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ON SOME APPLICATIONS OF FUZZY SETS AND COMMUTATIVE HYPERGROUPS TO EVALUATION IN ARCHITECTURE AND TOWN-PLANNING[1]

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SUNTO: In alcune recenti leggi sulla determinazione dei canoni di affitto e dei valori catastali degli alloggi è richiesta una "divisione in zone omogenee" della città in base ad assegnati criteri. In molte città, come risulta anche dai quotidiani, è stata a tale scopo effettuata una classificazione crisp dell'insieme degli alloggi. In alcuni nostri lavori abbiamo osservato che, dato il modo "non brusco" in cui variano le caratteristiche degli alloggi, sembra più opportuna una classificazione fuzzy.
In questo lavoro, partendo dal concetto di join space associato ad un insieme fuzzy, indaghiamo sulle relazioni fra partizioni fuzzy ed ipergruppi commutativi. Più in generale, introduciamo i concetti di "insieme fuzzy qualitativo lineare" e mostriamo le relazioni fra le famiglie di tali insiemi, le partizioni fuzzy e gli ipergruppi commutativi. Lo scopo del lavoro è di mostrare come la teoria degli ipergruppi commutativi possa essere un utile strumento di lavoro per affrontare problemi di valutazione in urbanistica. In particolare, per mezzo dei blocchi associati ad un opportuno ipergruppo commutativo si individuano aree "quasi omogenee", per le quali si possono determinare le oscillazioni di affitti e valori catastali.

ABSTRACT: In some recent laws about the determination of the rents and of the estimated income of properties, it is required a subdivision of the municipal area in homogeneous zones on the basis of assigned criteria. For this reason in many cities, as it also appeared in some daily newspapers, it has been made a crisp classification of the set of buildings. In some our papers we have observed that the peculiarities of the buildings change in a "not sharp" way and so a fuzzy classification seems more suitable.
In this paper, starting from the concept of join space associated to a fuzzy set, we study the relations between fuzzy partitions and commutative hypergroups. More in

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general, we introduce the concepts of "qualitative linear fuzzy set" and we show the relations among the families of these sets, fuzzy partitions and commutative hypergroups. The aim of the study is to show that the commutative hypergroups are a useful tool to study problems on the evaluation in town-planning. In particular, from the blocks associated to a suitable commutative hypergroup, we single out "almost homogeneous" areas and we can determine, in such areas, the fluctuation of the rents and of the values of the buildings.

KEYWORDS: Commutative hypergroups, Fuzzy classifications, Evaluation in Architecture and in Town-planning, Qualitative fuzzy sets

1. CLASSIFICATION FOR THE EVALUATION IN TOWN-PLANNING

Many problems about the evaluation in town-planning lead to a classification of a city or of a territory: for example the organization of the taxation of the building, the evaluation of the soil or the distribution of the mail and of the shops.
In recent times, many Italian newspapers show the classifications of some cities made by municipal governments for the definition of the rents. Every city is considered as a set Ω with elements the buildings and it is divided in a fixed number of subsets, called microzones, that are a partition of Ω. Such classifications depend on a set of criteria fixed by the national law.
The classifications considered by the municipal authorities are all of crisp type, that is any element of Ω belongs totally only to a class. Besides, by reading of the laws and of the reportages of the newspapers it would seem that such classifications are not obtained with precise statistical methods but rather in some empirical ways.
In some our recent papers, [6], [7], we note that for the characteristics of a city a fuzzy classification seems to be more appropriate than a crisp one. In fact the variations of the characteristics of the buildings are not crisp but they are always variable almost with continuity and so it is not possible to consider "walls" that divide the microzones from one to other. In [7] we propose some algorithms of fuzzy classification that we think suitable for the formation of the microzones.
In this paper we investigate about the relations between fuzzy classifications and commutative hypergroups. We think that the hypergroups are a very useful tool to individuate, both by an algebraic and a geometric point of view, homogeneous zones in the city. We consider also a generalization of the concept of fuzzy set, the "qualitative linear fuzzy set" that is a more natural function than fuzzy sets in problems of Architecture in which we can give judgments but not precise measures about the degree in which a building belongs to a given class or has a particular characteristic.

