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CLOSURE SYSTEMS AND CLOSURE HYPERGROUPS

Domenico Lenzi*

SUNTO - Dato un sistema di chiusura $(S,\mathfrak{C})$, sull'insieme S si può definire un'iperoperazione $\cdot$ ponendo, per ogni $a,b \in S$, $a \cdot b = \langle a,b \rangle_{\mathfrak{C}}$, dove $\langle a,b \rangle_{\mathfrak{C}}$ è il minimo elemento di $\mathfrak{C}$ a cui appartengono sia a che b.
In questo lavoro noi studiamo questo tipo di iperoperazione, evidenziando diverse proprietà significative. Nei casi in cui l'iperoperazione in questione è associativa essa attribuisce ad S una struttura di ipergruppo, che noi chiamiamo *ipergruppo chiusura*.

ABSTRACT - If $(S,\mathfrak{C})$ is a closure system, then one can define on the set S a hyperoperation $\cdot$ by setting, for any $a,b \in S$, $a \cdot b := \langle a,b \rangle_{\mathfrak{C}}$, where $\langle a,b \rangle_{\mathfrak{C}}$ is the minimum element of $\mathfrak{C}$ containig a and b.
In this paper we study such a type of hyperoperations and prove several interesting properties. Whenever the above hyperoperation is associative, then it gives S a hypergroup structure that we shall call *closure hypergroup*.

1. PRELIMINARIES AND RECALLS

A function $\cdot : S \times S \to \mathcal{P}(S)$ is said a partial (binary) hyperoperation on S. If $x \cdot y \neq \varnothing$ $(x \cdot y := \cdot(x,y))$ for any $x,y \in S$, then one speaks of hyperoperation. For any $X,Y \subseteq S$ one can set $X \cdot Y := \bigcup_{x \in X, y \in Y} x \cdot y$ (hence $\varnothing \cdot Y = \varnothing = X \cdot \varnothing$). Thus one has also a binary operation on $\mathcal{P}(S)$.
If $a \in S$ and $B \subseteq S$, then one usually writes respectively $a \cdot B$ and $B \cdot a$ instead of $\{a\} \cdot B$ and $B \cdot \{a\}$. It is obvious that $\bigcup_{x \in X} x \cdot Y = X \cdot Y = \bigcup_{y \in Y} X \cdot y$.
It is easy to verify that a partial hyperoperation on S is associative or commutative - with an obvious meaning of these terms - if and only if the corresponding operation on $\mathcal{P}(S)$ is associative or commutative.

* Dipartimento di Matematica dell'Università. 73100 Lecce (Italy).

Now we recall that a closure system on a set S is a subset $\mathfrak{C}$ of the power set $\mathcal{P}(S)$ which is closed under the arbitrary set intersection (in particular, $S = \cap\varnothing \in \mathfrak{C}$). One says also that $(S,\mathfrak{C})$ is a closure system.
For any $X \subseteq S$ one can consider the so called closure of X under $\mathfrak{C}$, given by the intersection of the elements of $\mathfrak{C}$ including X, and represented by $<X>_{\mathfrak{C}}$. If $x_1, x_2, \dots, x_n$ are elements of S, then one writes $<x_1, x_2, \dots, x_n>_{\mathfrak{C}}$ instead of $<\{x_1, x_2, \dots, x_n\}>_{\mathfrak{C}}$ and says that $<x_1, x_2, \dots, x_n>_{\mathfrak{C}}$ is finitely generated. The elements of $\mathfrak{C}$ of the type $<x>_{\mathfrak{C}}$ are said principal; moreover, if every element of $\mathfrak{C}$ is principal, then $(S,\mathfrak{C})$ is said principal.

2. BINARY CLOSURE SYSTEMS

Through a closure system $(S,\mathfrak{C})$ one can define a commutative (binary) hyperoperation $\cdot$ on S by setting, for any $a,b \in S$, $a\cdot b := <a,b>_{\mathfrak{C}}$. We shall say that $\cdot$ is a *(binary) closure hyperoperation*.

