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M-POLYSYMMETRICAL HYPERRINGS

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ABSTRACT - In this paper a special case of canonical polysymmetrical hyperrings, the M-Polysymmetrical hyperrings, are studied and investigated in detail. Certain methods of their construction from rings are presented and a new class of hyperstructures, the M-Polysymmetrical superrings, is introduced.

1. INTRODUCTION

Mittas, the first one of the co-authors, in his paper [6] (in the French Academy of Sciences) has introduced two new classes of hypercompositional structures along with some of their fundamental properties. These classes are special cases of canonical polysymmetrical hypergroups and hyperrings (MITTAS [9], [10]). Yatras, the second co-author of this paper, has studied the first of the above classes. He has named the hypercompositional structures that appear in it, *M-Polysymmetrical hypergroups* and he has already presented the first results in [11]. Going on with his study, he has also presented results on the subhypergroups of the M-polysymmetrical hypergroup (YATRAS [12]) and on the homomorphisms of the M-polysymmetrical hypergroup (YATRAS [13]). Here we give the initial results that we have reached during our study of the hypercompositional structures of the second of the above classes. Likewise the M-Polysymmetrical hypergroups, we have given them the name ***M-Polysymmetrical hyperrings***.
We give the relevant definitions:

Definition 1.1
A set H is called a ***M-polysymmetrical hypergroup*** *(M-P-H) if it is endowed with a hyperoperation x+y that satisfies the following axioms:*

1. $(x+y)+z = x+(y+z)$ for every $x,y,z\in H$
2. $x+y = y+x$ for every $x,y\in H$
3. $(\exists 0\in H)\,(\forall x\in H)\ [x\in 0+x]$
4. $(\forall x\in H)\,(\exists x'\in H)\,[x+x' = 0]$

(x′ is an **opposite** or **symmetrical** of x, with regard to the considered 0, and the set of the opposites $S(x)=\{x'\in H : x+x'=0\}$ is the **symmetrical** set of x).

5. For every $x,y,z\in H$, $x'\in S(x)$, $y'\in S(y)$, $z'\in S(z)$ we have: $z\in x+y \Rightarrow z'\in x'+y'$.

We remind that in such a hypergroup, when x runs in H the sets $C(x) = 0+x$ form a partition of H, which is notified by **mod(0)**, or simply (0) and for which we have $x\equiv y\ (0) \Leftrightarrow 0+x = 0+y \Leftrightarrow C(x) = C(y)$. Also, for every $x\in H$, $x'\in S(x)$ we have $S(x) = C(x')$ and the set of classes, $H/(0) = G(H)$ is an abelian group (MITTAS [6], YATRAS [11]).

Definition 1.2
A set A, endowed with a hyperoperation x+y (addition) and an operation xy (multiplication), is called ***M-polysymmetrical hyperring*** *(M-P.HR) (and with abuse of definition, hyperring, since it is not such in the sense of the papers* KRASNER [2], MASSOYROS [3], MITTAS [5], [7]) *if it satisfies the axioms:*

I. *The hyperstructure* (A,+) *is a M-P.H.*
II. *The structure* (A,·) *is a semigroup.*
III. *The multiplication is bilaterally distributive over addition i.e.*
$(x+y)z = xz+yz$, $z(x+y) = zx+zy$ for every $x,y,z \in A$.

As an example of those structures we give the one that appears in MITTAS [6], which is also the one that motivated their definition.

Example 1.1

Let (K,+,·) be a commutative algebraically closed field with characteristic p. Also let $n \neq 1$ be a comprime to p number and $\in_n$ be the multiplicative group of the n-th roots of the unity of K. We define the following hyperoperation in K:

$$x \dotplus y = \left\{ z \in K : x^n + y^n = z^n \right\}$$

The $x \dotplus y$ is a class modulo $\in_n$ in K and it holds that:

$$x \dotplus y = (x \in_n) \dotplus (y \in_n)$$

It can easily be proved that K, endowed with the above hyperoperation is a M-P.H. This M-P.H combined with the multiplication in K gives the hyperstructure $(K, \dotplus, \cdot)$ which is a M-P.HR and especially a M-polysymmetrical hyperfield as it will appear in the following.

We proceed now to a general study of the M-P.HR.

2. GENERALITIES

We can easily notice that the neutral element 0 of the additive hypergroup, is a *b i l a t e r a l l y a b s o r b i n g* element of the multiplicative semigroup, that is:

$$a0 = 0a = 0 \qquad \text{for every } a \in A$$

Remarks 2.1

a) Obviously we have $A \neq \emptyset$.
b) Every ring is a M-P.HR. A M-P.HR, which is not a ring, is called **proper**.
c) If there exists a scalar element $s \in A$, then M-P.H (A,+) is an abelian group (YATRAS [11]) and thus A is a ring.
d) If $W = \{x \in A : 0+x = x\}$, then W is a multiplicatively permissible subset (DUBREIL-JACOTIN [1]) of A. Since $x \in W \Rightarrow 0+x = x$ for every $a \in A$ we have:

$$a(0+x) = a0+ax = 0+ax = ax$$

thus $ax \in W$, i.e. $AW \subseteq W$. In the same way $WA \subseteq W$.

In the following, we will accept that all the terms which have been defined in the theory of the rings and which are dependent only on the multiplicative structure e.g. divisors, (units in the sense that every element which has a reverse with regard to the multiplicative unity element), divisors of zero, associate elements, unitary rings, commutative rings, integral rings etc. will have the same meaning for the M-P.HR, as it is for the hyperrings MITTAS [5], [7]. The only exception is the term «domain» (integral), that will be substituted with the term «hyperdomain». As a result in the case under consideration it will be named **M-polysymmetrical hyperdomain** (M-P.HD), that is, an integral (i.e. a commutative for which the cancellation law for multiplication holds) and unitary M-P.HR. Another special form of M-P.HR is the **M-polysymmetrical hyperfield** (M-P.HF), that is, a M-P.HR whose set $A^* = A-\{0\}$ is a group with regard to the multiplication. We also denote that, obviously, all the properties of the rings which are dependent only on the multiplication and which are being proved without the intervention of the addition, also hold in the theory of M-P.HR (as well as in the theory of the hyperrings MASSOUROS - MITTAS [4], MITTAS [7]).

