

Lower and Upper Approximations in H_ν -groups

B. Davvaz

Department of Mathematics

Yazd University

Yazd, Iran

E-mail: davvaz@vax.ipm.ac.ir

Abstract

The purpose of this paper is to introduce and discuss the concept of lower and upper approximations. A simple and straightforward way for interpreting rough sets is to use membership functions. We investigate the similarity between rough membership function and conditional probability. We also consider the fundamental relation β^* defined on an H_ν -group H and interpret the lower and upper approximations as subsets of the group H/β^* and give some properties of such subsets.

1991 AMS Subject Classification: 20N20.

Key words and Phrases: Hypergroup; H_ν -group; Fundamental group; Interval set; Rough set; Lower and upper approximations; Conditional probability.

1 Introduction

The notion of rough sets has been introduced by Pawlak [11] in 1982 and subsequently the algebraic approach of rough sets has been studied by some authors, for example, Bonikowaski [2], Iwinski [8], Pomykala and Pomykala [12], Gehrke and Walker [7]. Recently, Biswas and Nanda [1] introduced the notion of rough subgroups. Kuroki and Wang gave some properties of the lower and upper approximations with respect to the normal subgroups in [9].

The concept of hypergroup was introduced in 1934 by Marty [10] and has been studied in the following decades and nowadays by many mathematicians among whom, Krasner, Prenowitz, Mittas, Corsini, Sureou, Comer, Jantosciak, Vougiouklis.

The last of these, at the fourth A.H.A congress, Xanthi (1990), introduced the definitions of H_v -group.

The principal notions of hypergroup theory can be found in [3]. The basic results of H_v -groups are in [13].

In this paper we apply the concept of rough sets theory in the theory of algebraic hyperstructures. We consider the fundamental relation β^* defined on an H_v -group H and interpret the lower and upper approximations as subsets of the fundamental group H/β^* and obtain some results in this connection. In particular, we show that if X is an H_v -subgroup of H then upper approximation of X is a subgroup of H/β^* .

2 Interval sets

Given two subsets $A_1, A_2 \subseteq U$ with $A_1 \subseteq A_2$, we define the following closed interval set:

$$[A_1, A_2] = \{X \in \mathcal{P}(U) \mid A_1 \subseteq X \subseteq A_2\}$$

which is a subset of $\mathcal{P}(U)$. The set A_1 is called the lower bound, and A_2 the upper bound. That is, members of an interval set are subsets of the universe U . An interval set consists of all those subsets that are bounded by two particular elements of the Boolean algebra $\mathcal{P}(U)$. Let $I(\mathcal{P}(U))$ denote the set of all closed interval sets.

Set-theoretic operators on interval sets can be defined based on set operators on their members. For two interval sets $\mathcal{A} = [A_1, A_2]$ and $\mathcal{B} = [B_1, B_2]$, interval set intersection, union, and difference are defined by

$$\mathcal{A} \cap \mathcal{B} = \{X \cap Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\},$$

$$\mathcal{A} \cup \mathcal{B} = \{X \cup Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\},$$

$$\mathcal{A} \setminus \mathcal{B} = \{X - Y \mid X \in \mathcal{A}, Y \in \mathcal{B}\}.$$

The above defined operators are closed on $I(\mathcal{P}(U))$, namely, $\mathcal{A} \cap \mathcal{B}$, $\mathcal{A} \cup \mathcal{B}$, and $\mathcal{A} \setminus \mathcal{B}$ are interval sets. They can be explicitly computed by

$$\mathcal{A} \cap \mathcal{B} = [A_1 \cap B_1, A_2 \cap B_2],$$

$$\mathcal{A} \cup \mathcal{B} = [A_1 \cup B_1, A_2 \cup B_2],$$

$$\mathcal{A} \setminus \mathcal{B} = [A_1 - B_2, A_2 - B_1].$$

The interval set complement \neg is defined by $[U, U] \setminus [A_1, A_2]$. This is equivalent to $[U - A_2, U - A_1] = [\sim A_2, \sim A_1]$. Clearly, we have $\neg[\emptyset, \emptyset] = [U, U]$ and $\neg[U, U] = [\emptyset, \emptyset]$.

