1 \odot ...... \odot b_m \ll x \odot y$. Hence there exists $s \in x \odot y$ such that $a_1 \odot ...... \odot a_n \odot b_1 \odot ...... \odot b_m \ll \{s\}$ and so $x \odot y \cap F \neq \emptyset$. Thus F is a weak filter of A. For all $d \in D$ we have $\{d\} \ll \{d\}$, and so $d \in F$. Therefore F is a weak filter of A containing D.

$\square$

Note: In the following Example we will show that the equation, $[D)_w = F$ is not true, in general, where

$$F = \{x \in A | \exists\, a_1, ..., a_n \in D, \text{ s.t. } a_1 \odot .... \odot a_n \ll \{x\}\}$$

Example 4.16. In Example 3.3(v), if we take $D = \{b\}$ then it follows that $F = \{1, b, c\}$, that is a weak filter containing D, but $[D)_w = \{1, b\}$. Hence in this Example $[D)_w \neq F$.

Theorem 4.17. *If $\emptyset \neq D \subseteq A$, then*

$$[D) = \{x \in A | \exists\, a_1, ..., a_n \in D, \text{ s.t. } a_1 \odot .... \odot a_n \ll \{x\}\}$$

Proof. Let

$$F = \{x \in A | \exists\, a_1, ..., a_n \in D, \text{ s.t. } a_1 \odot a_2 \odot ...... \odot a_n \ll \{x\}\}$$

Let $x \leq y$ and $x \in F$, for $x, y \in A$. Then there exist $a_1, ..., a_n \in D$, such that,

$$a_1 \odot ...... \odot a_n \ll \{x\}$$

Since $x \leq y$, by (HHA24), $a_1 \odot ...... \odot a_n \ll \{y\}$ and so $y \in F$. Hence F is an upset. Now, let $x, y \in F$. Then there exist $a_1, ..., a_n, b_1, ..., b_m \in D$, such that, $a_1 \odot ...... \odot a_n \ll x$ and $b_1 \odot ...... \odot b_m \ll \{y\}$. For all $u \in x \odot y$, $x \odot y \ll \{u\}$. Then by(HHA10), $\{x\} \leq y \to u$. Since $a_1 \odot ...... \odot a_n \ll \{x\}$ and $\{x\} \leq y \to u$ by (HHA24), $a_1 \odot ...... \odot a_n \ll y \to u$. Since $b_1 \odot ...... \odot b_m \ll y$ by (HHA26), $y \to u \leq (b_1 \odot ...... \odot b_m) \to u$. Hence

$$a_1 \odot ...... \odot a_n \ll y \to u \leq (b_1 \odot ...... \odot b_m) \to u$$

and so by (HHA22), $a_1 \odot ...... \odot a_n \ll (b_1 \odot ...... \odot b_m) \to u$. Then by (HHA11), $(a_1 \odot ...... \odot a_n) \odot (b_1 \odot ...... \odot b_m) \ll \{u\}$ and so $u \in F$. Therefore $x \odot y \subseteq F$ and so F is a filter. Since $d \ll d$, for all $d \in D$, we have $d \in F$ and so F is a filter of A containing D. Let $D \subseteq C$ and C be a filter of A. For all $x \in F$, there exist $a_1, ..., a_n \in D$, such that

$$a_1 \odot .... \odot a_n \ll \{x\}$$

Then there exists $v \in a_1 \odot .... \odot a_n$, such that $v \leq x$. By $a_1, ..., a_n \in D \subseteq C$ and C is a filter, it follows that $a_1 \odot ..... \odot a_n \subseteq C$ and so $v \in C$. Since C is an upset we have $x \in C$ and so $F \subseteq C$. Therefore $[D) = F$.

□

Definition 4.18. Let A be bounded. Then $D \subseteq A$ is said to have *the finite intersection property* if $a_1 \odot a_2 ...... \odot a_n \cap \{0\} = \emptyset$, for all $a_1, ..., a_n \in D$.

Theorem 4.19. *Let A be bounded and $D \subseteq A$. Then $[D)$ is a proper filter of A if and only if D has the finite intersection property.*

Proof. Let $[D)$ be a proper filter of A and D has not the finite intersection property, by the contrary. Then there exist $a_1, ..., a_n \in D$ such that $0 \in a_1 \odot a_2 ...... \odot a_n$. Hence $a_1 \odot a_2 ...... \odot a_n \ll \{0\}$ and so by Theorem 4.17, $0 \in [D)$. Since $0 \leq x$, for all $x \in A$ and $[D)$ is a filter, we have $x \in [D)$ and so $[D) = A$, which is a contradiction. Hence D has the finite intersection property.
Conversely, let D has the finite intersection property and $[D)$ is not a proper filter, by the contrary. Then $[D) = A$ and so $0 \in [D)$. Then by Theorem 4.17, there exist $a_1, ..., a_n \in D$ such that $a_1 \odot a_2 ...... \odot a_n \ll \{0\}$ and so $0 \in a_1 \odot a_2 ...... \odot a_n$. Then D has not the finite intersection property, which is a contradiction. Hence $[D)$ is a proper filter.

□

Theorem 4.20. *If F is a filter of A and $a \in A$, then*

$$[F \cup \{a\}) = \{x | x \in A, \exists n \in \mathbb{N}, s.t., a^n \to x \cap F \neq \emptyset\}$$

Proof. Suppose that $x \in [F \cup \{a\})$. By Theorem 4.17, there exist $b_1, ..., b_m \in F$ and $n \in \mathbb{N}$ such that

$$b_1 \odot ..... \odot b_m \odot a^n \ll \{x\}$$

By (HHA11), we have $b_1 \odot ..... \odot b_m \ll a^n \to x$. Then there exists $u \in b_1 \odot ..... \odot b_m$ and $v \in a^n \to x$ such that $u \leq v$. Since F is a filter and $b_1, ..., b_m \in F$, we get $b_1 \odot ..... \odot b_m \subseteq F$ and so $u \in F$. Now, since F is a filter, we get $v \in F$. Hence $a^n \to x \cap F \neq \emptyset$.
Conversely, let there exists $n \in \mathbb{N}$ such that $a^n \to x \cap F \neq \emptyset$. If $s \in a^n \to x \cap F$, then $1 \in s \to (a^n \to x)$. Hence by (HHA4), $1 \in (s \odot a^n) \to x$. Therefore, $s \odot a^n \ll \{x\}$ and so by Theorem 4.17, $x \in [F \cup \{a\})$.

□

5 Conclusion

In this paper, we applied the hyper structure theory to the hoop algebras and introduced the notion of (quasi) hyper hoop algebra which is a generalization of hoop-algebra. Then we studied some properties and filter theory of this structure. Topological and categorical properties, quotient structures and relation with the other hyperstructures can be studied for the future researches.

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