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Ideals in a semihypergroup and Green's relations

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Abstract:

The concept of ideal in a right (left) semihypergroup is defined. Then some connections between ideals and the hyper versions of Green's relations are discussed.

1.Introduction

Marty in 1934[2] Introduced the notion of hypergroup.

We begin by recalling some definitions from [1].

A hyperopration of a non-empty set H, is a function from $H \times H$ into $P^*(H) = P(H)\backslash\{\emptyset\}$.

If "$*$" is a hyperoperation on H, then $(H, *)$ is called a hypergroupoid.

Let $(H, *)$ be a hypergroupoid and A, B two non-empty subsets of H, then $A * B$ is defined by

$$A * B = \bigcup_{a \in A, b \in B} a * b$$

By $x * A$, and $A * x$ we mean $\{x\} * A$ and $A * \{x\}$ respectively, for all $x \in H$, $\emptyset \neq A \subseteq H$.

2.Main results

Definition 2.1. Let $(H,*)$ be a hypergroupoid. Then H is said to be a right (left) semihypergroup (or r.s (l.s)) if

$$(x*y)*z \subseteq x*(y*z), \forall x,y,z \in H$$

$$(x*(y*z) \subseteq (x*y)*z, \forall x,y,z \in H).$$

An hypergroupoid is called a semihypergroup if it is both a left and a right semihypergroup.

Definition 2.2[2]. Let $(H,*)$be a semihyprgroup. Then H is called a hypergroup if $x*H = H*x = H$, for all $x \in H$.

Definition 2.3. Left $(H*)$ be a hypergroupoid and $A \in P^*(H)$. Then A is called

(i) a right ideal in H if

$$x \in A \Longrightarrow x*y \subseteq A, \forall y \in H$$

(ii) a left ideal in H if

$$x \in A \Longrightarrow y*x \subseteq A, \forall y \in H$$

(iii) an ideal in H if it is both a left and a right ideal in H.

Example 2.4. If H is a totally ordered set and the hyperoperation "$*$" on H is defined by

$$x * y = y * x = \begin{cases} \{z \in H : x \leq z\} & \text{if } \ y \leq x \\ \{z \in H : y \leq z\} & \text{if } \ x \leq y, \end{cases}$$

for all $x, y \in H$. Then we can show that $(H, *)$ is a semihypergroup. Infact if $x, y, z \in H$, and $w \in (x * y) * z$ are arbitrary, then we have $w \in a * z$, for some $a \in x * y$. If $x \leq y$, then $y \leq a$. Now we have two cases.

Cases 1: Let $z \geq a$. Then since $w \in a * z$, we have $w \geq z$. On the other hand, since $y \leq a$, we obtain that $x \leq y \leq z$. In other words $z \in y * z$ and $w \in x * z$. Now we get that $w \in x * z \subseteq x * (y * z)$.

Case 2: Let $a > z$. Then, since $w \in a * z$, we conclude that $w \geq a > z$. Now if $z \geq y$, then we have

$$w \geq a > z \geq y \geq x.$$

Hence $w \in x * z \subseteq x * z$ and $z \in y * z$. Thus

$$w \in x * z \subseteq x * (y * z).$$

If $y > z$, then we have

$$z < y \leq a \leq w, \text{ since } w \in a * z.$$

consequently

$w \in x * y$, since $y \leq w$ and $x \leq y \subseteq x * (y * z)$, since $y \in y * z$. Therefore $(x * y) * z \subseteq x * (y * z)$, if $x \leq y$. Now since $x * y = y * x$, we have $(x * y) * z \subseteq x * (y * z)$.

Note that, since $x * B = B * x$. For all $x, y, z \in H$. Thus $(H, *)$ is a semihypergroup. Now Let $A = \{x \in H : a \leq x\}$, where $a \in H$. Then we shall show that A is an ideal of H. To do this let $x \in A, y \in H$ and $z \in x * y$. Then if $x \leq y$, we have

$$z \geq y \geq x \geq a.$$

Hence $z \in A$. If $y \leq x$, we have

$$z \geq x \geq a.$$

That is $z \in A$. Consequently $x * y \subseteq A$.

Definition 2.5. Let $(H, *)$ be a hypergroupoid. For every $a \in H$ we define

$$\begin{aligned} aH &= (a * H) \cup \{a\}; \\ Ha &= (H * a) \cup \{a\}; \\ HaH &= ((H * a) * H) \cup Ha \cup aH. \end{aligned}$$

The hyper versions of Creen's relations are the equivalence relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and $\mathcal{K}$ defined for all $a, b \in H$ by

$$a\mathcal{R}b \Leftrightarrow aH = bH;$$

$$a\mathcal{L}b \Leftrightarrow Ha = Hb;$$

$$a\mathcal{I}b \Leftrightarrow HaH = HbH;$$

$$\mathcal{K} = \mathcal{L} \cap \mathcal{R}.$$

We shall also consider the relations $\leq (\mathcal{R})$, $\leq (\mathcal{L})$ and $\leq (\mathcal{I})$ defined for all $a, b \in H$ by

$$a \leq (\mathcal{R})b \Leftrightarrow aH \subseteq bH;$$

$$a \leq (\mathcal{L})b \Leftrightarrow Ha \subseteq Hb;$$

$$a \leq (\mathcal{I})b \Leftrightarrow HaH \subseteq HbH;$$

(See[1,page 29]).