Degenerate interval sets of the form $[A, A]$ are equivalent to ordinary sets. For degenerate interval sets, the proposed operators \cap, \cup, \setminus , and \neg reduce to set operators. Interval set operators obey most properties of set operators. For example, idempotence, commutativity, associativity, and distributivity laws hold for \cap and \cup ; De Morgan's and double negation laws hold for \neg . Thus, the system $(I(\mathcal{P}(U)), \cap, \cup)$ is a complete distributive lattice, with zero element

$[\emptyset, \emptyset]$ and unit element $[U, U]$. The associated order relation is called interval set inclusion. It can be defined using the set inclusion relation:

$$\mathcal{A} \sqsubseteq \mathcal{B} \iff A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2.$$

The system $(I(\mathcal{P}(U)), \sqcap, \sqcup, \neg, [\emptyset, \emptyset], [U, U])$ is called an interval set algebra.

3 Lower and upper approximations

Let ρ be an equivalence relation defined on the set U and $[x]_\rho$ equivalence class of the relation ρ generated by an element $x \in U$.

Any finite union of equivalence classes of U is called a definable set in U . Let A be any subset of U . In general, A is not a definable set in U . However, the set A may be approximated by two definable set in U . The first one is called a ρ -lower approximation of A in U , denoted by $\underline{\rho}(A)$ and defined as follows:

$$\underline{\rho}(A) = \{x \in U \mid [x]_\rho \subseteq A\}.$$

The second set is called a ρ -upper approximation of A in U , denoted by $\overline{\rho}(A)$ and defined as follows:

$$\overline{\rho}(A) = \{x \in U \mid [x]_\rho \cap A \neq \emptyset\}.$$

The ρ -lower approximation of A in U is the greatest definable set in U contained in A . The ρ -upper approximation of A in U is the least definable set in U containing A . The difference $\widehat{\rho}(A) = \overline{\rho}(A) - \underline{\rho}(A)$ is called the ρ -boundary region of A . In the case when $\widehat{\rho}(A) = \emptyset$ the set A is said to be ρ -exact.

Using ρ -lower and ρ -upper approximations, we define a binary relation on subsets of U :

$$X \approx Y \iff \underline{\rho}(X) = \underline{\rho}(Y) \text{ and } \overline{\rho}(X) = \overline{\rho}(Y).$$

It is an equivalence relation which induces a partition $\mathcal{P}(U)/\approx$ of $\mathcal{P}(U)$. An equivalence class of \approx is called a ρ -rough set. Therefore a ρ -rough set of X is

the family of all subsets of U having the same ρ -lower and the same ρ -upper approximations of X . More specifically, a ρ -rough set is the following family of subsets of U :

$$\langle A_1, A_2 \rangle = \{X \in \mathcal{P}(U) \mid \underline{\rho}(X) = A_1, \bar{\rho}(X) = A_2\}.$$

A set $X \in \langle A_1, A_2 \rangle$ is said to be a member of the ρ -rough set.

Rough set intersection \sqcap , union \sqcup , and complement \neg are defined by set operators as follows: for two ρ -rough sets $\langle A_1, A_2 \rangle$ and $\langle B_1, B_2 \rangle$,

$$\begin{aligned} \langle A_1, A_2 \rangle \sqcap \langle B_1, B_2 \rangle &= \{X \in \mathcal{P}(U) \mid \underline{\rho}(X) = A_1 \cap B_1, \bar{\rho}(X) = A_2 \cap B_2\} \\ &= \langle A_1 \cap B_1, A_2 \cap B_2 \rangle, \end{aligned}$$

$$\begin{aligned} \langle A_1, A_2 \rangle \sqcup \langle B_1, B_2 \rangle &= \{X \in \mathcal{P}(U) \mid \underline{\rho}(X) = A_1 \cup B_1, \bar{\rho}(X) = A_2 \cup B_2\} \\ &= \langle A_1 \cup B_1, A_2 \cup B_2 \rangle, \end{aligned}$$

$$\begin{aligned} \neg \langle A_1, A_2 \rangle &= \{X \in \mathcal{P}(U) \mid \underline{\rho}(X) = \sim A_2, \bar{\rho}(X) = \sim A_1\} \\ &= \langle \sim A_2, \sim A_1 \rangle. \end{aligned}$$

The results are also ρ -rough sets. The induced system $(\mathcal{P}(U)/\approx, \sqcap, \sqcup, \neg, [\emptyset]_{\approx}, [U]_{\approx})$ is called a ρ -rough set algebra.