2. FUZZY CLASSIFICATION AND HYPERGROUPS

We recall some fundamental definition

Definition 2.1 A *fuzzy set* with universe Ω is a function $\varphi:\Omega\to[0,1]$. If $\varphi(\Omega) = \{h\}$, $h\in[0,1]$ φ is called *constant fuzzy set*. In particular for h=1 is called *null fuzzy set* and for h=1 *unitary fuzzy set*. A *fuzzy partition* is a finite o countable family of non null fuzzy sets $\{\varphi_i, i\in I\}$ such that, $\forall x\in\Omega$, $\Sigma_{i\in I}\ \varphi_i(x) = 1$.
A fuzzy set $\varphi:\Omega\to[0,1]$ with $\varphi(\Omega)\subseteq\{0, 1\}$ is called *crisp set* and a finite o countable family of non null crisp sets $\{\varphi_i, i\in I\}$ such that, $\forall x\in\Omega$, $\Sigma_{i\in I}\ \varphi_i(x) = 1$ is said to be a *crisp partition*.

If we consider the bijection Ψ: $S\in\wp(\Omega)\to(\varphi_S: \Omega\to\{0, 1\} \ / \ \varphi_S^{-1}(1) = S)$ we can identify every subset S of Ω with the crisp set φ_S with universe Ω. Then a crisp partition is a usual partition of Ω.

Definition 2.2 A hypergroupoid $H = (\Omega, \sigma)$ is said to be a *hypergroup* if we have the following properties

(1) *associative* $\forall a,b,c\in\Omega$, $(a\sigma b)\sigma c = a\sigma(b\sigma c)$;
(2) *reproducibility* $\forall a,b\in\Omega$, $\exists x,y\in\Omega$: $b \in a\sigma x \cap y\sigma a$.

A hypergroup is said to be *commutative* if $\forall a,b\in\Omega$, $a\sigma b = b\sigma a$. In a commutative hypergroup we define "in a natural way" the division "/" if we put

$$\forall a,b\in\Omega,\ b/a = \{x\in\Omega: b\in a\sigma x\}.$$

Definition 2.3 Let $H = (\Omega, \sigma)$ be a commutative hypergroup. It is

- *open* if $\forall a,b\in\Omega$, $a\neq b$, $a\bullet b\cap\{a,b\}=\varnothing$;
- *closed* if $\forall a,b\in\Omega$, $a\bullet b\supseteq\{a,b\}$;
- *geometric* if $\forall a\in\Omega$, $a\bullet a = \{a\} = a/a$;
- *join space* if we have the following incidence property:
$$\forall a, b, c, d \in\Omega,\ a/b\cap c/d\neq\varnothing\Rightarrow a\bullet d\cap b\bullet c\neq\varnothing.$$

Let $\varphi:\Omega\to[0,1]$ be a fuzzy set with universe Ω. By a result of Corsini, [3], if we put

$$\forall a, b\in\Omega,\ a\sigma b = \{z\in\Omega: \min\{\varphi(a), \varphi(b)\}\leq \varphi(z)\leq \max\{\varphi(a), \varphi(b)\}\}$$

then $H = (\Omega, \sigma)$ is a *closed join space*, called *associated* to φ.

Let I be a finite or countable set and let $\forall i\in I$, $\varphi_i:\Omega\to[0,1]$ be a fuzzy set with universe set Ω. If we put,

$$\forall i\in I,\ a\sigma_i b = \{z\in\Omega: \min\{\varphi_i(a), \varphi_i(b)\} \leq \varphi_i(z)\leq \max\{\varphi_i(a), \varphi_i(b)\},$$

we have the family of the closed join spaces $\{H_i = (\Omega, \varphi_i), i\in I\}$ associated to the family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$.