Remark 2.2

In the ring or (canonical) hyperring theory, satisfying the cancellation law, is equivalent to not having divisors of zero. However, the same does not hold for the M-P.HR, where obviously satisfying the cancellation law does not imply that there exist divisors of zero ($x \neq 0$ and $xy = 0 \Rightarrow xy = x0 \Rightarrow y = 0$, thus x is a non-divisor of zero element), while for the converse we generally have that if $x \in A^*$ a non-divisor of zero (e.g. from the left) then $xy = xz \Rightarrow$ $xy+xz' = xz+xz' \Rightarrow x(y+z') = x(z+z') \doteq 0 \Rightarrow 0 \in y+z' \Rightarrow y+z' = 0 \Rightarrow$ $y+z+z' = 0+z \Rightarrow 0+y = 0+z \Rightarrow z \equiv y \bmod(0)$.

And so the proposition:

Proposition 2.1

If A is a M-P.HR and $x \in A^*$ is a non-divisor of zero (from one side for instance) then

$$xy = xz \Rightarrow z \equiv y \bmod(0)$$

Examples 2.1

1. The set resulting from the set C of the complex numbers after the subtraction of the non zero real numbers and the pure imaginary numbers i.e. $A = C-(R^* \cup iR^*)$, endowed with the hyperaddition

$$x \dot{+} y = \begin{cases} \{z \in A : \operatorname{Re} z = \operatorname{Re} x + \operatorname{Re} y, \operatorname{Im} z \in R^*\}, & \text{if } \operatorname{Re} x + \operatorname{Re} y \neq 0 \\ 0 & , \quad \text{if } \operatorname{Re} x + \operatorname{Re} y = 0 \end{cases}$$

and with multiplication

$$x \circ y = \text{Re}x \cdot \text{Re}y + i \cdot \text{Im}x \cdot \text{Im}y$$

is a M-P.HR. This can be proved with the verification of the axioms. We remark that, if $z \in C$, $\text{Re}z = a$, $a \in R^*$ then the class $C(z)$ mod(0) is

$$C(z) = X_a = \{a+iy,\ y \in R^*\}$$

(i.e. it is a line parallel to the imaginary axis), while for the symmetrical of z it is $S(z) = \{-a+iy,\ y \in R^*\}$. Obviously for $a,x,y \in A$, $a \neq 0$, $a \circ x = a \circ y \Rightarrow x = y$, i.e. the cancellation law is valid and as a result the M-P.HR $(A, \dot{+}, \circ)$ has no divisors of zero. The number $e = 1+i$ is the unitary element. Also, assuming that the commutative law of the multiplication is valid, then the M-P.HR $(A, \dot{+}, \circ)$ is a M-P.HD and moreover a commutative one. Obviously, for every $z = x+iy \in A^*$ the $w = \frac{1}{x} + \frac{i}{y}$ is the multiplicative inverse of z $(z \circ w = 1+i)$.

2. It is obvious that, if we consider the Im instead of Re and $\text{Re}z \in R^*$ instead of $\text{Im}z \in R^*$ in the above definition of the hyperoperation, then the deriving hyperstructure is also a commutative M-P.HF.

3. We easily conclude that the previous examples can give more M-P.HRs which become M-P.HDs if instead of C, we consider its subset E of the Gaussian integers and instead of R, the set Z of integers. In other words, we consider the set $B = E-(Z^* \cup i\ Z^*)$, enriched with similar hyperoperations and operation to the ones that we have previously used in A.

4. If we equipe the set G of an abelian group (G,+) with the multiplication $xy=0$ for every $x,y \in G$, then it becomes a ring, the *z e r o r i n g*. Likewise, in the case of the hyperstructures if we equipe a M-P.H (H,+) with the zero multiplication as above, then it becomes a M-P.HR, the *z e r o M-P.HR.* Obviously in this M-P.HR every $x \in H^*$ is a zero divisor [see proposition 2.14].

In the following we will see others M-P.HRs which are neither M-P.HDs, and thus nor M-P.HFs. Based on the above information regarding the M-P.HF we have the following propositions which are presented without proof, since their proof depends only on the multiplication.

Proposition 2.2

Every commutative M-P.HF is a M-P.HD.

Proposition 2.3

Every integral M-P.HR having a finite number of elements is a M-P.HF.

Corollary 2.1

Every M-P.HD having a finite number of elements is a M-P.HF.

Regarding the M-P.HR we initially give the following lemma and theorem from MITTAS [6].

Lemma 2.1

If $x,y \in A$, ***then:***

i) $xC(y) = C(x)y = C(xy) = C(x_1y_1)$

ii) $C(x)C(y) = C(xy) = C(x_1y_1)$

for every $x_1 \in C(x)$, $y_1 \in C(y)$.

Theorem 2.1

The set $A/(0)=\{C(x), x\in A\}$ ***of equivalent classes*** mod(0), ***is a ring with respect to the addition*** $C(x)+C(y)$ $(=x+y)$ ***and the multiplication*** $C(x)C(y)$ $(=C(xy))$, *which is called* ***ring of reduction*** *or simply* ***ring*** *of M-P.HR* $(A,+,\cdot)$, *symbolized* R(A).

This theorem, combined with the study that has been presented in MITTAS [6], YATRAS [11] concerning the relation of the M-P.H to the abelian groups, shows that, generally, the M-P.HRs are closely related to the rings.

Obviously, from the relation $C(x)C(y) = C(x_1y_1)$, for every $x,y\in A$, $x_1\in C(x)$, $y_1\in C(y)$ derives that the equivalence mod(0) in A is compatible to the multiplication and consequently a *n o r m a l* equivalence relation in M-P.HR $(A,+,\cdot)$. According to the relevant theory (MITTAS [7]) the quotient set $R(A) = A/(0)$ is a hyperring, which, as it is mentioned in the above theorem, is a ring in the case of the M-P.HR.