Remark 1. If $\cdot$ is the above hyperoperation, since $a\cdot b = <a,b>_{\mathfrak{C}}$ and $a\cdot b \subseteq <a>_{\mathfrak{C}}\cdot b \subseteq <a>_{\mathfrak{C}}\cdot<b>_{\mathfrak{C}} \subseteq <a,b>_{\mathfrak{C}}$ for any $a, b \in S$, then one has:

1) $a\cdot b = <a>_{\mathfrak{C}}\cdot b = a\cdot<b>_{\mathfrak{C}} = <a>_{\mathfrak{C}}\cdot<b>_{\mathfrak{C}}$.

Consequently, if $a \in S$ and $B \subseteq S$, then one has:

2) $a\cdot B = \cup_{b\in B}\, a\cdot b = \cup_{b\in B}<a>_{\mathfrak{C}}\cdot b = <a>_{\mathfrak{C}}\cdot B$ $(= B\cdot<a>_{\mathfrak{C}})$. In particular, if $\cdot$ is an associative hyperoperation and $x_1, x_2, \dots, x_n \in S$, with $n > 2$, then $x_1\cdot x_2\cdot \dots \cdot x_n = <x_1>_{\mathfrak{C}}\cdot<x_2>_{\mathfrak{C}}\cdot \dots \cdot<x_n>_{\mathfrak{C}}$.

Now, if $\mathfrak{C}$ is a closure system on S, let $\mathfrak{C}_2$ be the set of the parts X of S which are closed under the hyperoperation $\cdot$ associated to $\mathfrak{C}$ (i.e.: $X\cdot X \subseteq X$). Thus we shall say that the set $\mathfrak{C}_2$ is a *binary* (or *linear*) *closure system* on S and call *binary subspace* (or *pseudo-linear subspace*) of $(S,\mathfrak{C})$ every element of $\mathfrak{C}_2$. Obviously, $\varnothing$ is a binary subspace of $(S,\mathfrak{C})$; moreover $\mathfrak{C} \subseteq \mathfrak{C}_2$.
Now we recall that a closure system $(S,\mathfrak{C})$ is said algebraic if, for any subset X of S and for any $x \in <X>_{\mathfrak{C}}$, there is a finite subset F of X such that $x \in <F>_{\mathfrak{C}}$. It is known that $(S,\mathfrak{C})$ is algebraic if and only if $\mathfrak{C}$ is closed under the set union of the elements of any subset of $\mathfrak{C}$, which is upper directed (in particular, which is a chain) with respect to $\subseteq$. Therefore one can easily verify that $\mathfrak{C}_2$ is an algebraic closure system.
If $+$ is a partial hyperoperation on S, then it is clear that the set of the parts of S which are closed under $+$ is a binary closure system on S.

Remark 2. Let $(S,\mathfrak{C})$ and $(S,\mathfrak{C}')$ be closure systems. If $<x, y>_{\mathfrak{C}'} \subseteq <x, y>_{\mathfrak{C}}$ for any $x, y \in S$, then $\mathfrak{C}_2 \subseteq \mathfrak{C}'_2$. Consequently, if $\mathfrak{C} \subseteq \mathfrak{C}'$, then $\mathfrak{C}_2 \subseteq \mathfrak{C}'_2$.

Remark 3. 1) If $x,y \in S$, then $<x,y>_{\mathfrak{C}_2} \subseteq <x,y>_{\mathfrak{C}}$. Moreover by definition of $\mathfrak{C}_2$, since $x,y \in <x,y>_{\mathfrak{C}_2}$, one has $<x,y>_{\mathfrak{C}} \subseteq <x,y>_{\mathfrak{C}_2}$. As a consequence one gets:

(a) $<x, y>_{\mathfrak{C}} = <x, y>_{\mathfrak{C}_2}$ (in particular, $<x>_{\mathfrak{C}} = <x>_{\mathfrak{C}_2}$). Therefore $\mathfrak{C}$ and $\mathfrak{C}_2$ define the same binary closure hyperoperation, and hence $\mathfrak{C}_2 = (\mathfrak{C}_2)_2$.