Now we introduce some hypergroups associated to the fuzzy partitions. We begin by considering the particular case of the crisp partitions.
Let $\Phi = \{\varphi_i, i\in I\}$ be a crisp partition with universe Ω and, $\forall a\in\Omega$, let $C(a)$ the class of a. If we put $\forall a, b\in\Omega$, $a*b = C(a)\cup C(b)$ we can consider the hypergroupoid $H=(\Omega, *)$, that we call *hypergroupoid associated* to Φ.

We have the following
Theorem 2.4 Let $\Phi = \{\varphi_i, i\in I\}$ be a crisp partition with universe Ω and let $H=(\Omega, *)$ be the hypergroupoid associated to Φ. Then H is a commutative closed hypergroup. Moreover we have $\forall a, b\in\Omega$, $a*b = \cap_{i\in I} a\sigma_i b$.
Proof The associative and commutative properties are consequences of the ones of the union. Since $\{a, b\} \subseteq a*b$ we have the closure and the reproducibility and so H is a commutative hypergroup.
Let x an element Ω. We have that
$(x\in a*b) \Leftrightarrow (x\in C(a)\cup C(b)) \Leftrightarrow (\forall i\in I, \varphi_i(x) = \varphi_i(a) \text{ or } \varphi_i(x) = \varphi_i(b)) \Leftrightarrow$
$\Leftrightarrow (\forall i\in I, x\in a\sigma_i b) \Leftrightarrow (x\in\cap_{i\in I} a\sigma_i b)$.

In general, let $\Phi = \{\varphi_i, i\in I\}$ be a fuzzy partition with universe Ω. If we wish to extend the results achieved for the crisp partitions to the fuzzy ones, we have to consider the hyperoperation $*: (x, y)\in\Omega\to\cap_{i\in I} a\sigma_i b$. But, for the applications to the architecture and town-planning it is convenient to examine a concept more general than the one of fuzzy set, that we call *qualitative linear fuzzy set*.
In fact, if Ω is a set of objects (e. g. the buildings of a city) to evaluate as to a criterion K, in general we do not have a numeric function $\varphi_K: \Omega\to[0,1]$ such that, $\forall x\in\Omega$, $\varphi_K(x)$ gives the *measure* in which x satisfies the criterion K. We are happy if we can find a set totally ordered S of the "possible judgements" about the truth or the falsity of the proposition "x satisfies K". For example S may be "false, not all false, partially false, partially true, not all true, true". S is called *set of qualitative values*. We suppose that S has a minimum F and a maximum T, that are, respectively, the qualitative values "false" and "true". For any judgement g of S we call *the opposite of g* the judgement g^c obtained by g with the change between the words "true" and "false". We call *the opposite of S* the set S^c of the opposites of the elements of S.

Definition 2.5 Let S be a set of qualitative values. An application $\alpha: \Omega\to S$ is called *qualitative linear fuzzy set (qlfs)*. A qlfs $\alpha^c: \Omega\to S^c$ that to any $x\in\Omega$ associates $(\alpha(x))^c$ is called *the opposite* of α.

Definition 2.6 Let S be a set of qualitative values. We call *numerical evaluation* on S any function $v: S\to[0,1]$ such that $v(F) = 0$, $v(V)=1$, $\forall z, t\in S$, $z\le t \Rightarrow v(z)\le v(t)$.

We say that v: $S \to [0,1]$ is a *strong numerical evaluation* on S if it is a numerical evaluation on S and $\forall z, t \in S$, $v(z) \le v(t) \Rightarrow z \le t$.

We can note that, if v: $S \to [0,1]$ is a numerical evaluation on S then v^c: $S^c \to [0,1]$ such that, $\forall z \in S$, $v^c(z^c) = 1 - v(z)$ is a numerical evaluation on S^c such that $\forall z, t \in S$, $z \le t \Rightarrow v^c(z^c) \ge v^c(t^c)$. If v is strong then also v^c is strong. We call v^c *the opposite of v*. We have the relations $(S^c)^c = S$, $(\alpha^c)^c = \alpha$ and $(v^c)^c = v$.