More specifically, we have the relevant to the above theorem proposition:

Proposition 2.4

If the M-P.HR $(A,+,\cdot)$ ***is M-P.HF, then the set*** $A/(0)$ ***of the equivalent classes mod(0) is a field*** *called* ***field of reduction*** *or simply* ***field*** *of M-P.HF* $(A,+,\cdot)$ *symbolized* F(A).

Taking into consideration the previous theorem the proof becomes easy. Obviously the unity element of F(A) is the class C(1). Every class C(x) has its reverse $C(x^{-1})$ (1 symbolizes the multiplication unity of A where there is no danger of confusion).

Next we present the following propositions showing the relation of the various M-P.HRs (A,+,·) to the corresponding resulting rings of reduction A/(0) without their proofs for reasons of simplicity and keeping this text short.

Proposition 2.5

If A is a commutative M-P.HR, then A/(0) is also a commutative ring.

Conversely, we have the proposition:

Proposition 2.6

If the ring of reduction A/(0) ***of M-P.HR*** (A,+,·) ***is commutative then:***

$$xy \equiv yx \mod(0) \quad \textit{for every} \quad x, y \in A$$

and conversely.

Proposition 2.7

If the cancellation law for multiplication holds in A then it holds in A/(0) as well.

Proposition 2.8

If A is a unitary M-P.HR, then A/(0) is a unitary ring.

Proposition 2.9

If the M-P.HR A has zero divisors, then the ring A/(0) has also zero divisors and reversely. Thus if one of them has no zero divisors, neither has the other.

Therefore we have the following proposition based on the above ones:

Proposition 2.10

If A an integral M-P.HR, then A(0) is an integral ring, and thus, if A is a M-P.HD, then A/(0) is an integral domain.

Based on lemma 2.1 we are led to the following proposition which is significant, especially for the construction of the M-P.HRs from the rings. Such constructions are presented in the following.

Proposition 2.11

If A is a M-P.HR for which the cancellation law for multiplication holds (*especially,* ***if A is an integral, or a M-P.HD or even a M-P.HF) then the***

classes mod(0) have the same cardinality i.e. $|C(x)|=|C(y)|$ ***for every*** $x,y\in A^*$.

Proof

Obviously $xy\neq 0$. According to lemma 2.1 we have

$$C(x)y=xC(y)$$

From the above relation derives that for every $x_1\in C(x)$ there is $y_1\in C(y)$. So $x_1y=xy_1$, and because of the cancellation law, there exists only one y_1 like this. So, for every choice of x,y from their classes, a mapping is defined:

$$\varphi_{xy} : C(x) \longrightarrow C(y)$$

which is bijective as it can easily be concluded. And so the proposition.

Thus the classes mod(0) in example 1.1 have the same finite cardinality n, while in the 1st and 2nd of examples 2.1, they have the cardinality of the continium. Also in the 3rd of examples 2.1 those classes have the cardinality of the countable. Moreover in the 4th (of examples 2.1), where the condition of the above proposition does not hold, there exist zero M-P.HRs with classes having the same cardinality and also others with a different one.

Remark 2.3

Generally if A is an arbitrary M-P.HR and if for two elements $x,y\in A^*$ (for every $x_1,x_2\in C(x)$, $y_1,y_2\in C(y)$) we have:

$$xy_1 = xy_2 \Rightarrow y_1 = y_2 \text{ and } x_1y = x_2y \Rightarrow x_1 = x_2,$$

then $|C(x)|=|C(y)|$.

Next we give the following properties of M-P.HRs omitting the proofs:

Properties 2.1

First we introduce the definition of a multiplication of an integer n with an element $x\in H$ (where H is a M-PH) as follows:

$$n\cdot x=\begin{cases} x+x+...+x & \text{n times for } n>0, n\neq 1 \\ 0 & \text{for } n=0 \\ x'+x'+...+x' & \text{n times for } n<0, n\neq -1 \end{cases}$$

for n=1 we define $1\cdot x=0+x$

for n=-1 we define $(-1)\cdot x=0+x'$, $x'\in S(x)$ arbitrary, (YATRAS [11]).

So we have

1st. i) For every $x,y,z,t\in A$

$$(x+y)(z+t) = xz+yz+xt+yt$$

ii) For every $m,n\in Z$ and $x\in A$

$$(m\cdot x)(n\cdot x) = (mn)x^2$$

And continuing we have:

2nd. $x'y \in S(xy)$, $xy' \in S(xy)$ for every $x,y \in A$ and thus $x'y \equiv xy' \bmod(0)$.

3rd. For every $x,y \in A$ we have:

$$S(xy) = xS(y) = S(x)y$$

4th. For every $x \in A$ and $n \in N^*$ we have:

$$S^n(x) = \begin{cases} C^n(x) & \text{if n even} \\ C^{n-1}(x)S(x) & \text{if n odd} \end{cases}$$

5th. For every $x \in A$ and $n \in N^*$ is valid that:

$$C^n(x) = C(x^n)$$

6th. For every $x,y,z \in A$ we have:

$$C(x)(y+z) = C(x)C(y)+C(x)C(z)$$

and

$$(y+z)\cdot C(x) = C(y)C(x)+C(z)C(x)$$

7th. For every $x,y,z \in A$ we have:

$$xC(y+z) = xC(y)+xC(z)$$

and

$$C(y+z)x = C(y)x+C(z)x$$

Remarks 2.4

a) If the M-P.HR $(A,+,\cdot)$ is unitary and 1 is its unitary element, then 1 is unique (and obviously $1 \neq 0$ for $A \neq \{0\}$).

b) If the M-P.HR $(A,+,\cdot)$ is unitary, then the set U of its *u n i t s* (i.e. of its elements which are inverse with regard to $\cdot$), is obviously a group and $U \subseteq A^*$, as in the case of rings.

