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Superior eccentric domination polynomial

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Abstract

Superior distance involves the path which travels through the closed neighbourhood of both the vertices and the shortest path between them. This unique distance led to the advent of superior dominating sets and superior eccentric dominating sets. The former has a superior neighbourin its compliment for every vertex in itself and the latter has a superior eccentric vertex in itself for every vertex in its compliment. The domination polynomials disuss the idea of total number of dominating sets and dominating sets of specific cardinality. This inspired us to conceptualise the idea of superior eccentric domination polynomial.In this paper, we introduce the superior eccentric domination polynomial $SED(G, \phi) = \sum_{l=\gamma_{sed}(G)}^{\beta} |sed(G, l)|\phi^l$ where $|sed(G, l)|$ is the number of all distinct superior eccentric dominating sets with cardinality l and $\gamma_{sed}(G)$ is superior eccentric domination number. We find SED(G,) for family of wheel graphs and different standard graphs. The correlation between the coefficients of different SED polynomials are stated and proved. The motivation for this paper is to find a domination polynomial using distance concept in graphs. Eccentricity is a distance and eccentric dominating set was already existing.

Keywords: Superior distance, superior eccentricity, superior eccentric domination polynomial.
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1 Introduction

The shortest path between any two vertices is known as geodesic. The concept of distance in graphs always yields to cater the needs of applications in technology. There are many variants of distances in graphs. The shortest, longest, path involving degree of vertices and chords. Kathiresan et al. [2007] introduced the superior distance in graphs. Let the path $D_{pq} = N[p] \cup N[q]$. The shortest superior distance between p to q is $d_D(p, q)$. Superior eccentricity of $e_D(p) = max\{d_D(q, p) : p, q \in V\}$ Superior neighbour of p is $d_D(p) = min\{d_D(p, q) : q \in V - \{p\}\}$. A vertex p(q) is a superior neighbour of p if $d_D(p, q) = d_D(p)$. The superior distance involves the shortest path between two vertices which travels through all their closed neighbourhoods.

Using the superior distance the same authors Kathiresan and Marimuthu [2008] introduced the superior domination(SD-set).A set $\subseteq V$ is called a SD-set if every vertex in S has superior neighbour in S-D. The SD-number is the cardinality of the minimum SD-set, denoted by $\gamma_{Sed}(G)$. The superior eccentric vertex p is given by $d_D\ (p, q) = e_D(p)$. Here the adjacency between the vertices in S and its compliment is not mandatory. Bhanumathi and Abhirami [2017] introduced the superior eccentric domination(SED-set) in graphs. A set $\subseteq V$ is an SED-set if every vertex in S-Dhas a superior eccentric vertex in S. The SED-number is the cardinality of the minimum SED-set, denoted by $\gamma_{Sed}(G)$. Along with being a superior dominating set if the same set has a superior eccentric vertex in itself for every vertex in the compliment of S then it becomes a superior eccentric dominating set. These two conditions play a vital role in the formation of a SED-set. Alikhani and Peng [2009] conceptualized the idea of domination polynomial, a domination polynomial consists of a coefficients which gives the number of dominating sets and the power of the variable denotes the cardinality of the dominating set which varies between one and the vertex cardinality of graph. They discussed and proved certain properties which speaks of the corelation between the dominating sets.

The motivation for this paper is to find a domination polynomial using distance concept in graphs. Eccentricity is a distance and eccentric dominating set was already existing. Inspired by this work Ismayil and Tejaskumar [2020] introduced the eccentric domination polynomial. The eccentric dominating polynomial gives the idea about the number of eccentric dominating sets with different cardinality and the symmetry in the coefficients of their polynomials werediscussed and proved. Superior distance, superior eccentric domination existed but there was a gap in the literature we did not have a formula which could find the total number of SED in a graph or a SED of specific cardinality, henceforth the same authors Ismayil and Tejaskumar extended the idea of domination polynomial to superior eccentric domination polynomial. In this paper, we discuss the concept of superior eccentric domination polynomial with an apt example, this concept was mainly introduced to find the total number of SED-sets of any graph. We found the formula which yields a SED polynomial for the family of wheel graphswhich helps us to easily find the total number of SED-sets of any cardinality at a given point of time for a wheel graph.We obtain the formulas and discuss the corelation between the coefficients of different SED polynomials. We tabulate the SED polynomials and their roots of different standard graphs. For all the undefined terminologies refer the book Graph Theory by Harary [2001].

