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Linear predictor and autocorrelation for noisy and delayed digital signal

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Abstract

This paper deals with the association between the linear prediction and digital signal modeling and ends up with the suitable ways to predict the signal by considering a stationary signal y_n. The linear prediction of signal modeling based on the finite past and the solutions are arrived in a recursive manner. Further we analyzed the wiener filter along with spectral theorem and autocorrelation in terms ofpredictive analysis. This estimates the gap function along with delay and noise. The delayed signal'sproperties are analyzed like causal, stability and applied these into optimum filtering. Finally the predicted error is compared with linear predictor and Wiener filter. Then transfer function is applied to estimate the interval function and gap function along with delay.

Keywords: Linear Predictor, Weiner filter, gapped function, delay, autocorrelation, orthogonal.

2020 AMS subject classifications: 39 A10, 39 A45. [1]

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1 Introduction

Due to recent developments in Digital signal processing and communication technology, and its subfields of spectrum estimation, real-time adaptive signal processing and prediction algorithms comes into an attention of many researchers Chen et al. [2006], Dogariu et al. [2021] and Gland and Oudjane [2003]. The unified extension of this digital signal processing is developed in terms of analysis of geometrical point of view, linear estimator and applied various algorithms like Gram-Schmidt orthogonalizations, lattice realizations and so on. Further these concepts deals with the autoregressive extensions and singular autocorrelation matrices and their sinusoidal representations Mao et al. [2017]. This motivates us to proceed further with linear prediction Makhoul [1975] and Pituk [2004].

This paper is organized as follows. Section II focuses the linear prediction and digital signal modeling Welch et al. [2006]. Section III applies the autoregressive models into the prediction coefficients. In section IV Linear Predictions and Levinson's Formula are applied in the random input signal. Finally, section V concludes the paper.

2 Linear predictor

The linear prediction and digital signal modeling and ends up with the suitable ways to predict the signal by considering a stationary signal y_n. This rules the signal pattern as follows

$$S_{yy}(z) = \sigma_\epsilon^2 B(z)B(z^{-1})\epsilon_n \to (B(z)) \to y_n \tag{1}$$

Here $B(z)$ be any filter as bounded, ϵ_n be a sequence of noise term by spectral factoring theorem. Let $R_{yy}(k)$ be the autocorrelation of y_n:

$$R_{yy}(k) = E[y_n + ky_n]$$

This is used to predict the present value through the past values by using $Y_{n-1} = \{y_i, -\infty < i \leq n-1\}$. If $y_1(n) = y_{n-1}$, then the linear prediction is identified and compared with the optimum Wiener filtering and estimated the signal $y_1(n)$. Now we identify $Y_1(z) = z^{-1}Y(z)$ with the spectral value $B(z)$. Now define the optimum filter $H(z)$ as

$$H(z) = \frac{1}{\sigma_\epsilon^2 B(z)}[\frac{\sigma_\epsilon^2 B(z)B(z^{-1})}{B(z^{-1})}], \tag{2}$$

Here $B(z)$ be a causal and stable filter, and extend the causal and stale filter as $zB(z)$ is then

$$zB(z) = z(b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots)$$

The optimum filter $H(z)$ is then

$$H(z) = z[1 - \frac{1}{B(z)}] \tag{3}$$

$$y_n \to (z^{-1}) \longrightarrow^{y_1(n)} (H(z)) \to \hat{y}_{n/n-1}$$

This filter output is $y_1(n)$ and the consequent output is predicted by $y_{n/n-1}$. The predicted error is defined as ϵ_n. In the figure 1, the indicator line separates the