c) If the M-P.HR $(A,+,\cdot)$ is commutative, then the binomial formula for $(x+y)^n$ holds, i.e.:

$$(x+y)^n = x^n + nx^{n-1}y + \ldots + \binom{n}{k} x^{n-k}y^k + \ldots + y^n \quad \text{for every } x,y \in A$$

The binomial formula generally holds for every M-P.HR for x,y commutative elements. In addition since $z \in x+y \Rightarrow x+y = 0+z$, we have:

$$(x+y)^n = (0+z)^n = 0 \dotplus z^n$$

which also is obviously valid in the case of the non commutative M-P.HR.

d) If $(A,+,\cdot)$ M-P.HF then:

i) $xA^* = A^*x = A^*$ for every $x \in A^*$

ii) $A^* = U$ and conversely (if A unitary M-P.HR and $A^* = U$, then A M-P.HF).

iii) for $a \neq 0$ each one of the equations $ax=b$, $xa=b$ has one and only one solution in A and conversely (that is, if A a unitary M-P.HR, $A \neq \{0\}$ and the equations $ax=b$, $xa=b$ have one and only one solution in A each, then A is M-P.HF).

e) If A unitary M-P.HR and $1\in W$, then A is a ring. (Indeed, since $AW\subseteq W$, $WA\subseteq W$ [remark 2.1,d] and $1\in W$, then A=W and since 0+x=x for every $x\in W$, the structure (A,+) is a group and thus $(A,+,\cdot)$ is a ring).

Let R(A) = A/(0) be the ring of reduction of A and let $(\tilde{G},\dot{+})$ be a group of choice of the additive hypergroup (A,+) of M-P.HR $(A,+,\cdot)$ (YATRAS [11]). We define in $\tilde{G}$ the following operation starting from the group of reduction G of (A,+) and through the already known bijective mapping (YATRAS [11])

$$f : G \longrightarrow \tilde{G} \text{ with } f(C) = x_c \in \tilde{G}$$

(where as it is known the x_c is arbitrarily chosen in every class mod(0) of (A,+) element),

$$x\circ y = f[C(x)C(y)] = f[C(xy)]$$

for every $x,y\in A$ and thus for the mapping f we also have

$$f[C(x)C(y)] = f[C(x)]\circ f[C(y)]$$

for every $x,y\in A$.
This operation, in combination with the addition

$$x\dot{+}y = f[C(x)+C(y)]$$

in $\tilde{G}$, endows $\tilde{G}$ with the structure of a ring.

Indeed:

a) $(x\circ y)\circ z = [f(C(x)C(y))]\circ f(C(z)) = [f(C(xy))]\circ f(C(z)) =$
$f(C(xy)C(z)) = f(C(xyz)) = f(C(x)C(yz)) =$
$= f(C(x))\circ[f(C(yz))] = f(C(x))\circ[f(C(y)\,C(z))] = x\circ(y\circ z)$
for every $x,y,z\in\tilde{G}$.

b) $x\circ(y\dot{+}z) = f(C(x))\circ f(C(y)+C(z)) = f(C(x))\circ f(C(y+z)) =$
$f(C(x)C(y+z)) = f(C(x)C(y)+C(x)C(z)) =$
$= f(C(xy)+C(xz)) = f(C(xy))\dot{+}f(C(xz)) = x\circ y\dot{+}x\circ z$
for every $x,y,z\in\tilde{G}$ and from the property 2.1, 6^{th}.
In the same way we have $(y\dot{+}z)\circ x = y\circ x\dot{+}z\circ x$ for every $x,y,z\in\tilde{G}$

So, as a result we have for the M-P.HR the following proposition which is analogous to the proposition of the M-P.H.

Proposition 2.12

***In every M-P.HR* (A,+,·) *there exists a subset* $\widetilde{R}$ *of A which has the structure of a ring, isomorphic to the ring of reduction* A/(0).**

We call the ring ($\widetilde{R}$,+,∘), **ring of choice** of (A,+,·).

Example 2.2

In the M-P.HF (C,+,·) of example 1.1 (where C is the set of complex numbers) the set F(C) = C/(0) with elements $C(x) = 0+x = x\mathcal{E}_n$, x∈C, equipped with the operations

$$C(x)+C(y)=(x\mathcal{E}_n)+(y\mathcal{E}_n)$$

$$C(x)C(y)=(x\mathcal{E}_n)(y\mathcal{E}_n)=xy\mathcal{E}_n$$

for every x,y∈C and C(x),C(y)∈C/(0) is obviously its field of reduction. Also if:

i) we consider as a group of choice $\widetilde{R}$ of the additive hypergroup (C,+) the one which derives as follows: we take as a distinct element of every class C(x), the element $x_c \in x\mathcal{E}_n$ which has a principal value of argument the minimum of the principal values of the arguments of the numbers $z\in x\mathcal{E}_n$.

ii) Through the bijective mapping

$$f : x\mathcal{E}_n = C(x) \longrightarrow x_c$$

we define the operation

$$x \circ y = f[C(x)C(y)] = f[C(xy)]$$

for every x,y∈C.

iii) we consider the addition of the group $\widetilde{R}$

$$x \dot{+} y = f[C(x)+C(y)]$$

for every x,y∈C.

Then ($\widetilde{R}$,+,·) becomes a field (field of choice) which is isomorphic to the field of reduction F(C).

Remarks 2.5

a) Obviously we will have more rings of choice, for more choices of elements from classes mod(0) of A as distinct. All of these are isomorphic to each other since they are also isomorphic to the ring of choice.

b) Every ring of choice $\widetilde{R}$ is obviously a proper subset of A when A is a proper M-P.HR.

c) More rings with the same operations as the ones mentioned can result from one M-P.HR if we consider the mapping $g : R(A) \longrightarrow A$ bijective in A, instead of the mapping f, without necessarily having $g(C) \in C$ for $C \in R(A)$.