Teorem 2.6. Let H be a r.s and $\emptyset \neq A \subseteq H$. Then A is a right ideal iff for every $x, y \in H$.

$$x \leq (\mathcal{R})y \text{ and } y \in A \Rightarrow x \in A. \tag{1}$$

Proof. Let H be a r.s, $A \in P^{*}(H)$ and (1) hold. Then for all $x \in A$ and $y \in H$, we will prove that $x * y \subseteq A$. To do this let $z \in x * y$, and $w \in zH$ are abitrary.Then $w = z$ or $w \in z * t$, for some $t \in H$. If $w = z$, then since $z \in x * y$ we have $w \in x * y$ and hence $w \in xH$. If $w \in z * t$, then since $z \in x * y$ we get that $w \in (x * y) * t \subseteq x * (y * t) \subseteq xH$. Therefore $zH \subseteq xH$. In other words $z \leq (\mathcal{R})x$.Hence $z \in A$. That is $x * y \subseteq A$.

Conversly, let A be a right ideal in H, $x \leq (\mathcal{R})y$ and $y \in A$. Then $yH \subseteq A$,since A is a right ideal. Hence

$$x \in xH \subseteq yH \subseteq A.$$

In other words $x \in A$.

Theorem 2.7. Let H be a r.s $a \in H$. Then aH is the smallest right ideals containing a. Right ideals of this form are called principal right ideals.

Proof. The proof is easy.

Theorem 2.8. If H is a r.s and $a, b \in H$, then the following are equivalent:

(1) $a \leq (\mathcal{R})b$

(2) $a \in bH$

(3) $b \in J \Rightarrow a \in J$ for all principal right ideals J in H,

(4) $b \in J \Rightarrow a \in J$ for all right ideals J in H.

Proof. Clearly (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (2). By Theoem 2.7 we have (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (4).

Corollary 2.9. If H is a r.s and $a, b \in H$, Then the following are equivalent:

(1) $a\mathcal{R}b$,

(2) $b \in aH$ and $a \in bH$,

(3) $a \in J \Leftrightarrow b \in J$ for all principal right ideals J in H,

(4) $a \in J \Leftrightarrow b \in J$ for all right ideals J in H.

Proof. The proof follows from Definition 2.5 and Theorem 2.8.

Definition 2.10. Let H be a hypergroupoid. A set F of functions on H is separating if, for all distinct x and y in H, there is an $f \in F$ with $f(x) \neq f(y)$.

Notation. Let H be a hypergroupoid $x \in H$. Then $\mathcal{R}$-class, $\mathcal{L}$ - class, $\mathcal{I}$-class and $\mathcal{K}$-class of x are denoted by $x_{\mathcal{R}}$, $x_{\mathcal{L}}$, $x_{\mathcal{I}}$, and $x_{\mathcal{K}}$ respectively.

Corollary 2.11. Let H be *a* r.s. Then the following are equivalent:

(1) the relation $\leq(\mathcal{R})$ is an order on H.

(2) $x_{\mathcal{R}} = x, \forall x \in H$,

(3) The set of all characteristic function of principal right ideals in H is separating,

(4) The set of all charactcristic functions of right ideals in H is separating .

Proof. Obviously $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$. Firstly we shall prove $(2) \Rightarrow (3)$.Let x and y be two distinct elements in H. Then by Corollary 2.9,we have $y \notin xH$ or $x \notin$.Hence $\chi_{xH}(x) \neq \chi_{xH}(y)$ or $\chi_{yH}(y) \neq \chi_{yH}(x)$.

It is now sufficient to show that $(4) \Rightarrow (2)$.Let $x \leq (\mathcal{R})y$ and $y \leq (\mathcal{R})x$. Then $x \in yH$ and $y \in xH$. From Corollary 2.9 we have

$$x \in J \Leftrightarrow y \in J \text{ for all right ideals } J.$$

Hence $x = y$, by (4). Clearly $\leq(\mathcal{R})$ is reflexive and transitive.

Remark 2.12. For a l.s H there are corresponding theorems and corollaries connecting the relation $\mathcal{L}$ with left ideals.Moreover for a semihypergroup H there are corresponding theorems and corollaries connecting the relation $\mathcal{I}$ with ideals, we shall summarise a few of these results in the next theorem.

Theorem 2.13 Let H be a semihypergroup and $a, b \in H$.Then

(1) $a\mathcal{I}b$ iff

$a \in J \Leftrightarrow b \in J$ for all ideals J in H,

(2) $a\mathcal{K}b$ iff

$a \in J \Leftrightarrow b \in J$

whenever J is a left ideal or a right ideal in H.

Theorem 2.14. Let H be a semihypergroup. Then H is a hypergroup iff $x \in x * H$ and $x_{\mathcal{K}} = H, \forall x \in H$.

Proof. The proof is easy.

Definition 2.15. Let H,H' be two hypergroupoid and $f : H \longrightarrow H'$ a function. Then F is called a homomorphism if

$$f(x * y) = f(x) * f(y).$$

Theoerem 2.16. Let H,H' be two hypergroupoids and $f : H \longrightarrow H'$ an onto homomorphism. Then f preserves the relations $\leq (\mathcal{R})$,$\leq (\mathcal{L})$ and $\leq (\mathcal{I})$.Moreover $f(x_{\mathcal{R}}) = (f(x))_{\mathcal{R}}, f(x_{\mathcal{L}}) = (f(x))_{\mathcal{L}}, f(x_{\mathcal{I}}) = (f(x))_{\mathcal{I}}$. and $f(x_{\mathcal{K}}) = (f(x))_{\mathcal{K}}$.

Proof. The proof is easy.

REFERENCES

[1]K.H. Hofman and P.S.Mostert, Elements of compact Semigroups (Charles E. Merrill, Columbus,OH,1966).

[2]F.Marty, Sur une generalization de la notion de group, Actes d 8me Congres des Mathematiciens Scandinaves. Stockholm (1934) 45-49.