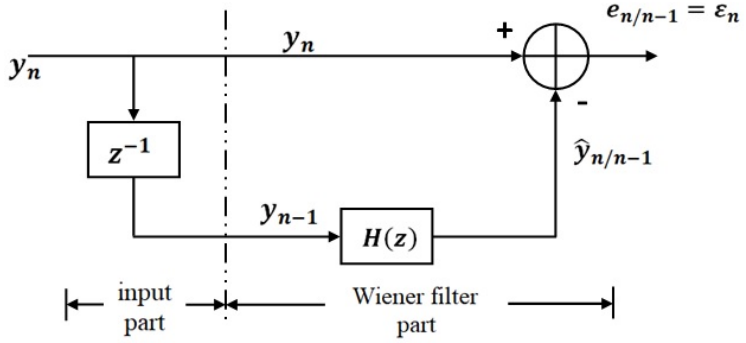

Figure 1: Error Predictor through Wiener Filter

linear predictor part and Wiener filter part in the signal error prediction [2, 7]. Apply the reduction equation (1) in terms of the predicted error filter $A(z)$ as,

$$S_{yy}(z) = \frac{\sigma_\epsilon^2}{A(z)A(z^{-1}} \tag{4}$$

and

$$A(z)S_{yy}(z) = \frac{\sigma_\epsilon^2}{A(z^{-1})} \tag{5}$$

this follows that

$$S_{\epsilon y}(z) = A(z)S_{yy}(z) \tag{6}$$

furthermore

$$R_{\epsilon y}(k) = E[\epsilon_n y_{n-k}] = \sum_{i=0}^{\infty} a_i R_{yy}(k-i) \tag{7}$$

which is recognized by the interval function [3]. Now construct ϵ_n from the orthogonal complement of $Y_{n-1} = y_{n-k}, k = 1, 2, \ldots$, and hence y_{n-k} is orthogonal to all $k = 1, 2, \ldots$. Therefore, the equation (7) implies

$$R_{\epsilon y}(k) = E[\epsilon_n y_{n-k}] = \sum_{i=0}^{\infty} a_i R_{yy}(k-i) = 0 \tag{8}$$

This result follows from the z-domain equation of (6) and interval function. Applying the symmetry property in (7) provided $k = 0$ and we get

$$\sigma_\epsilon^2 = E[\epsilon_n^2] = E[\epsilon_n y_n] = R_{yy}(0) + a_1 R_{yy}(1) + a_2 R_{yy}(2) + ... \tag{9}$$

Combined the equations (8) and (9),

$$\sum_{i=0}^{\infty} a_i R_{yy}(k-i) = \sigma_\epsilon^2 \delta(k), k \geq 0 \tag{10}$$

This normal equation is extended with the parameters $\{a_1, a_2, \ldots, \sigma_\epsilon^2\}$ based on the output signal? y_n and this is computed with $R_{yy}(k)$.

3 Autoregressive models

In general, the prediction coefficients are infinite since the predictor is predicated on the infinite past. When y_n is autoregressive, then the signal model $B(z)$ is defined as

$$B(z) = \frac{1}{(1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_p z^{-p})} \tag{11}$$

This shows that the prediction filter is polynomial

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_p z^{-p} \tag{12}$$

The output function y_n is defined for uncorrelated sequence ϵ_n, we get

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} + ... + a_p y_{n-p} = \epsilon_n \tag{13}$$

further optimum prediction of y_n is written like

$$\hat{y}_{n/n-1} = -[a_1 y_{n-1} + a_2 y_{n-2} + ... + a_p y_{n-p}] \tag{14}$$

Here most effective prediction of y_n is calculated based on the past p samples. The infinite set of equations (10) or (11) remains valid and the primary $p+1$ samples coefficients $\{1, a_1, a_2, \ldots, a_p\}$are nonzero [8, 14]. These primary past samples are a part of the equation (11) and these samples are enough to define the parameters of $\{a_1, a_2, \ldots, a_p; \sigma_\epsilon^2\}$:

$$\begin{bmatrix} R_{yy}(0) & R_{yy}(1) & \cdots & R_{yy}(p) \\ R_{yy}(1) & R_{yy}(0) & \cdots & R_{yy}(p-1) \\ R_{yy}(2) & R_{yy}(1) & \cdots & R_{yy}(p-2) \\ \vdots & \vdots & \vdots & \vdots \\ R_{yy}(p) & R_{yy}(p-1) & \cdots & R_{yy}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{15}$$