Now we will deal with the construction of the M-P.HRs from one ring. As it has been indicated in MITTAS [6] this is a more complex problem than the corresponding one of the construction of the M-P.Hs with a given abelian group as a group of reduction, as it derives from proposition 2.3 of YATRAS [11], and from the consequence 3 of theorem 1 of §1 of MITTAS [6]. Indeed, we start from a set E with the structure of a multiplicative semigroup which has an absorbing element and which also has as a subset a ring $(A,+,\cdot)$, such that:

i) Its multiplication being the restriction of the corresponding of the semigroup $(E,\cdot)$ in A.

ii) The zero (0) of A being the absorbing element of the semigroup $(E,\cdot)$.

Apart from these conditions we assume that:

iii) There is a partition R of E with the property:

$$xC_R(y) = C_R(x)y = C_R(xy) \quad \text{for every } x, y \in E$$

iv) There is also a bijective mapping f of the quotient set E/R on A such that:

$$f^{-1}(x) = C_R(x) \quad \text{for every } x \in A.$$

[where $C_R(x)$ is the class of E mod (R) that contains the element x].

v) $C_R(0) = \{0\}$

We also consider the hyperoperation:

$$x \oplus y = f^{-1}[f(C_R(x)+f(C_R(y))] \quad \text{for every } x,y \in E$$

defined in E through the additive group $(A,+)$ of the ring A, and the operation:

$$x \circ y = xy \quad \text{for every } x,y \in E$$

As it is already known (YATRAS [11]) the hyperoperation $x \oplus y$ makes E a M-P.H whose group of reduction E/(0) coincides with E/R. Also the operation $x \circ y$ is associative in E since $(E,\cdot)$ is a semigroup. Furthermore for every $x,y,z \in E$ we have:

$$x \circ (y \oplus z) = x(y \oplus z) = x(y_1 \oplus z_1) = xf^{-1}(y_1 + z_1) = xC_R(y_1 + z_1) =$$
$$= xC_R(w_1) = C_R(xw_1) = C_R(x)w_1 = C_R(x_1)w_1 = C_R(x_1w_1) =$$
$$= f^{-1}(x_1w_1) = f^{-1}[x_1(y_1 + z_1)] = f^{-1}(x_1y_1 + x_1z_1) = C_R(x_1y_1 + x_1z_1) =$$
$$= x_1y_1 \oplus x_1z_1 = C_R(x_1y_1) \oplus C_R(x_1z_1) = x_1C_R(y_1) \oplus x_1C_R(z_1) =$$
$$= x_1C_R(y) \oplus x_1C_R(z) = C_R(x_1)y \oplus C_R(x_1)z = C_R(x)y \oplus C_R(x)z =$$
$$= C_R(xy) \oplus C_R(xz) = xy \oplus yz = x \circ y \oplus y \circ z$$

and in the same manner $(y \oplus z) \circ x = y \circ x \oplus z \circ x$.

In other words the multiplication is bilaterally distributive over the hyperoperation.
Consequently the hyperstructure $(E,\oplus,\circ)$ is a M-P.HR.
Thus we have the proposition:

Proposition 2.13

Let E be a set with the structure of a multiplicative semigroup that has an absorbing element, having as a subset, a ring $(A,+,\cdot)$, ***whose multiplication is the restriction of the corresponding one of the semigroup*** $(E,\cdot)$ ***in A and the zero (0) of A is the absorbing element of the semigroup*** $(E,\cdot)$. ***Then if***

i) ***there is a partition R of E having the property***

$$xC_R(y) = C_R(x)y = C_R(xy) \quad \text{for every } x,y\in E$$

ii) ***there is a bijective mapping of the quotient set E/R on A such that for every x∈A***

$$f^{-1}(x) = C_R(x)$$

[where $C_R(x)$ *is the class of E mod(R) that contains the element x]*
and

iii) $C_R(0)=\{0\}$

the hyperoperation $x\oplus y = f^{-1}[f(C_R(x))+f(C_R(y))]$ ***defined on E through the group (A,+) of the ring and the operation*** $x\circ y = xy$ *(that is the operation of the semigroup and the ring)* ***make E a M-P.HR whose ring of reduction E/(0) coincides with E/R.***

More specifically if E is an arbitrary set and A its subset with the structure of a zero ring (i.e. a ring every element of which is its zero divisor, that is for every $x,y\in A$ and $x\neq 0$, $y\neq 0$ $xy=0$ is valid) and R is a partition of E and f is a bijective mapping of the quotient set E/R on A such that for every $x\in A$, $f^{-1}(x) = C_R(x)$ and $C_R(0) = \{0\}$, then the previous hyperoperation $x\oplus y$, and the operation $x\circ y = f(C_R(x))\cdot f(C_R(y)) = x_1y_1$ for every $x,y\in E$ with $x_1,y_1\in A$ make E a M-P.HR.
Indeed

$$\begin{aligned}
&x\circ(y\oplus z) = x\circ f^{-1}(y_1+z_1) = x\circ f^{-1}(w_1) = x\circ C_R(w_1) = \\
&= x_1w_1 = x_1(y_1+z_1) = x_1y_1 + x_1z_1 = 0 = C_R(0) = C_R(x_1y_1+x_1z_1) = \\
&= f^{-1}(x_1y_1+x_1z_1) = x_1y_1\oplus x_1z_1 = x\circ y\oplus x\circ z
\end{aligned}$$

and in the same way $(y\oplus z)\circ x = y\circ x\oplus z\circ x$ for every $x,yz\in E$.
Additionally for every $x,y,x^*,y^*\in E$ we have:

$$x\oplus y = x\oplus C_R(y) = C_R(x)\oplus y = C_R(x)\oplus C_R(y)$$

$$x\circ y = x\circ C_R(y) = C_R(y)\circ x = C_R(x)\circ C_R(y)$$

$x \oplus y = x^* \oplus y^*, x \circ y = x^* \circ y^*$ with $x \equiv x^* \bmod(R)$ and $y \equiv y^* \bmod(R)$.