(b) Since $\mathfrak{C}_2 = (\mathfrak{C}_2)_2$, $\mathfrak{C}$ is a binary closure system if and only if $\mathfrak{C} = \mathfrak{C}_2$.

2) $\mathfrak{C}_2$ is the lower binary closure system on S including $\mathfrak{C}$. Indeed, if $\mathfrak{C}'$ is a binary closure system and $\mathfrak{C} \subseteq \mathfrak{C}'$, then $\mathfrak{C}_2 \subseteq \mathfrak{C}'_2 = \mathfrak{C}'$.

3) If X is an element of $\mathfrak{C}_2$, then $X \subseteq X{\cdot}X$, and hence $X = X{\cdot}X$.

Theorem 4. If $(S,\mathfrak{C})$ is a binary closure system, let C be a fixed element of $\mathfrak{C}$, and let $\mathfrak{C}' = \{Y \in \mathfrak{C} \mid Y = \varnothing \text{ or } C \subseteq Y\}$. Then $(S,\mathfrak{C}')$ is a binary closure system.

Proof. It is obvious that $(S,\mathfrak{C}')$ is a closure system. Thus, by Remark 3 (see (b) of property 1)), it is sufficient to prove that $\mathfrak{C}'_2 \subseteq \mathfrak{C}'$.

To this end, let X be a non empty element of $\mathfrak{C}'_2$. Thus, for every $x, y \in X$, one has $C \subseteq <x, y>_{\mathfrak{C}'} \subseteq X$ and $<x, y>_{\mathfrak{C}} \subseteq <x, y>_{\mathfrak{C}'} \subseteq X$. Therefore $X \in \mathfrak{C}'$. ∎

2. PARA-NORMAL CLOSURE SYSTEMS

Now, in order to extend some interesting properties of the normal subgroups of a group, in this paragraph let us assume that $\mathfrak{N}$ is a <u>binary</u> closure system on S, $+$ is a hyperoperation on S and $\mathfrak{C}$ is the closure system of the subsets of S closed under $+$. Thus we shall indicate respectively with $\cdot_{\mathfrak{N}}$ and $\cdot_{\mathfrak{C}}$ the closure hyperoperations associated to $\mathfrak{N}$ and to $\mathfrak{C}$.

We shall say that $(S,\mathfrak{N})$ is *para-normal* with respect to $+$ whenever the following condition holds:

$$(°) \qquad \forall\, x,y \in S\colon <x>_{\mathfrak{N}} + <y>_{\mathfrak{N}} = <x, y>_{\mathfrak{N}} .$$

Hence, since if $x,y \in S$ we have $x+y \subseteq \langle x\rangle_{\mathfrak{N}} + \langle y\rangle_{\mathfrak{N}} = \langle x, y\rangle_{\mathfrak{N}} = x \cdot_{\mathfrak{N}} y$, then the following property holds:

$$(^{\circ\circ}) \qquad \forall\, x,y \in S\colon\ x+y \subseteq x \cdot_{\mathfrak{N}} y.$$

And now let us assume that the binary closure system $(S, \mathfrak{N})$ is para-normal with respect to the hyperoperation $+$. Then we have the following theorems.

Theorem 5. for any $A,B \in \mathfrak{N}$ one has:

$$(*) \qquad A+B = A \cdot_{\mathfrak{C}} B = A \cdot_{\mathfrak{N}} B.$$

Furthermore $\mathfrak{N}$ is included in $\mathfrak{C}$.