These equations are solved efficiently through Levinson's algorithm and this algorithm needs $O(p^2)$ operations and $O(p)$ memory locations. $O(p^3)$ and $O(p^2)$ which is necessary to calculate the inverse of the autocorrelation matrix R_{yy}. The parameters $\{a_1, a_2, \ldots, a_p; \sigma_\epsilon^2\}$ completely determines y_n. By considering $z = ej\omega$ in the equation (5) we determine

$$S_{yy}(\omega) = \frac{\sigma_\epsilon^2}{|A(\omega)|^2} = \frac{\sigma_\epsilon^2}{|1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega} + \ldots + a_p e^{-j\omega p}|} \tag{16}$$

The normal equations (16) build is used to approximate and estimates the parameters $\{a_1, a_2, \ldots, a_p; \sigma_\epsilon^2\}$. There are many various ways to extract the estimates and the parameters. Here are the few methods

1. Yule-Walker methodology
2. Variance methodology and
3. Burg's methodology.

Autocorrelations $R_{yy}(k)$ of equation (16) is wrriten based on the Yule-Walker methodology, is

$$R_{yy}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} y_{n+k} y_n \tag{17}$$

The primary $p+1$ changes are required in (16) as like $p \leq N-1$ based on the parameters $\{\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_p; \hat{\sigma}_\epsilon^2\}$. This represents the block of N samples and filter parameters (i.e. $p+1$). To synthesize the random samples, variance $\hat{\sigma}_\epsilon^2$ would be generated and pass through the generator filter whose coefficients are calculated like,

$$\hat{B}(Z) = \frac{1}{\hat{A}(Z)} = \frac{1}{|1 + \hat{a}_1 z^{-1} + \hat{a}_2 z^{-2} + +\hat{a}_p z^{-p}|^2} \tag{18}$$

4 Linear predictions and Levinson's Formula

In this section, we come accross that if the autoregressive random input signal is of order p, then the optimum linear predictor reduces to a predictor of order p. A geometrical method to perceive this property could be extended in to the projection of y_n onto the topological subspace based on the output signal $\{y_{n-i}, 1 \leq i < \infty\}$ and the same could be reduced based on past samples; i.e. $\{y_{n-i}, 1 \leq i \leq p\}$. This generates the output function y_n.

Consider a stationary series (based on time) y_n with the autocorrelation function $R(k) = E[y_{n+k}y_n]$. For any given p, the output function takes the following new form Consider a stationary series (based on time) y_n with the autocorrelation function $R(k) = E[y_{n+k}y_n]$. For any given p, the output function takes the following new form

$$\hat{y}_n = -[a_1 y_{n-1} + a_2 y_{n-2} + ... + a_p y_{n-p}] \tag{19}$$

The prediction coefficients are chosen to reduce the mean square error as

$$\varepsilon = E[e_n^2] \tag{20}$$

where e_n is the predicted error and define e_n as follows

$$e_n = y_n - \hat{y}_n = y_n + a_1 y_{n-1} + a_2 y_{n-2} + ... + a_p y_{n-p} \tag{21}$$

$$E[e_n y_{n-i}] = 0, \tag{22}$$

By substituting (21) in the equation (22), we get p linear equations

$$\sum_{j=0}^{p} a_j E[y_{n-j} y_{n-i}] = \sum_{j=0}^{p} R(i-j) a_j = 0 \tag{23}$$

By (22), we found the reduced value as

$$\sigma_\epsilon^2 = E[e_n y_n] \tag{24}$$

Equations (23) and (24) may be combined into the matrix equation like $(p+1) \times (p+1)$,

$$\begin{bmatrix} R_{yy}(0) & R_{yy}(1) & \cdots & R_{yy}(p) \\ R_{yy}(1) & R_{yy}(0) & \cdots & R_{yy}(p-1) \\ R_{yy}(2) & R_{yy}(1) & \cdots & R_{yy}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_{yy}(p) & R_{yy}(p-1) & \cdots & R_{yy}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_\epsilon^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{25}$$

which is identical for equation (16) for the autoregressive case.