Consequently we have:

Proposition 2.14

Every zero ring can give a M-P.HR, *which is called* ***zero M-P.HR.***

Remark 2.6

From the above it derives that we can construct all the M-P.HRs $(A,+,\cdot)$ whose ring of reduction is isomorphic to a given ring $(\Delta,+,\cdot)$. We consider arbitrary semigroups $(A,*)$ with absorbing element $0 \in A$ and surjective mappings $\varphi : A \longrightarrow \Delta$ such that, if we set

$$\varphi^{-1}(\varphi(x)) = C_R(x) \quad \text{for every } x,y \in A$$

such that

i) $\varphi^{-1}(\varphi(0)) = C_R(0) = \{0\}$ (singleton, symbolized again 0)

ii) $\varphi(x * y) = \varphi(x)\varphi(y)$

and

iii) $x * C_R(y) = C_R(x) * y = C_R(x * y)$

Starting from every such mapping we consider the hyperoperation $x \dot{+} y = \varphi^{-1}(\varphi(x) + \varphi(y))$ and the multiplication $x * y$. Under the above hyperoperation and multiplication the structure $(A, \dot{+}, *)$ is a M-P.HR with ring of reduction isomorphic to $(\Delta,+,\cdot)$. As it is already known (MITTAS [6], YATRAS [11]) the structure $(A, \dot{+})$ is a M-P.H and furthermore it holds:

$$x * (y \dot{+} z) = x * \varphi^{-1}(\varphi(y) + \varphi(z)) = x * \varphi^{-1}(\varphi(w)) =$$
$$x * C_R(w) = C_R(x * w) = \varphi^{-1}(\varphi(x * w)) =$$
$$= \varphi^{-1}(\varphi(x) \cdot \varphi(w)) = \varphi^{-1}[\varphi(x) \cdot (\varphi(y) + \varphi(z))] = \varphi^{-1}(\varphi(x)\varphi(y) + \varphi(x)\varphi(z)) =$$
$$= \varphi^{-1}(\varphi(x * y) + \varphi(x * z)) = x * y \dot{+} x * z .$$

Similarly for $(y \dot{+} z) * x$.

Hence the structure $(A, \dot{+}, *)$ is a M-P.HR with the ring $(\Delta,+,\cdot)$ as a ring of reduction.

Examples 2.3

1. Having as a starting point the example 1 of YATRAS [11] we consider the set of the rational numbers Q and the partition $O \cup X$ where $O = \{0\}$ and $X = Q^*$. Then the set $H = \{0\} \cup Q^* = Q$ equipped with the hyperoperation $x \dot{+} y$ of the

previously mentioned example, that is $0+0=0$, $0+x = x+0 = Q^*$, $x+y = y+x = 0$ for every $x,y \in Q^*$, and multiplication the usual multiplication of the rational numbers is a M-P.HF.
The set $F(Q) = \{C(0),C(x)\} = \{O,X\}$ is the field of reduction, with addition:

$$O+O=O, \quad O+X=X+O=X, \quad X+X=O$$

and multiplication

$$OX=XO=O, \quad XX=X$$

$F(Q)$ is obviously isomorphic to the field of integers mod 2. The set $\widetilde{R}=\{0,q\}$ with q any arbitrary element of Q^*, is the field of choice that is defined through the mapping:

$$f:H \longrightarrow \widetilde{R} \text{ with } f(0) = 0 \text{ and } f(x) = q, \ x \in Q^*$$

Generally we can consider X to be an arbitrary multiplicative group and as a result we have the construction of all the M-P.HFs that arise from a multiplicative group whose field of reduction is isomorphic to the field of integers mod 2.

2. Having as a starting point example 2 of YATRAS [11] we take again the set of rational numbers Q and we define in it the partition $O \cup X \cup Y$, where $O = \{0\}$, $X = Q^*_+$ and $Y = Q^*_-$. Then the set $H = \{0\} \cup Q^*_+ \cup Q^*_+ = Q$ equipped with the hyperoperation x+y of the above mentioned example and operation the usual multiplication of the rational numbers becomes a M-P.HF. The field of reduction is the set $F(Q) = \{C(0),C(x),C(y)\} = \{O,X,Y\}$ with addition

$O+O = O$, $O+X = X+O = X$, $X+Y = Y+X = O$, $X+X = Y$, $Y+Y = X$

and multiplication

$OX = XO = O$, $OY = YO = O$, $XX = X$, $YY = X$, $XY = YX = Y$

which is obviously isomorphic to the field of integers mod 3.

The set $\widetilde{R}=\{0,p,q\}$, with $p \in Q^*_+$ and $q \in Q^*_-$ arbitrarily chosen, is the field of choice that is defined through the mapping.

$$f:H \longrightarrow \widetilde{R} \text{ with } f(0)=0, \ f(x)=p, \ x \in Q^*_+ \text{ and } f(y)=q, \ y \in Q^*_-$$

We reach an analogous M-P.HF if we consider the set R of the real numbers and the partition $\{0\} \cup R^*_+ \cup R^*_-$. More generally we can consider an arbitrary multiplicative group and its partition $X \cup Y$. In this way we have the construction of all the M-P.HFs which derive from a multiplicative group and whose field of reduction is isomorphic to the field of integers mod 3.