If α: $\Omega \to S$ is a qualitative linear fuzzy set and v: $S \to [0,1]$ is a numerical evaluation then the function $\varphi = v \circ \alpha$ is a fuzzy set with universe Ω. Then, from a qlfs on Ω, if we give a suitable numerical evaluation, we can have also a fuzzy set on Ω. Let $\varphi^c = v^c \circ \alpha^c$. It is easy to prove that $\varphi^c = 1 - \varphi$.

We can prove that the theorem of Corsini can be extended also to the qualitative linear fuzzy set. In fact the proof of this theorem utilizes only the properties of the total order relation on [0, 1]. So, if α: $\Omega \to S$ is a qualitative linear fuzzy set, if we put

$$\forall a, b \in \Omega,\ a\sigma b = \{z \in \Omega: \min\{\alpha(a), \alpha(b)\} \le \alpha(z) \le \max\{\alpha(a), \alpha(b)\}\},$$

with $\le$ order relation on S, we have that $H = (\Omega, \sigma)$ is a closed join space, that we call *join space associated* to α.

We note that, if we consider the opposite α^c of α, the order relation $\le^c$ on S^c is the opposite of $\le$ on S. Then we have $\{z \in \Omega: \min\{\alpha(a),\alpha(b)\} \le \alpha(z) \le \max\{\alpha(a),\alpha(b)\}\} = \{z \in \Omega: \max\{\alpha^c(a),\alpha^c(b)\} \ge \alpha^c(z) \ge \min\{\alpha^c(a),\alpha^c(b)\}\}$. So the *join space associated* to α^c is equal to the one associated to α.

Let $\Psi = \{\alpha_i, i \in I\}$ a family of qualitative linear fuzzy sets and let, $\forall i \in I$, $H_i = (\Omega, \sigma_i)$ the join space associated to α_i. Let "*" the hyperoperation such that to any pair (a, b) of elements of Ω associates $a*b = \cap_{i\in I}\, a\sigma_i b$. We have the following

Theorem 2.7 The pair $H=(\Omega, *)$ is a closed commutative hypergroup, called *associated to the family of the qualitative linear fuzzy sets* $\Psi = \{\alpha_i, i \in I\}$.

Proof Since $a*b = \cap_{i\in I}\, a\sigma_i b$, by definitions it follows that H is a closed and commutative hypergroupoid and since $b \in a*b$, $\forall a, b \in \Omega$, the reproducibility holds. Therefore, it remain to prove that * is associative. We have, $\forall a,b,c \in \Omega$, $(a*b)*c = \cap_{i\in I}\, (a*b)\sigma_i c$. Let $m_i = \min\{\alpha_i(a), \alpha_i(b), \alpha_i(c)\}$, $M_i = \max\{\alpha_i(a), \alpha_i(b), \alpha_i(c)\}$. Since $a*b \subseteq a\sigma_i b$, $\forall x \in a*b$, $\alpha_i(x)$ belongs to the closed interval with extremes $\alpha_i(a)$ and $\alpha_i(b)$. Then, we have $(a*b)\sigma_i c = \{z \in \Omega: m_i \le \alpha_i(z) \le M_i\}$ and so $(a*b)*c = \cap_{i\in I} \{z \in \Omega: m_i \le \alpha_i(z) \le M_i\}$. Similarly, we prove that $a*(b*c) = \cap_{i\in I} \{z \in \Omega: m_i \le \alpha_i(z) \le M_i\}$ and so the associative property holds.

Since a family of fuzzy sets $\Phi = \{\varphi_i, i \in I\}$ with universe Ω is also a family of qualitative linear fuzzy sets with universe Ω and with [0, 1] as set of judgements, the previous theorem is valid also if Ψ is a family of fuzzy sets. In particular, if Ψ is a fuzzy partition, the commutative hypergroup $H=(\Omega, *)$ generalizes the one considered in theorem 2.4 for the crisp partitions.