The corresponding order is called ρ -rough set inclusion and is given by

$$\langle A_1, A_2 \rangle \sqsubseteq \langle B_1, B_2 \rangle \iff A_1 \subseteq B_1 \text{ and } A_2 \subseteq B_2.$$

The proof of the following theorem is similar to the Proposition 2.2 of Pawlak [11] and Theorem 2.1 of Kuroki [9]. We shall give a proof for completeness.

Theorem 1. Let ρ be an equivalence relation on a set U . If A and B are non-empty subsets of U , then the following hold:

- 1) $\underline{\rho}(A) \subseteq A \subseteq \bar{\rho}(A)$;
- 2) $\bar{\rho}(A \cup B) = \bar{\rho}(A) \cup \bar{\rho}(B)$;

$$3) \rho(A \cap B) = \rho(A) \cap \rho(B);$$

$$4) A \subseteq B \text{ implies } \rho(A) \subseteq \rho(B);$$

$$5) A \subseteq B \text{ implies } \bar{\rho}(A) \subseteq \bar{\rho}(B);$$

$$6) \rho(A \cup B) \supseteq \rho(A) \cup \rho(B);$$

$$7) \bar{\rho}(A \cap B) \subseteq \bar{\rho}(A) \cap \bar{\rho}(B);$$

Proof. (1) If $a \in \rho(A)$, then $a \in [a]_\rho \subseteq A$. Hence $\rho(A) \subseteq A$. Next, if $a \in A$, then, since $a \in [a]_\rho$, we have $[a]_\rho \subseteq A \neq \emptyset$, and $a \in \bar{\rho}(A)$. Thus $A \subseteq \bar{\rho}(A)$.

$$\begin{aligned} (2) \ a \in \bar{\rho}(A \cup B) &\iff [a]_\rho \cap (A \cup B) \neq \emptyset \iff ([a]_\rho \cap A) \cup ([a]_\rho \cap B) \neq \emptyset \\ &\iff [a]_\rho \cap A \neq \emptyset \text{ or } [a]_\rho \cap B \neq \emptyset \iff a \in \bar{\rho}(A) \text{ or } a \in \bar{\rho}(B) \\ &\iff a \in \bar{\rho}(A) \cup \bar{\rho}(B) \end{aligned}$$

Thus $\bar{\rho}(A \cup B) = \bar{\rho}(A) \cup \bar{\rho}(B)$.

$$\begin{aligned} (3) \ a \in \rho(A \cap B) &\iff [a]_\rho \subseteq A \cap B \iff [a]_\rho \subseteq A \text{ and } [a]_\rho \subseteq B \\ &\iff a \in \rho(A) \text{ and } a \in \rho(B) \iff a \in \rho(A) \cap \rho(B). \end{aligned}$$

Thus $\rho(A \cap B) = \rho(A) \cap \rho(B)$.

(4) Since $A \subseteq B$ iff $A \cap B = A$, by (3) we have

$$\rho(A) = \rho(A \cap B) = \rho(A) \cap \rho(B).$$

This implies that $\rho(A) \subseteq \rho(B)$.

(5) Since $A \subseteq B$ iff $A \cup B = B$, by (2) we have

$$\bar{\rho}(B) = \bar{\rho}(A \cup B) = \bar{\rho}(A) \cup \bar{\rho}(B).$$

This implies that $\bar{\rho}(A) \subseteq \bar{\rho}(B)$.

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (4) we have

$$\rho(A) \subseteq \rho(A \cup B) \text{ and } \rho(B) \subseteq \rho(A \cup B),$$

which yields

$$\rho(A) \cup \rho(B) \subseteq \rho(A \cup B).$$

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (5) we have

$$\bar{\rho}(A \cap B) \subseteq \bar{\rho}(A) \text{ and } \bar{\rho}(A \cap B) \subseteq \bar{\rho}(B),$$

which yields

$$\bar{\rho}(A \cap B) \subseteq \bar{\rho}(A) \cap \bar{\rho}(B). \square$$

4 Probabilistic rough sets

The notion of conditional probability is a basic tool of probability theory, and it is unfortunate that its great simplicity is somewhat obscured by a singularly clumsy terminology.