2 Superior eccentric domination polynomial

Definition 2.1. *The superior eccentric domination polynomial* $SED(G,\phi) = \sum_{l=\gamma_{sed}(G)}^{\beta} |sed(G,l)|\phi^l$ *where* $|sed(G,l)|$ *is the number of distinct superior eccentric dominating sets (SED-sets) with cardinality* l, $\beta \in \mathbb{N}$ *and* $\gamma_{sed}(G)$ *is superior eccentric domination number.*

Example 2.1. .

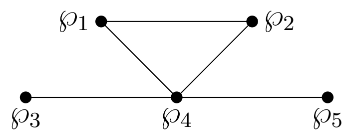


Figure 1: Cricket graph

Vertex	Superior eccentricity	Superior eccentric vertex $e_D(\wp)$
$\wp_1$	2	$\wp_2$
$\wp_2$	2	$\wp_1$
$\wp_3$	2	$\wp_5$
$\wp_4$	6	$\wp_1, \wp_2, \wp_3, \wp_5$
$\wp_5$	2	$\wp_3$

Here we see the cricket graph has a SED-set $\{\wp_4\}$ *of cardinality 1,* $\{\wp_1, \wp_4\}$, $\{\wp_2, \wp_4\}$, $\{\wp_3, \wp_4\}$, $\{\wp_4, \wp_5\}$ *SED sets of cardinality 2,* $\{\wp_1, \wp_2, \wp_4\}$, $\{\wp_1, \wp_3, \wp_4\}$, $\{\wp_1, \wp_4, \wp_5\}$, $\{\wp_2, \wp_3, \wp_4\}$, $\{\wp_2, \wp_4, \wp_5\}$, $\{\wp_3, \wp_4, \wp_5\}$ *SED sets of cardinality 3,* $\{\wp_1, \wp_2, \wp_3, \wp_4\}$, $\{\wp_1, \wp_2, \wp_4, \wp_5\}$, $\{\wp_1, \wp_3, \wp_4, \wp_5\}$, $\{\wp_2, \wp_3, \wp_4, \wp_5\}$ *SED sets of cardinality 4 and* $\{\wp_1, \wp_2, \wp_3, \wp_4, \wp_5\}$ *SED sets of cardinality 5.*
Therefore $SED(G,\phi) = \phi^5 + 4\phi^4 + 6\phi^3 + 4\phi^2 + \phi$.

3 Superior eccentric domination polynomial of wheel graph

Definition 3.1. *Superior eccentric domination polynomial of a wheel graph* W_β *is given by* $SED(W_\beta, \phi) = \sum_{l=\gamma_{sed}(W_\beta)}^{\beta} |sed(W_\beta, l)|\phi,^l$ *where* $|sed(W_\beta, l)|$ *is the number of distinct SED-sets with cardinality* l *and* $\gamma_{sed}(W_\beta)$ *is SED-number of wheel.*

Observation 3.1. .

1. $SED(W_\beta, \phi) = (1 + \phi)^\beta - 1$, *for* $\beta = 4, 5$.
2. $SED(W_\beta, \phi) = (1 + \phi)^\beta$, *for* $\beta = 6$.

Theorem 3.1. *For a wheel graph* W_β *of order* β,

1. $|sed(W_\beta, l)| = |sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|$ *where* $l \leq \beta$ *and* $\beta \geq 7$.
2. $SED(W_\beta, \phi) = \phi\ SED(W_{\beta-1}, \phi) + SED(W_{\beta-1}, \phi)$.
3. $SED(S_\beta, \phi) = \phi\,(\phi + 1)^{\beta-1}$, *for all* $\beta \geq 7$.

Proof: Let $V(W_\beta) = \{\wp_1, \wp_2, \ldots \wp_\beta\}$.

1. Since $|sed(W_\beta, l)| =^{\beta-1} C_{l-1}$, $|sed(W_\beta, l-1)| =^{\beta-2} C_{l-2}$ and $|sed(W_{\beta-1}, l)| =^{\beta-2} C_{l-1}$. But $^{\beta-1}C_{l-1} =^{\beta-2} C_{l-2} +^{\beta-2} C_{l-1}$.
   Therefore $|sed(W_\beta, l)| = |sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|$.