Now we consider the following problem: given a closed commutative hypergroup $H=(\Omega,*)$, we wish to find, if it exists, a fuzzy partition $\Phi = \{\varphi_i, i \in I\}$ such that $H=(\Omega, *)$ is the commutative hypergroup associated. For this aim, we introduce the following

Definition 2.8 We say that a hypergroup $H=(\Omega,*)$ is *fuzzy decomposable* if it is a closed commutative hypergroup and there exists a finite or countable family $\{H_i=(\Omega, \sigma_i)\}_{i\in I}$ of closed commutative hypergroups, called *fuzzy decomposition* of H, such that

(FD1) $\forall i \in I$, H_i is associated to a qualitative non-constant linear fuzzy set α_i;

(FD2) $\forall$ a, $b \in \Omega$, $a*b = \cap_{i\in I} a\sigma_i b$.

If $H=(\Omega, *)$ is a *fuzzy decomposable* hypergroup and $\{H_i=(\Omega, \sigma_i)\}_{i\in I}$ is a fuzzy decomposition of H, then, for any $i \in I$, there exists a qualitative linear non-constant fuzzy set $\alpha_i:\Omega \to S_i$ such that H_i is associated to α_i and so H_i is associated also to α_i^c.
Let v_i be a strong numerical evaluation of S_i. Then $\varphi_i = v_i \circ \alpha_i$ and $\varphi_i^c = v_i^c \circ \alpha_i^c$ are two non-constant fuzzy sets associated to H_i, and such that $\varphi_i^c = 1 - \varphi_i$. Moreover, $\forall i \in I$ and $\forall \lambda_i \in (0, 1]$ also $\lambda_i \varphi_i$ and $\lambda_i \varphi_i^c$ are non-constant fuzzy sets associated to H_i.
We say that a family of fuzzy sets $\Phi = \{\varphi_i, i \in I\}$ is associated to H if, $\forall i \in I$, φ_i is a non-constant fuzzy set associated to H_i.
We say that two fuzzy sets φ and ψ are similar if there exists a $\lambda \in (0, 1]$ such that $\psi=\lambda\varphi$ or $\psi=\lambda\varphi^c$. Moreover, we say that two family of fuzzy sets $\Phi = \{\varphi_i, i \in I\}$ and $\Psi = \{\psi_i, i \in I\}$ are similar if, $\forall i \in I$, φ_i and ψ_i are similar.
Now we consider the following problem: given a family of fuzzy sets $\Phi = \{\varphi_i, i \in I\}$ associated to a fuzzy decomposable hypergroup H to find a family of fuzzy sets $\Psi = \{\psi_i, i \in I\}$ similar to Φ and fuzzy partition of Ω. We consider the case in which I is finite. We prove the following

Theorem 2.9 Let $H=(\Omega, *)$ be a fuzzy decomposable hypergroup, and let $\Phi = \{\varphi_i, i \in I\}$ be a family of fuzzy sets associated to (H, v). Then there exists a fuzzy partition of Ω $\Psi=\{\psi_i, i \in I\}$ similar to Φ if and only if, $\exists k \in R$ and $\forall i \in I$ $\exists \beta_i \neq 0$ such that $\Sigma_{i\in I} \beta_i\varphi_i = k$.
Proof Let $\Phi = \{\varphi_i, i \in I\}$ be a family of fuzzy sets associated to H and suppose $\Psi = \{\psi_i, i \in I\}$ is a fuzzy partition similar to Φ. Then, $\forall i \in I$, there exists a $\lambda_i \in (0, 1]$ such that $\psi_i=\lambda_i\varphi_i$ or $\psi_i=\lambda_i\varphi_i^c$ and $\Sigma_{i\in I} \psi_i=1$.
Let $P=\{i \in I: \psi_i=\lambda_i\varphi_i\}$ and let $Q==\{i \in I: \psi_i=\lambda_i\varphi_i^c\}$.
We have $\Sigma_{i\in P} \lambda_i\varphi_i + \Sigma_{i\in Q} \lambda_i(1-\varphi_i) = 1$ and so $\Sigma_{i\in P} \lambda_i\varphi_i + \Sigma_{i\in Q} (-\lambda_i)\varphi_i = 1-\Sigma_{i\in Q} \lambda_i$. If we put $\beta_i = \lambda_i$ for $i \in P$, $\beta_i = -\lambda_i$ for $i \in Q$ and $1-\Sigma_{i\in Q} \lambda_i = k$, the first part of theorem is proved.