3. Based on the example of YATRAS [12] we consider the M-P.H $(H, \dot{+})$ with $H = \bigcup_{i \in Z} X^i$ which appears in it and which is defined through the partition R of H into classes $X^i, i \in Z$ (the set of integers) with $X^0 = \{x_a^0\} = \{0\}$ and with hyperoperation

$$x_a^i \dot{+} x_b^j = X^{i+j}$$

Next the set of classes $H/R = H/(0) = \{X^i, i \in Z\}$ becomes a semigroup isomorphic to Z based on the isomorphism $Z \ni i \longrightarrow X^i \in H/(0)$ with multiplication

$$X^i X^j = X^{ij}$$

Consequently the structure $(H/(0),+,\cdot)$ becomes a ring isomorphic to $(Z,+,\cdot)$. Now, in order to make the M-P.H $(H, \dot{+})$ a M-P.HR we define an associate multiplication xy in H taking into consideration that since the ring $(H/(0),+,\cdot)$ is an integral domain, it does not have divisors of zero with regard to this multiplication. If we further assume that it also verifies the cancellation law according to proposition 2.11, we have that the classes X^i $(i \neq 0)$ have the same cardinality. Moreover applying lemma 2.1 in class $X^1 = C(1)$ which contains the unity element $1 \in Z$, we have that $xC(1) = C(1) \cdot 1 = C(1)C(1) = C(1)$, that is $xX^1 = X^1$ for every $x \in X^1$ and consequently we have that the class X^1 is a group.

Furthermore denoting the elements of H as x_a^i with $x_a^i \in X^i$ and $a \in I$, where I is a set of indexes, which as it easily results from the consideration of $x_a^i \in X^1$, has the structure of a group we have for the multiplication of H the following:

$$x_a^i \, x_b^j = x_{ab}^{ij}$$

It derives that $1 = x_1^1 \in X^1$ is the unitary element of H under this multiplication (because $1 \cdot x_a^i = x_1^1 \, x_a^i = x_a^i$). The hypercompositional structure $(H, \dot{+}, \cdot)$ defined in the above mentioned way, is an integral M-P.HR and more precisely a M-P.HD. We can easily see that what is mentioned in remark 2.6 is satisfied through the surgective mapping

$$\varphi : H \longrightarrow H/R \text{ with } \varphi(x_a^i) = X^i.$$

As a partial case we can consider the classes X^i with cardinality 2, that is $X^i = \{x_1^i, x_2^i\}$ with the set of indexes a group isomorphic to the group $\{\overline{1}, \overline{2}\}$ of the classes $(\neq \overline{0})$ of the integers mod 3.

4. Obviously the above construction of M-P.HR can also be applied if instead of Z we consider any integral ring. Thus we construct M-P.HRs that are also M-P.HFs, if we consider as ring Δ the field of classes of integers mod(p) with $p \in N$ (the set of naturals) prime number. E.g. for p=5, $\Delta = \{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4}\}$, so $H = X^0 \cup X^1 \cup X^2 \cup X^3 \cup X^4$ and e.g. $X^i = \{x_1^i, x_2^i\}$ or more generally $X^i = \{x_1^i,...,x_{q-1}^i\}$ where $I = \{1,...,q-1\}$ isomorphic to the multiplicative group of the classes of integers mod(q) with $q \in N$ prime number.

5. Finally we give one more example in which the M-P.HR that results according to the above way, does not have classes mod(0) with the same cardinality. Indeed let the ring $\Delta = \{\overline{0},\overline{1},\overline{2},\overline{3}\}$ of classes mod 4 and $H = X^0 \cup ... \cup X^3$. Here in the ring H/(0) which is isomorphic to Δ, we obviously have $X^2X^2 = X^0$ and thus for every $x_a^2 x_b^2 \in X^2$ we will have $x_a^2 x_b^2 = 0$. Trying to construct M-P.HRs with classes that have different cardinalities, we consider (due to simplicify): $X^1 = \{x_1^1, x_2^1\} = \{1, a\}$, $X^2 = \{x_1^2\} = \{b\}$, $X^3 = \{x_1^3, x_2^3\} = \{c, d\}$ (while $X^0 = \{x_a^0\} = \{0\}$) and the set of indexes I = {1,2} for the multiplication to be a group isomorphic to the group $\{\overline{1},\overline{2}\}$ mod 3. The equality of certain products of elements is necessary for the validity of lemma 2.1, e.g. in order to be

$$x_a^1 X^2 = X^1 x_1^2$$

it must be $$x_a^1 X^2 = x_1^1 x_1^2 = x_2^1 x_1^2 = x_1^2 = x_2^2$$

and $$X^1 x_1^2 = \{x_1^1, x_2^1\} x_1^2 = \{x_1^2, x_2^2\}$$

since it is also $$X^1 X^2 = X^2$$

So we get the following table, after having checked all the cases of products in H:

·	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	1	b	d	c
b	0	b	b	0	b	b
c	0	c	d	b	1	a
d	0	d	c	b	a	1

and the corresponding one for the addition, with the use of the same symbolism for the elements is:

+	0	1	a	b	c	d
0	0	{1,a}	{1,a}	b	{c,d}	{c,d}
1	{1,a}	b	b	{c,d}	0	0
a	{1,a}	b	b	{c,d}	0	0
b	b	{c,d}	{c,d}	0	{1,a}	{1,a}
c	{c,d}	0	0	{1,a}	b	b
d	{c,d}	0	0	{1,a}	b	b

As it can be seen from the above tables the cancellation law for the multiplication does not hold in this M-P.HR.

Remark 2.7

According to the above examples 4 and 5, we can generalize the method and proceed to the construction of all the M-P.HRs with ring of reduction isomorphic to the ring of classes of integers mod(p), with the same cardinality when p is not prime. However, in this case, since the ring of reduction does have zero divisors, the M-P.HRs which have this one as ring of reduction do not satisfy the cancellation law. This fact proves that the fundamental proposition 2.11 provides only a sufficient condition for the same cardinality of the classes mod(0) of the M-P.HR.

Lastly, if we consider the multiplication as a hyperoperation in a M-P.HRs we are led to a new polysymmetrical hyperstructure analogous to the superrings (MITTAS [8]). So, based on the introductory example of the hyperpolynomials (i.e. polynomials with coefficients from a hyperring), which appears in MITTAS [8] we can consider in an analogous way the **M-hyperpolynomials**, which due to the polysymmetrical of their addition, give a polysymmetrical M-hyperstructure.