Let X be an event with positive probability. For an arbitrary event A we shall write

$$P(A|X) = \frac{P(A \cap X)}{P(X)}.$$

The quantity so defined will be called the conditional probability of A on the hypothesis X (or for given X). When all sample points have equal probabilities, $P(A|X)$ is the ratio $\frac{|A \cap X|}{|X|}$ of the number of sample points common to A and X , to the number of points in X . All theorems on probabilities are valid also for conditional probabilities with respect to any particular hypothesis X . For example, the fundamental relation for probability of the occurrence of either A or B or both takes on the form

$$P(A \cup B|X) = P(A|X) + P(B|X) - P(A \cap B|X).$$

For any $A \subseteq U$, a rough membership function is defined by

$$\mu_A(x) = \frac{|A \cap [x]_\rho|}{|[x]_\rho|}.$$

By definition, elements in the same equivalence class have the same degree of membership. One can see the similarity between rough membership function and conditional probability. The rough membership value $\mu_A(x)$ may be interpreted as the probability of x belonging to A given that x belongs to an equivalence class. Under this interpretation, one obtains the notion of probabilistic rough sets. By the laws of probability, the intersection and union of probabilistic rough sets are not truth-functional. Nevertheless, we have

- 1) $\mu_A(x) = 1 \iff x \in \rho(A)$,
- 2) $\mu_A(x) = 0 \iff x \in \rho(A^c)$,
- 3) $0 < \mu_A(x) < 1 \iff x \in \widehat{\rho(A)}$,
- 4) $\mu_A(x) = 1 - \mu_{A^c}(x)$,
- 5) $\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_{A \cap B}(x)$,
- 6) $\max\{\mu_A(x), \mu_B(x)\} \leq \mu_{A \cup B}(x) \leq \min\{1, \mu_A(x) + \mu_B(x)\}$,
- 7) $\mu_{A \cap B} \leq \min\{\mu_A(x), \mu_B(x)\}$,
- 8) for any pairwise disjoint collection P of subsets

$$\mu_{\cup P}(x) = \sum\{\mu_Y(x) \mid Y \in P\}.$$

They follow from the properties of probability.

With the rough membership function, One may view a probabilistic rough set as a special type of fuzzy set. By drawing such a link between these two theories, the non-truth-functionality of the operators on probabilistic rough sets may provide more insights into the definition of fuzzy set operators.

The notion of probabilistic rough sets may be related to ρ -rough set algebra $(\mathcal{P}(U/\approx, \cap, \cup, \neg, [\emptyset]_{\approx}, [U]_{\approx})$. For two members of the same membership function, i.e., $A \approx B$, they may not be characterized by the same membership

function, i.e., $\mu_A \neq \mu_B$. Let $c(\mu_A)$ and $s(\mu_A)$ denote the core and support of μ_A defined by

$$c(\mu_A) = \{x \mid \mu_A(x) = 1\},$$

$$s(\mu_A) = \{x \mid \mu_A(x) > 0\}.$$

By properties (1) and (2), one can verify that if $A \approx B$, then $c(\mu_A) = c(\mu_B)$, and $s(\mu_A) = s(\mu_B)$. In other words, a ρ -rough set is a family of probabilistic rough sets with the same core and support.

5 Algebraic hyperstructures

A hyperstructure is a set H together with a function $\cdot : H \times H \rightarrow \mathcal{P}^*(H)$ called hyperoperation, where $\mathcal{P}^*(H)$ denotes the set of all the non-empty subsets of H . According to [10] Marty defined a hypergroup as a hyperstructure (H, \cdot) such that the following axioms hold: (i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y, z in H , (ii) $a \cdot H = H \cdot a = H$ for all a in H . The second axiom is called reproduction axiom. In the above definition if $A, B \subseteq H$ and $x \in H$ then we define

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b, \quad x \cdot B = \{x\} \cdot B, \quad A \cdot x = A \cdot \{x\}.$$

An H_v -group (cf. [4,5,13,14,15]) is a hyperstructure (H, \cdot) such that (i) $(x \cdot y) \cdot z \cap x \cdot (y \cdot z) \neq \emptyset$ for all x, y, z in H , (ii) $a \cdot H = H \cdot a = H$ for all a in H . The first axiom is called weak associativity. If (H, \cdot) satisfies only the first axiom, then it is called an H_v -semigroup. A subset K of H is called an H_v -subgroup if (K, \cdot) is itself an H_v -group.