On the converse, suppose that $\exists k\in R$, $\forall i\in I$ $\exists\beta_i\neq 0$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = k$. Let $P=\{i\in I: \beta_i>0\}$ and let $Q=\{i\in I: \beta_i<0\}$. We have $\Sigma_{i\in P}\,\beta_i\varphi_i + \Sigma_{i\in Q}\,(-\beta_i)(1-\varphi_i) = k-\Sigma_{i\in Q}\,(-\beta_i)$. Let $h = k-\Sigma_{i\in Q}\,(-\beta_i)$. We have that $h>0$. If we put $\psi_i = \beta_i\varphi_i/h$ for $i\in P$, $\psi_i = -\beta_i(1-\varphi_i)/h$ for $i\in Q$ then we have $\Sigma_{i\in I}\,\psi_i=1$ and so $\{\psi_i, i\in I\}$ is a fuzzy partition similar to Φ.

If I and $\Omega=\{O_j\}_{j\in J}$ are finite, $|I|=c$, $|J|=n$, a family of fuzzy sets $\Phi = \{\varphi_i, i\in I\}$ is represented by a matrix A_Φ of type [c, n], with generic element $a_{ij}=\varphi_i(O_j)$, called *first matrix* of Φ. Denote by A_Φ^*, called *second matrix* of Φ, the matrix obtained by A_Φ by adding a vector row, denoted u, with n columns and with any element equal to 1. By previous theorem we have the following

Corollary 2.10 Let $\Phi = \{\varphi_i, i\in I\}$ be a family of non-constant fuzzy sets associated to H and let A_Φ^* be the second matrix of Φ. If there exists a fuzzy partition $\Psi = \{\psi_i, i\in I\}$ similar to Φ then the rank of A_Φ^* is not superior to $c=|I|$. On the converse, if the rank of A_Φ^* is not superior to c then there exists a $I^*\subseteq I$ and a fuzzy partition $\Psi^*=\{\psi_i^*, i\in I^*\subseteq I\}$ that is similar to the subfamily, of Φ, $\Phi^* = \{\varphi_i, i\in I^*\}$.

Proof In fact, if there exists a fuzzy partition of Ω $\Psi = \{\psi_i, i\in I\}$ similar to Φ, for the previous theorem $\exists k\in R$, $\forall i\in I$ $\exists\beta_i\neq 0$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = k$. If $k=0$ the φ_i, $i\in I$ are linearly dependent and, if $k\neq 0$, u is linearly dependent on the φ_i, $i\in I$. In both the cases the rank of A_Φ^* is not superior to c.

On the converse, if the the rank of A_Φ^* is not superior to c we have that $\forall i\in I$, $\exists\beta_i\in R$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = 0$ if the φ_i, $i\in I$, are linearly dependent and $\forall i\in I$, $\exists\beta_i\in R$ such that $\Sigma_{i\in I}\,\beta_i\varphi_i = u$ if the φ_i, $i\in I$, are linearly independent. In both the cases, let $I^*=\{i\in I: \beta_i\neq 0\}$. By theorem 2.9 we have that from the subfamily of fuzzy sets $\Phi^* =\{\varphi_i, i\in I^*\}$ we can find a family of fuzzy sets $\Psi^* =\{\psi_i, i\in I^*\}$ that is a fuzzy partition of Ω.