Proof. Preliminarily let us remark that $A+B = A \cdot_{\mathfrak{N}} B$. In fact the following equalities are trivial:

$$A+B = \bigcup_{a\in A, b\in B} \langle a\rangle_{\mathfrak{N}} + \langle b\rangle_{\mathfrak{N}} = \bigcup_{a\in A, b\in B} \langle a, b\rangle_{\mathfrak{N}} =$$

$$= \bigcup_{a\in A, b\in B} a \cdot_{\mathfrak{N}} b = A \cdot_{\mathfrak{N}} B.$$

In particular, one has $A+A = A \cdot_{\mathfrak{N}} A = A$. Thus $\mathfrak{N}$ is included in $\mathfrak{C}$ and hence $A \cdot_{\mathfrak{C}} B \subseteq A \cdot_{\mathfrak{N}} B$. As a consequence - since it is obvious that $A+B \subseteq A \cdot_{\mathfrak{C}} B$ - we get $A+B = A \cdot_{\mathfrak{C}} B = A \cdot_{\mathfrak{N}} B$. ∎

Theorem 6. Let $A \in \mathfrak{C}$ be a union of elements of $\mathfrak{N}$. Then $A \in \mathfrak{N}$ [1].

Proof. It is sufficient to prove that $A \cdot_{\mathfrak{N}} A \subseteq A$. Indeed one has:

$$A \cdot_{\mathfrak{N}} A \subseteq \bigcup_{a,a'\in A} \langle a\rangle_{\mathfrak{N}} \cdot_{\mathfrak{N}} \langle a'\rangle_{\mathfrak{N}} = \bigcup_{a,a'\in A} \langle a, a'\rangle_{\mathfrak{N}} =$$

$$= \bigcup_{a,a'\in A} \langle a\rangle_{\mathfrak{N}} + \langle a'\rangle_{\mathfrak{N}} \subseteq A+A \subseteq A. \qquad ∎$$

[1] See the case of a subgroup which is a union of normal subgroups.

Remark 7. Let us point out that if the binary closure system $(S, \mathcal{N})$ is para-normal with respect to an associative hyperoperation $+$ then, as in the case of normal subgroups of a group, also $\cdot_{\mathcal{N}}$ is associative. Indeed, by Remark 1 and by Theorem 5, for any $a, b, c \in S$ we have:

$$a\cdot_{\mathcal{N}}(b\cdot_{\mathcal{N}}c) = <a>_{\mathcal{N}}\cdot_{\mathcal{N}}(<b, c>_{\mathcal{N}}) = <a>_{\mathcal{N}}+(<b>_{\mathcal{N}}+<c>_{\mathcal{N}}) =$$

$$= (<a>_{\mathcal{N}}+<b>_{\mathcal{N}})+<c>_{\mathcal{N}} = (<a, b>_{\mathcal{N}})\cdot_{\mathcal{N}}<c>_{\mathcal{N}} = (a\cdot_{\mathcal{N}}b)\cdot_{\mathcal{N}}c.$$

In the meantime $\cdot_{\mathcal{C}}$ can be not associative, as in most groups $(S,+)$ in which $+$ is a non commutative operation.

3. ASSOCIATIVE CLOSURE SYSTEMS

Now let $\cdot$ be the hyperoperation associated to a given closure system $(S,\mathcal{C})$; hence, for any $x, y \in S$, x and y belong to the hyperproduct $x\cdot y$. As a consequence, if $\cdot$ is associative, then $\cdot$ gives S a structure of commutative hypergroup (in the sense of [1], p. 8). Therefore we shall say that $(S,\mathcal{C})$ is *associative* and $(S,\cdot)$ is a *closure hypergroup*.
Furthermore, we shall say that $(S,\mathcal{C})$ is 3-*strong associative* if, for any $x, y, z \in S$, $x\cdot(y\cdot z) = <x, y, z>_{\mathcal{C}}$ $(= z\cdot(x\cdot y) = (x\cdot y)\cdot z)$. In such a case, since $x\cdot(y\cdot z) \subseteq <x, y, z>_{\mathcal{C}_2} \subseteq <x,y, z>_{\mathcal{C}}$, then one has $<x, y, z>_{\mathcal{C}} = <x, y, z>_{\mathcal{C}_2}$.
More generally, given a natural numbers $n \geq 2$, we shall say that $(S,\mathcal{C})$ is n-*strong associative* if it is associative and $x_1\cdot x_2\cdot \dots \cdot x_n = <x_1, x_2, \dots, x_n>_{\mathcal{C}}$ for any $x_1, x_2, \dots, x_n \in S$. Furthermore, we shall say that $(S,\mathcal{C})$ is *finitely strong associative* if it is n-strong associative for any natural numbers $n \geq 2$.
Obviously, a closure system $(S,\mathcal{C})$ is 2-strong associative if and only if it is associative. Furthermore, if m and n are natural numbers such that $2 \leq m < n$ and $(S,\mathcal{C})$ is n-strong associative, then $(S,\mathcal{C})$ is m-strong associative. In fact, if one set $x_m = x_{m+1} = \dots = x_n$, then (see 1) of Remark 1) one gets [2]:

$$x_1\cdot x_2\cdot \dots \cdot x_{m-1}\cdot x_m = x_1\cdot x_2\cdot \dots \cdot x_{m-1}\cdot <x_m>_{\mathcal{C}} =$$

[2] We recall that $<x>_{\mathcal{C}}\cdot<x>_{\mathcal{C}} = <x>_{\mathcal{C}}$ for any $x \in S$.

$$= x_1 \cdot x_2 \cdot \ldots \cdot x_{m-1} \cdot <x_m>_{\mathfrak{C}} \cdot <x_{m+1}>_{\mathfrak{C}} \cdot \ldots \cdot <x_n>_{\mathfrak{C}} =$$

$$= x_1 \cdot x_2 \cdot \ldots \cdot x_m \cdot x_{m+1} \cdot \ldots \cdot x_n = <x_1, x_2, \ldots, x_n>_{\mathfrak{C}} =$$

$$= <x_1, x_2, \ldots, x_m>_{\mathfrak{C}} .$$

Theorem 8. Let n be a natural number greather than 1. Then a closure system $(S,\mathfrak{C})$ is n-strong associative if and only if the following property holds [3]:

(*) $\forall\, x_1, x_2, \ldots, x_n \in S: <x_1, x_2, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot <x_2, \ldots, x_n>_{\mathfrak{C}}$.

Proof. The assertion is obvious if $n = 2$ or $n = 3$. Thus let $n > 3$. If $(S,\mathfrak{C})$ is n-strong associative then, since it is also $(n-1)$-strong associative, we have:

$$<x_1, x_2, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot x_2 \cdot \ldots \cdot x_n = x_1 \cdot <x_2, \ldots, x_n>_{\mathfrak{C}} .$$

Conversely, let the condition (*) hold. Then it holds also with n replaced by a natural number m such that $2 < m < n$. In fact we can set $x_m = x_{m+1} = \ldots = x_n$. Thus, by setting $m = 3$, we have that $(S,\mathfrak{C})$ is associative.
As an immediate consequence, by induction, we get $<x_1, x_2, \ldots, x_n>_{\mathfrak{C}} = x_1 \cdot x_2 \cdot \ldots \cdot x_n$. ∎

Theorem 9. Let $(S,\mathfrak{C})$ be a closure system and let $a_1, a_2, \ldots, a_n, b \in S$. If $<a_2, \ldots, a_n>_{\mathfrak{C}} = <b>_{\mathfrak{C}}$, then $a_1 \cdot <a_2, \ldots, a_n>_{\mathfrak{C}} = <a_1, a_2, \ldots, a_n>_{\mathfrak{C}}$.

Proof. Indeed (see 1) of Remark 1), $a_1 \cdot <a_2, \ldots, a_n>_{\mathfrak{C}} = a_1 \cdot <b>_{\mathfrak{C}} = a_1 \cdot b = <a_1, b>_{\mathfrak{C}} = <a_1, a_2, \ldots, a_n>_{\mathfrak{C}}$; whence the thesis. ∎

Remark 10. If the finitely generated and non empty elements of a closure system $\mathfrak{C}$ are principal, then (by Theorem 6) the condition (*) of Theorem 8 is true for any natural number n, and hence $(S,\mathfrak{C})$ is finitely strong associative. In particular, the closure system $\mathfrak{I}$ of the ideals of a semilattice $(S,\cup)$ [4] is a finitely strong associative closure system. In fact one can immediately verify that, for any $x_1, x_2, \ldots, x_n \in S$, the ideal generated by $x_1, x_2, \ldots, x_n$ is equal to $<x_1+x_2+ \ldots +x_n>_{\mathfrak{I}}$.