Let (H, \cdot) be an H_v -group. The relation β^* is the smallest equivalence relation on H such that the quotient H/β^* , the set of all equivalence classes, is a group. β^* is called the fundamental equivalence relation on H . This relation is studied by Corsini [3] concerning hypergroups, see also [6], [13] and [16].

According to [13] if \mathcal{U} denotes the set of all the finite products of elements

of H , then a relation β can be defined on H whose transitive closure is the fundamental relation β^* . The relation β is as follows: for x and y in H we write $x\beta y$ if and only if $\{x, y\} \subseteq u$, for some $u \in \mathcal{U}$. We can rewrite the definition of β^* on H as follows:

$a\beta^*b$ iff there exist $z_1, \dots, z_{n+1} \in H$ with $z_1 = a$, $z_{n+1} = b$ and $u_1, \dots, u_n \in \mathcal{U}$ such that

$$\{z_i, z_{i+1}\} \subseteq u_i \quad (i = 1, \dots, n).$$

Suppose $\beta^*(a)$ is the equivalence class containing $a \in H$. Then the product \odot on H/β^* is defined as follows: $\beta^*(a) \odot \beta^*(b) = \{\beta^*(c) \mid c \in \beta^*(a) \cdot \beta^*(b)\}$ for all a, b in H . It is proved in [13] that $\beta^*(a) \odot \beta^*(b)$ is the singleton $\{\beta^*(c)\}$ for all $c \in \beta^*(a) \cdot \beta^*(b)$. In this way H/β^* becomes a hypergroup. If we put $\beta^*(a) \odot \beta^*(b) = \beta^*(c)$, then H/β^* becomes a group.

Let ρ be an equivalence relation on an H_v -group H . If $\{A, B\} \subseteq \mathcal{P}^*(H)$, we write $A\bar{\rho}B$ to denote that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $b \in B$, there exists $a \in A$ such that $a\rho b$.

We write $A\bar{\rho}B$ if for every $a \in A$ and for every $b \in B$, one has $a\rho b$.

Definition 2.(cf. [3]). An equivalence relation ρ on an H_v -group H is called regular to the right if for every $(x, y) \in H \times H$, one has

$$x\rho y \implies x \cdot a\bar{\rho}y \cdot a \text{ for all } a \in H.$$

We say that ρ is strongly regular to the right if for every $(x, y) \in H \times H$, the implication

$$x\rho y \implies x \cdot a\bar{\rho}y \cdot a \text{ for all } a \in H$$

is valid.

Analogously we define the regularity (strong regularity) of an equivalence relation to the left. A regular equivalence (strongly regular) relation to the right and to the left is called regular (strongly regular).

The following corollary is exactly obtained from above definitions.

Corollary 3. β^* is a strongly regular relation.

Definition 4. Let (H_1, \cdot) and $(H_2, *)$ be H_v -groups. A mapping T from H_1 into H_2 is called a strong homomorphism if

$$\bigcup_{c \in a \cdot b} T(c) = T(a) * T(b)$$

for all $a, b \in H$. The set $K = \{(a, b) \in H_1 \times H_1 \mid T(a) = T(b)\}$ is called the kernel of T .

Proposition 5. Let $T : H_1 \rightarrow H_2$ be a strong homomorphism of the H_v -groups (H_1, \cdot) and $(H_2, *)$. Then K is a regular relation on H_1 .

Proof. The proof is straightforward and omitted. \square

6 On the fundamental relation β^*

Throughout this section we let H be an H_v -group.

The lower and upper approximations can be presented in an equivalent form as shown below. Let X be a non-empty subsets of H . Then

$$\underline{\beta^*(X)} = \{\beta^*(x) \in H/\beta^* \mid \beta^*(x) \subseteq X\}$$

and

$$\overline{\beta^*(X)} = \{\beta^*(x) \in H/\beta^* \mid \beta^*(x) \cap X \neq \emptyset\}.$$

Now, we discuss these sets as subsets of the fundamental group H/β^* .