So we have the definition:

Definition 2.2

We name **M-polysymmetrical superring** (M-P.SR) a set S equipped with two hyperoperations x+y - addition - and xy - multiplication - which verify the following conditions:

I. S is a M-P.H under the addition (called **additive** hypergroup of the M-P.SR).

II. S is a semihypergroup under the multiplication (called **multiplicative** semihypergroup of the M-P.SR) the 0 of which is a bilateral absorbing element.

i.e.: i) $xy \subseteq S$

ii) $(xy)z = x(yz)$

iii) $x0 = 0x = 0$ for every $x,y,z \in S$

III. The multiplication is bilaterally distributive over addition i.e.
$(x+y)z = xz+yz$, $z(x+y) = zx+zy$ for every $x,y,z \in S$

Remark 2.8

As in the case of rings and of M-P.HRs we can define various kinds of M-P.SRs such as commutative M-P.SRs, unitary M-P.SRs etc. Another special type of M-P.SRs is the **M-polysymmetrical superfield** (M-P.SF), that is a M-P.SR whose set $S^*=S-\{0\}$ is a multiplicative hypergroup.
It is obvious that by starting with the M-P.SRs we are led to various interesting hyperstructures, as it happens for instance with the consideration of the multiplicative semihypergroup S, as a M-P.H under the multiplication.
Regarding the M-P.SRs we set the following proposition which is analogous to 2.13 and the theorem that follows, while their special study will be the subject of a different paper.

Proposition 2.15

Let E be a set and A its subset with the structure of a ring without divisors of zero. Also, let 0 to be its neutral element and let for every $x \in A$, $-x$ to be its opposite. Then if:

i) ***there exists a partition R of E and a bijective mapping of the quotient set E/R on A such that $x \in A$, $f^{-1}(x) = C_R(x)$ for every $x \in A$ and***
ii) $C_R(0) = \{0\}$

then the hyperoperations $x \oplus y = f^{-1}[f(C_R(x)) + f(C_R(y))]$ ***and*** $x \circ y = f^{-1}[f(C_R(x)) \cdot f(C_R(y))]$ ***defined in E through the ring A make E a M-P.SR.***

Proof

Firstly the hyperoperation $x \oplus y$ makes E a M-P.H (YATRAS [11]). Next we notice that for every $x,y \in E$ the hyperproduct $x \circ y$ is a class mod(R) of E. We also have that

$$x \circ y = x \circ z \Leftrightarrow y \equiv z \bmod(R)$$

Indeed, if we set $f(C_R(x)) = x_1$ etc. we have

$$x \circ y = x \circ z \Leftrightarrow f^{-1}(x_1 y_1) = f^{-1}(x_1 z_1) \Leftrightarrow x_1 y_1 = x_1 z_1 \Leftrightarrow y_1 = z_1 \Leftrightarrow$$
$$C_R(y) = C_R(z) \Leftrightarrow y \equiv z \bmod(R)$$

Also, we obviously have:

$$x \circ y = x^* \circ y^* \quad \text{for every} \quad x^* \in C_R(x),\ y^* \in C_R(y)$$

Proceeding with the proof we see that the associative law for the multiplication is valid. Indeed if we set $x \circ y = C_R(w)$ we have:

$$f[(x\circ y)\circ z]=f(C_R(w)\circ z)=\{f(w^*\circ z),\ w^*\in C_R(w)\}=f(w\circ z)=$$
$$f(C_R(w))\cdot f(C_R(z))=w_1z_1=(x_1y_1)z_1$$

[since $f(x\circ y)=f(C_R(w))=w_1=x_1y_1=f(C_R(x))\cdot f(C_R(y))$].

But in A $(x_1y_1)z_1=x_1(y_1z_1)$. Thus $f[(x\circ y)\circ z]=f[x\circ(y\circ z)]$ and consequently the associate law holds.

Also

$$x\circ 0=f^{-1}[\ f(C_R(x))\cdot f(C_R(0))]=f^{-1}(x_1\cdot 0)=f^{-1}(0)=C_R(0)=0$$

In the same way $0\circ x=0$.

Yet, if we set $y\oplus z=C_R(w)$ we have:

$$f[x\circ(y\oplus z)]=f[x\circ C_R(w)]=\{f(x\circ w^*),w^*\in C_R(w)\}=f(x\circ w)=$$
$$f(C_R(x))\cdot f(C_R(w))=x_1w_1=x_1(y_1+z_1)$$

[since $f(y\oplus z)=f(C_R(w))=w_1=y_1+z_1=f(C_R(y))+f(C_R(z))$].

However, in the set A we have $x_1(y_1+z_1)=x_1y_1+x_1z_1$ and so

$$x_1y_1+x_1z_1=f(C_R(x))f(C_R(y))+f(C_R(x))f(C_R(z))=$$
$$=f(x\circ y)+f(x\circ z)=f(x\circ y\oplus x\circ z).$$

Thus

$f[x\circ(y\oplus z)]=f[x\circ y\oplus x\circ z]$ and since f is a bijective mapping it holds

$$x\circ(y\oplus z)=x\circ y\oplus x\circ z$$

In the same way we have $(y\oplus z)\circ x=y\circ x\oplus z\circ x$.

Consequently $(E,\oplus,\circ)$ is a M-P.SR.

Lastly, we can have the following theorem, which derives in an analogous way to the corresponding one 3.1 of YATRAS [11]:

Theorem 2.2

Let (A,+,·) be a ring without divisors of zero. If we correspond bijectively to A a family $\widetilde{A}$ of disjoint sets, from which the corresponding to the neutral element of A is a singleton, and if we make this family, (through the properly defined operations $X\oplus Y$, $X\circ Y$) a ring isomorphic to A, then the set $S=\bigcup_{X\in\widetilde{A}}X$ with hyperoperations $x\dotplus y=X\oplus Y$ and $x*y=X\circ Y$ for every $x,y\in S$, $(x,y)\in X\times Y$, $X,\ Y\in\widetilde{A}$ is a M-P.SR.

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