We can obtain, in the previous theorem, a fuzzy partition $\Psi^*=\{\psi_i^*, i\in I^*\subseteq I\}$ with I^* maximal. In fact, the rank of A_Φ^* is not superior to c if and only if the homogeneous system $\Sigma_{i\in I}\,x_i\varphi_i - x_{c+1}u$ has at least a not trivial solution. The set S of solutions is a vector space. If d is the dimension of S and $\{v_1, v_2,\ldots,v_d\}$ is a base of S, we can find a vector $v=\{\beta_1, \beta_2,\ldots,\beta_c, \beta_{c+1}\}\in S$ such that, $\forall j\in\{1, 2, \ldots,n+1\}$, β_j is null if and only if the component j of v_s is null for all $s\in\{1, 2,\ldots,d\}$. Since, $\forall w=\{w_1, w_2,\ldots,w_c, w_{c+1}\}\in\in S$ and $\forall j\in\{1, 2, \ldots,n+1\}$ if β_j is null then also w_j is null, the set $I_w^*=\{j\in I: w_j\neq 0\}$ is maximal for v. By theorem 2.9 by the subfamily of fuzzy sets $\Phi^* =\{\varphi_i, i\in I_v^*\}$ of Φ we obtain a maximal fuzzy partition $\Psi^* =\{\psi_i, i\in I_v^*\}$.

In the application to evaluation in Architecture and Town-Planning, we have a set Ω of objects to evaluate and a family $\Gamma=\{C_i, i\in I\}$ of classes. We suppose that, for any class C_i, we can find a total preorder relation ρ_i on Ω such that, $\forall x, y\in\Omega$, we have $x\rho_i y$ if and only if we think that the measure in which x belongs to C_i is not superior the one that y belongs to C_i. We suppose ρ_i is not trivial, that is that there are at least two elements a, b of Ω such that $a\rho_i b$ but not $b\rho_i a$. If for any $i\in I$, we put, $\forall a, b\in\Omega$, $a\sigma_i b = \{z\in\Omega$: ($a\rho_i z$ and $z\rho_i b$) or ($b\rho_i z$ and $z\rho_i a$)$\}$ and $a*b = \cap_{i\in I} a\sigma_i b$, we can prove that $H = (\Omega, *)$ is a commutative closed hypergroup and $\{H_i=(\Omega, \sigma_i), i\in I\}$ is a fuzzy decomposition of H. In fact, $\forall i\in I$, if we put, $\forall a, b\in\Omega$, $a \sim_i b$ if and only if ($a\rho_i b$ and $b\rho_i a$), $\sim_i$ is an equivalence relation on Ω. Let $E_i = \Omega/\sim_i$. Denote by $[a]_i$ the equivalence class of $a\in\Omega$. If we put, $\forall a, b\in\Omega$, $[a]_i<[b]_i \Leftrightarrow$ ($a\rho_i b$ and not $b\rho_i a$) we have a total order on E_i. Any element of E_i is a judgment about the truth of the proposition $P(x)$="the element $x\in\Omega$ belongs to C_i". If there is not an element of E_i that means "true" we add as "true" the symbol "T" and we assume $[a]_i<T$, $\forall a\in\Omega$. In an analogous way, if there is not an element of E_i that means "false" we add as "false" the symbol "F" and we assume $F<[a]_i<T$, $\forall a\in\Omega$. Let $S_i=E_i\cup\{V, F\}$. The function α_i: $x\in\Omega\rightarrow[a]_i\in S_i$ is a qualitative linear fuzzy set and $H_i=(\Omega,\sigma_i)$ is the associate closed commutative hypergroup. Then $\{H_i, i\in I\}$ is a fuzzy decomposition of H.

By theorem 2.9 and corollary 2.10 we can find, under suitable conditions, a fuzzy partition of Ω.

3. A REAL CASE OF STUDY

In this paragraph we describe the methodology we propose to classify an urban territory in homogeneous areas. This is a "multicriteria" procedure, since the clustering is made by considering a set of *criteria*.

We start our study by the law 431/98 that has recently pointed out the problem of a fairer regulation of the rents: it fixes new criteria to define the prices on extended metropolis and on towns with high density of population.

This act prescribes to establish the reference values of the rents in a municipal area on the basis of the quality of real estates, considering both the conditions of buildings and the conditions of services of the zones in which they are placed; sectors of fluctuation of the prices have to be singled out by dividing municipal areas in homogeneous zones, said *microzones*.