It was necessary to connect the order of the predictor associate with the previous one. Hence the lower order optimum predictors also are calculated. Consider the gap function as

$$g_p(k) = E[(\sum_{i=0}^{p} a_{pi} y_{n-i}) y_{n-k}] = \sum_{i=0}^{p} a_{pi} R(k-i) \tag{26}$$

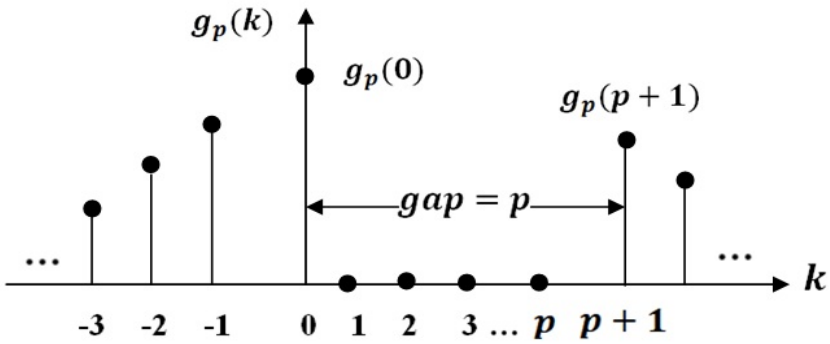

Figure 2: Gap conditions for the delay

These gap conditions are an equivalent because of the orthogonal equations (22) which is illustrated in figure 2. Utilizing $g_p(k)$ construct a new function with space $g_{p+1}(k)$ from the past $p+1$ hence we get, $g_p(k) \to g_p(-k)$. A delay of $(p+1)$ time can realigned and illustrated in the following figure. This shows the minimum of p and choosen the parameter γ_{p+1} and $g_{p+1}(k)$ adds an additional delay which deviates the length $p+1$ are illustrated in the figure 3.

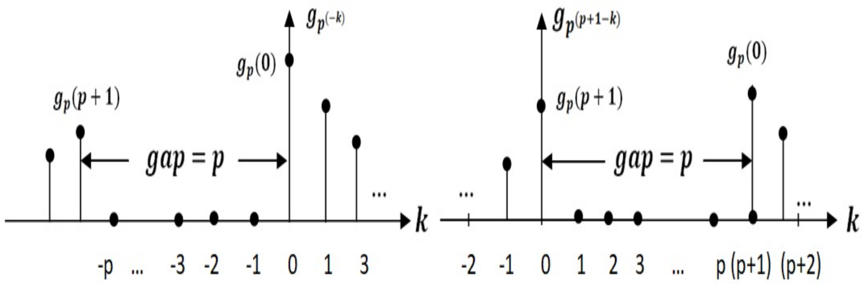

Figure 3: Gap (delay) conditions for $g_p(k)$ and $g_p(p+1-k)$

5 Conclusions

In this paper, linear prediction is predicted the present value through the past values. The linear prediction of signal modeling related to finite past and the solutions are arrived in a recursive manner. Further we analyzed the wiener filter along with spectral theorem and autocorrelation in terms of predictive analysis. This estimates the gap function along with delay and noise. There will be an indicator line which separates the linear predictor part and Wiener filter part in the signal error prediction. This normal equation is extended with the signal parameters based on the output signal y_n and this is computed with $R_{yy}(k)$. Then the infinite matrix equation is reduced to a finite form and, moreover, the $R_{yy}(k)$ is obviously measurable. Finally the predicted error is compared with linear predictor and Wiener filter. Then transfer function is applied to estimate the interval function and gap function along with delay. Finally the gapped function $g_p(k)$ and $g_p(p+1-k)$ possess the same value.

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