Proposition 6. Let X and Y are non-empty subsets of H , then the following hold:

- 1) $\overline{\beta^*(X \cup Y)} = \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}$;
- 2) $\underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}$;
- 3) $X \subseteq Y$ implies $\underline{\beta^*(X)} \subseteq \underline{\beta^*(Y)}$;
- 4) $X \subseteq Y$ implies $\overline{\beta^*(X)} \subseteq \overline{\beta^*(Y)}$;
- 5) $\underline{\beta^*(X \cup Y)} \supseteq \underline{\beta^*(X)} \cup \underline{\beta^*(Y)}$;
- 6) $\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \cap \overline{\beta^*(Y)}$.

Proof. (1)

$$\begin{aligned}
 \beta^*(x) \in \overline{\beta^*(X \cup Y)} &\iff \beta^*(x) \cap (X \cup Y) \neq \emptyset \\
 &\iff (\beta^*(x) \cap X) \cup (\beta^*(x) \cap Y) \neq \emptyset \\
 &\iff \beta^*(x) \cap X \neq \emptyset \text{ or } \beta^*(x) \cap Y \neq \emptyset \\
 &\iff \beta^*(x) \in \overline{\beta^*(X)} \text{ or } \beta^*(x) \in \overline{\beta^*(Y)} \\
 &\iff \beta^*(x) \in \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}.
 \end{aligned}$$

Thus $\overline{\beta^*(X \cup Y)} = \overline{\beta^*(X)} \cup \overline{\beta^*(Y)}$.

(2)

$$\begin{aligned}
 \beta^*(x) \in \underline{\beta^*(X \cap Y)} &\iff \beta^*(x) \subseteq X \cap Y \\
 &\iff \beta^*(x) \subseteq X \text{ and } \beta^*(x) \subseteq Y \\
 &\iff \beta^*(x) \in \underline{\beta^*(X)} \text{ and } \beta^*(x) \in \underline{\beta^*(Y)} \\
 &\iff \beta^*(x) \in \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}.
 \end{aligned}$$

Thus $\underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}$.

(3) Since $X \subseteq Y$ iff $X \cap Y = X$, by (2) we have

$$\underline{\beta^*(X)} = \underline{\beta^*(X \cap Y)} = \underline{\beta^*(X)} \cap \underline{\beta^*(Y)}.$$

This implies that $\underline{\beta^*(X)} \subseteq \underline{\beta^*(Y)}$.

(4) Since $X \subseteq Y$ iff $X \cup Y = Y$, by (1) we have

$$\overline{\beta^*(Y)} = \overline{\beta^*(X \cup Y)} = \overline{\beta^*(X) \cup \beta^*(Y)}.$$

This implies that $\overline{\beta^*(X)} \subseteq \overline{\beta^*(Y)}$.

(5) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, by (3) we have

$$\overline{\beta^*(X)} \subseteq \overline{\beta^*(X \cup Y)} \text{ and } \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cup Y)},$$

which yields

$$\overline{\beta^*(X) \cup \beta^*(Y)} \subseteq \overline{\beta^*(X \cup Y)}.$$

(6) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, by (4) we have

$$\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \text{ and } \overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(Y)},$$

which yields

$$\overline{\beta^*(X \cap Y)} \subseteq \overline{\beta^*(X)} \cap \overline{\beta^*(Y)}. \square$$

Theorem 7. If X is an H_ν -subgroup of (H, \cdot) , then $\overline{\beta^*(X)}$ is a subgroup of $(H/\beta^*, \odot)$.

Proof. The kernel of the canonical map $\varphi : H \rightarrow H/\beta^*$ is called the core of H and is denoted by ω_H . Here we also denote by ω_H the unit element of H/β^* .

First we show that $\omega_H \in \overline{\beta^*(X)}$. Since X is an H_ν -subgroup of (H, \cdot) , then for every $a \in X$ we have $a \cdot X = X$. Therefore $a \in a \cdot X$ and so there exists $b \in X$ such that $a \in a \cdot b$ which implies $\beta^*(a) = \beta^*(a \cdot b) = \beta^*(a) \odot \beta^*(b)$. Therefore $\beta^*(b) = \omega_H$ and so $b \in \omega_H \cap X$ which implies $\omega_H \cap X \neq \emptyset$. Therefore $\omega_H \in \overline{\beta^*(X)}$.