[3] If $n = 2$ then, by 1) of Remark 1, property (*) is true even if $(S,\mathfrak{C})$ is non associative.

[4] A semilattice is a structure $(S,\cup)$, where $\cup$ is an idempotent, commutative and associative binary operation; an ideal is a subset B of S closed under $\cup$ such that, for any $x \in S$ and $x' \in B$, if $x \leq x'$ (i.e.: $x \cup x' = x'$), then $x \in B$.

Theorem 11. Let $(S, \mathfrak{C})$ be an algebraic and associative closure system. Then $\mathfrak{C}$ is finitely strong associative if and only if $\mathfrak{C} \cup \{\emptyset\} = \mathfrak{C}_2$.

Proof. Let $(S, \mathfrak{C})$ be finitely strong associative. Thus, since $\mathfrak{C} \cup \{\emptyset\} \subseteq \mathfrak{C}_2$, in order to prove that $\mathfrak{C} \cup \{\emptyset\} = \mathfrak{C}_2$ it is sufficient to verify that if X is a non empty element of $\mathfrak{C}_2$, then $<X>_{\mathfrak{C}} = X$ (hence X is also an element of $\mathfrak{C}$).

Since $X \subseteq <X>_{\mathfrak{C}}$, we only have to verify that $<X>_{\mathfrak{C}} \subseteq X$. Thus let $x' \in$ $\in <X>_{\mathfrak{C}}$ and (by the hypothesis that $(S, \mathfrak{C})$ is algebraic) let us consider x_1, ..., $x_n \in X$ such that $x' \in <x_1, \dots, x_n>_{\mathfrak{C}}$. As $\mathfrak{C}$ is finitely strong associative, $<x_1, \dots, x_n>_{\mathfrak{C}} = x_1 \cdot \dots \cdot x_n \subseteq X$, and hence $x' \in X$.

On the contrary, let $\mathfrak{C} \cup \{\emptyset\} = \mathfrak{C}_2$. Hence $\mathfrak{C}$ and $\mathfrak{C}_2$ determine the same hyperoperation $\cdot$; moreover $<X>_{\mathfrak{C}_2} = <X>_{\mathfrak{C}}$ for any non empty subset X of S, hence $\mathfrak{C}$ is finitely strong associative if and only if $\mathfrak{C}_2$ is finitely strong associative. Thus let us verify that if $x_1, x_2, \dots, x_n \in S$, then $x_1 \cdot x_2 \cdot \dots \cdot x_n = <x_1, x_2, \dots, x_n>_{\mathfrak{C}_2}$.

Indeed, since $x_1, x_2, \dots, x_n \in x_1 \cdot x_2 \cdot \dots \cdot x_n \subseteq <x_1, x_2, \dots, x_n>_{\mathfrak{C}_2}$ and $<x_1, x_2, \dots, x_n>_{\mathfrak{C}_2}$ is the minimum element of $\mathfrak{C}_2$ containing $\{x_1, x_2, \dots, x_n\}$, it is sufficient to point out that, by associativity and by commutativity, one has (cf. Remark 1):

$$(x_1 \cdot x_2 \cdot \dots \cdot x_n) \cdot (x_1 \cdot x_2 \cdot \dots \cdot x_n) = x_1 \cdot x_1 \cdot x_2 \cdot x_2 \cdot \dots \cdot x_n \cdot x_n =$$

$$= <x_1>_{\mathfrak{C}_2} \cdot <x_2>_{\mathfrak{C}_2} \cdot \dots \cdot <x_n>_{\mathfrak{C}_2} = x_1 \cdot x_2 \cdot \dots \cdot x_n. \qquad \blacksquare$$

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