According to the aforesaid law the microzones, as sectors with similar peculiarities, are individuated by the municipality on the basis of the following elements:

1. *market price* of the area;
2. *infrastructures (transport, public parks and gardens, schools, health services, commercial equipments)*;
3. *kind of housing*, considering cadastral categories and classes;
4. *artistic quality* of the area;
5. presence of *urban decay*.

These parameters, or *criteria*, are useful also to define precise maximum and minimum values of prices within each sector. In order to assign the actual rent between this two aforesaid extreme values in each microzone the law prescribes to consider also the following elements: *typology of the house*; *maintenance state of the house and of the whole building*; *pertinences of the house* (box, car place, cellar, etc.); *presence of common spaces* (courtyard, open spaces and gardens, sport facilities, etc.); *technical fixtures and fittings* (lift, independent or centralized heating system, air conditioner, etc.); *possible equipment of forniture*.

The cities of Milan and Pescara conformed to the law, defining their microzones by observing peculiarities of the settlements. The first one has been divided in nine zones; each zone is subdivided in three strips: an economical sector, a middle sector and a luxury sector and each of these is characterized by a minimum value and a maximum value of the rent [11]. In the city of Pescara the present three cadastral zones, to which three different sectors of classification and cadastral rent correspond, are now replaced with ten new microzones [12].

As we have underlined in the paragraph 1, there are some perplexities about the trait of homogeneousness of the microzones: is it suitable the empirical clustering made in Milano and Pescara? Is it possible a so "hard" difference among peculiarities of near zones? We think the reality is more complex and that there is a gradual, soft passing from the traits of a certain zone to the traits of another zone. Really it seems that the microzones include elements with characteristics not to distinguish so sharply and that their frontiers have a fuzzy connotation: a building could belong even to more zones simultaneously. For this reason we deem it opportune the recourse to methods of statistical clustering that permit to obtain a subdivision of an urban territory in clusters having traits of major homogeneousness than clusters recently singled out in an empirical way. Moreover we think that the fuzzy classification is more suitable than the crisp one.

The aim of our study is the formulation of algorithms to define the grade in which a building, having defined peculiarities deduced by analyzing the context in which the building is located, belongs to various zones.

Our mathematical model of clustering is based on the attribution of *qualitative judgments* representing the grade of achievement of the set of the predefined criteria: this step corresponds to the formulation of a *qualitative matrix* in which the rows are the criteria, the columns are the objects and the element in the row i and column j is the judgment on the grade in which the object j achieves the criterion i.

Successively, through a suitable strong numerical evaluation $v: S \rightarrow [0,1]$, it is possible to parametrize the model, by substituting the qualitative judgments with numerical values deduced by analysing the urban context: this step corresponds to the formulation of a *quantitative matrix*.

At this step, for the formulation of the *clustering*, it is necessary a suitable algorithm, by fixing the number of classes and a *distance* among the objects.

In [7] we have formulated for the case of Pescara an algorithm of fuzzy classification obtained by considering a *distance* among the buildings based both on the geographical position (as urban metric) and on the difference among the grades of achievement of criteria fixed by the law.

This classification is represented by a matrix A = $[a_{ij}]$ such that $a_{ij} \in [0,1]$ represents the grade in which the building O_j belongs to the class C_i.
Each row of the matrix is a fuzzy set and it is possible to associate to this one a hypergroup of Corsini H_i. Moreover we can associate to the set of the rows of the matrix the commutative hypergroup considered in the theorem 2.7. Each block $a_1 * a_2 * \ldots * a_n$ of this hypergroup is the set of buildings having peculiarities, in particular rents and estimated incomes, constrained by the ones of $a_1, a_2, \ldots, a_n$.
By utilizing the theorem 2.9, the corollary 2.10 and the definition 2.8 we notice that an alternative criterion of clustering is obtained by fixing "a priori" the classes C_1, $C_2, \ldots, C_p$. For any class we assign a qualitative linear fuzzy set with universe the set Ω of the buildings. Finally with the methods considered in the previous paragraph we have a fuzzy classification on Ω.

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