Now, suppose $\beta^*(x), \beta^*(y) \in \overline{\beta^*(X)}$, we show that $\beta^*(x) \odot \beta^*(y) \in H/\beta^*$.

We have $\beta^*(x) \cap X \neq \emptyset$ and $\beta^*(y) \cap X \neq \emptyset$ then there exist $a \in \beta^*(x) \cap X$ and $b \in \beta^*(y) \cap X$. Thus $a \in \beta^*(x)$, $a \in X$, $b \in \beta^*(y)$, $b \in X$ and so

$$a \cdot b \subseteq \beta^*(x) \cdot \beta^*(y) \subseteq \beta^*(x \cdot y) = \beta^*(x) \odot \beta^*(y).$$

For every $c \in x \cdot y$ we have $\beta^*(c) = \beta^*(x) \cdot \beta^*(y)$. Therefore we get $a \cdot b \subseteq \beta^*(c)$ and $a \cdot b \subseteq X$.

Therefore $\beta^*(c) \cap X \neq \emptyset$ which yields $\beta^*(c) \in \overline{\beta^*(X)}$ or $\beta^*(x) \odot \beta^*(y) \in \overline{\beta^*(X)}$.

Finally, if $\beta^*(x) \in \overline{\beta^*(X)}$ then we show that $\beta^*(x)^{-1} \in \overline{\beta^*(X)}$. Since $\omega_H \cap X \neq \emptyset$ then there exists $h \in \omega_H \cap X$ and since $\beta^*(x) \cap X \neq \emptyset$ then there exists $y \in \beta^*(x) \cap X$. By reproduction axiom we get $h \in y \cdot X$ then there exists $a \in X$ such that $h \in y \cdot a$ which implies $\beta^*(h) = \beta^*(y) \odot \beta^*(a)$. Since $h \in \omega_H$ then $\beta^*(h) = \omega_H$. Therefore $\omega_H = \beta^*(y) \odot \beta^*(a)$ or $\omega_H = \beta^*(x) \odot \beta^*(a)$ which yields $\beta^*(a) = \beta^*(x)^{-1}$. Since $a \in X$ and $a \in \beta^*(a)$ then $\beta^*(a) \cap X \neq \emptyset$ and so $\beta^*(a) \in \overline{\beta^*(X)}$. Therefore $\overline{\beta^*(X)}$ is a subgroup of $H/\beta^*, \odot$. \square

Proposition 8. If X and Y are non-empty subsets of H , then

$$\overline{\beta^*(X)} \odot \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cdot Y)}.$$

Proof. We have

$$\begin{aligned} \overline{\beta^*(X)} \odot \overline{\beta^*(Y)} &= \{\beta^*(a) \odot \beta^*(b) \mid \beta^*(a) \in \overline{\beta^*(X)}, \beta^*(b) \in \overline{\beta^*(Y)}\} \\ &= \{\beta^*(a) \odot \beta^*(b) \mid \beta^*(a) \cap X \neq \emptyset, \beta^*(b) \cap Y \neq \emptyset\}. \end{aligned}$$

Therefore $(\beta^*(a) \cdot \beta^*(b)) \cap (X \cdot Y) \neq \emptyset$. Since $\beta^*(a) \cdot \beta^*(b) \subseteq \beta^*(a \cdot b)$. We obtain $\beta^*(a \cdot b) \cap (X \cdot Y) \neq \emptyset$. Thus $\beta^*(a \cdot b) = \beta^*(a) \odot \beta^*(b) \in \beta^*(X \cdot Y)$ and so $\overline{\beta^*(X)} \odot \overline{\beta^*(Y)} \subseteq \overline{\beta^*(X \cdot Y)}$. \square

Proposition 9. Let X and Y be two H_ν -subgroups of H and let $f : X \rightarrow Y$ be a strong homomorphism, then f induces a homomorphism $F : \overline{\beta^*(X)} \rightarrow \overline{\beta^*(Y)}$ by setting

$$F(\beta^*(x)) = \beta^*(f(x)), \quad \forall x \in X.$$

Proof. First we prove that F is well-defined. Suppose that $\beta^*(a) = \beta^*(b)$ then there exist $x_1, \dots, x_{m+1} \in H$ with $x_1 = a$, $x_{m+1} = b$ and $u_1, \dots, u_m \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$ ($i = 1, \dots, m$) which implies $\{f(x_i), f(x_{i+1})\} \subseteq f(u_i)$ ($i = 1, \dots, m$). Since f is a strong homomorphism and $u_i \in \mathcal{U}$ we get $f(u_i) \in \mathcal{U}$. Therefore $f(a)\beta^*f(b)$ or $F(\beta^*(a)) = F(\beta^*(b))$. On the other hand if $\beta^*(a) \in \overline{\beta^*(X)}$ then $\beta^*(a) \cap X \neq \emptyset$ and so there exists $b \in \beta^*(a) \cap X$. Thus $b\beta^*a$ and $b \in X$ which yield $f(b)\beta^*f(a)$ and $f(b) \in Y$. So $f(b) \in \beta^*(f(a))$ and $f(b) \in Y$ then $\beta^*(f(a)) \cap Y \neq \emptyset$ and so $\beta^*(f(a)) \in \overline{\beta^*(Y)}$ or $F(\beta^*(a)) \in \overline{\beta^*(Y)}$. Thus F is well-defined. Now we have

$$\begin{aligned} F(\beta^*(a) \odot \beta^*(b)) &= F(\beta^*(a \cdot b)) \\ &= \beta^*(f(a \cdot b)) \\ &= \beta^*(f(a) \cdot f(b)) \\ &= \beta^*(f(a)) \odot \beta^*(f(b)) \\ &= F(\beta^*(a)) \odot F(\beta^*(b)). \end{aligned}$$

Therefore F is a homomorphism. \square

References

- [1] R. Biswas and S. Nanda, *Rough groups and rough subgroups*, Bull. Polish Acad. Sci. Math., 42 (1994) 251-254.
- [2] Z. Bonikowaski, *Algebraic structures of rough sets*, in W.P. Ziarko Editor, *Rough sets, Fuzzy sets and Knowledge Discovery*, Springer-Verlag, Berlin, (1995) 242-247.
- [3] P. Corsini, *Prolegomena of hypergroup theory*, Second edition, Aviani editor, (1993).
- [4] B. Davvaz, *Fuzzy H_v -groups*, Fuzzy sets and systems, 101 (1999) 191-195.

- [5] B. Davvaz, *On H_v -subgroups and anti fuzzy H_v -subgroups*, Korean J. Comput. & Appl. Math., 5 (1998) 181-190.
- [6] B. Davvaz, *Weak Polygroups*, Proc. of the 28th Annual Iranian Math. Conference, (1997) 28-31.
- [7] M. Gehrke and E. Walker, *On the structure of rough sets*, Bull. Polish Acad. Sci. Math., 40 (1992) 235-245.
- [8] T. Iwinski, *Algebraic approach to rough sets*, Bull. Polish Acad. Sci. Math., 35 (1987) 673-683.
- [9] N. Kuroki and P.P. Wang, *The lower and upper approximation in a fuzzy group*, Information Science, 90 (1996) 203-220.
- [10] F. Marty, *Sur une generalization de la notion de group*, 8th Congress Math. Scandenaves, Stockholm, (1934) 45-49.
- [11] Z Pawlak, *Rough sets*, Int. J. Inf. Comp. Sci., 11 (1982) 341-356.
- [12] J. Pomykala and J.A. Pomykala, *The stone algebra of rough sets*, Bull. Polish Acad. Sci. Math., 36 (1988) 495-508.
- [13] T.Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Inc., Florida, (1994).
- [14] T.Vougiouklis, *A new class of hyperstructures*, J. Combin. inform. System Sciences, 20 (1995) 229-235.
- [15] T. Vougiouklis, *Convolutions on WASS hyperstructures*, Discrete Math., 174 (1997) 337-355.
- [16] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Proc.of the 4th Int. Congress on Algebraic Hyperstructures and Appl. (A.H.A 1990), Xanthi, Greece, World Sientific, (1991) 203-211.