2. By theorem-$wheelTHM01-(1)$ we have $|sed(W_\beta, l)| = |sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|$.
   When $l = 1$,

$$\begin{aligned} |sed(W_\beta, 1)| &= |sed(W_{\beta-1}, 0)| + |sed(W_{\beta-1}, 1)| \\ \implies \phi, |sed(W_\beta, 1)| &= \phi, |sed(W_{\beta-1}, 0)| + \phi, |sed(W_{\beta-1}, 1)|. \end{aligned}$$

When $l = 2$,

$$\begin{aligned} |sed(W_\beta, 2)| &= |sed(W_{\beta-1}, 1)| + |sed(W_{\beta-1}, 2)| \\ \implies \phi,^2 |sed(W_\beta, 2)| &= \phi,^2 |sed(W_{\beta-1}, 1)| + \phi,^2 |sed(W_{\beta-1}, 2)|. \end{aligned}$$

When $l = 3$,

$$\begin{aligned} |sed(W_\beta, 3)| &= |sed(W_{\beta-1}, 2)| + |sed(W_{\beta-1}, 3)| \\ \implies \phi^3 |sed(W_\beta, 3)| &= \phi^3 |sed(W_{\beta-2}, 1)| + \phi^3 |sed(W_{\beta-1}, 3)|. \end{aligned}$$

When $l = 4$,

$$\begin{aligned} |sed(W_\beta, 4)| &= |sed(W_{\beta-1}, 3)| + |sed(W_{\beta-1}, 4)| \\ \implies \phi^4|sed(W_\beta, 4)| &= \phi^4|sed(W_{\beta-1}, 3)| + \phi^4|sed(W_{\beta-1}, 4)|. \end{aligned}$$

$\vdots$

When $l = \beta - 1$,

$$\begin{gathered} |sed(W_\beta, \beta-1)| = |sed(W_{\beta-1}, \beta-2)| \\ +|sed(W_{\beta-1}, \beta-1)| \\ \implies \phi^\beta - 1|sed(W_\beta, \beta-1)| = \phi^\beta - 1|sed(W_{\beta-1}, \beta-2) \\ +\phi^\beta - 1|sed(W_{\beta-1}, \beta-1)|. \end{gathered}$$

When $l = \beta$,

$$\begin{gathered} |sed(W_\beta, \beta)| = |sed(W_{\beta-1}, \beta-1)| + |sed(W_{\beta-1}, \beta)| \\ \implies \phi^\beta|sed(W_\beta, \beta)| = \phi^\beta|sed(W_{\beta-1}, \beta-1)| + \phi^\beta|sed(W_{\beta-1}, \beta)|. \end{gathered}$$

Therefore $\phi\,|sed(W_\beta, 1)| + \phi^2|sed(W_\beta, 2)| + \phi^3|sed(W_\beta, 3)| + \phi^4|sed(W_\beta, 4)| + \cdots + \phi^{\beta-1}|sed(W_\beta, \beta-1)| + \phi^\beta|sed(W_\beta, \beta)|$
$= \phi\,|sed(W_{\beta-1}, 0)| + \phi\,|sed(W_{\beta-1}, 1)| + \phi^2|sed(W_{\beta-1}, 1)| + \phi^2|sed(W_{\beta-1}, 2)| + \phi^3|sed(W_{\beta-2}, 1)| + \phi^3|sed(W_{\beta-1}, 3)| + \phi^4|sed(W_{\beta-1}, 3)| + \phi^4|sed(W_{\beta-1}, 4)|$
$+ \cdots + \phi^{\beta-1}|sed(W_{\beta-1}, \beta-2)|$
$+ \phi^{\beta-1}|sed(W_{\beta-1}, \beta-1)| + \phi^\beta|sed(W_{\beta-1}, \beta-1)| + \phi^\beta|sed(W_{\beta-1}, \beta)|.$
$= \phi\,|sed(W_{\beta-1}, 0)| + \phi^2|sed(W_{\beta-1}, 1)| + \phi^3|sed(W_{\beta-2}, 1)| + \phi^4|sed(W_{\beta-1}, 3)|$
$+ \cdots + \phi^{\beta-1}|sed(W_{\beta-1}, \beta-2)| + \phi^\beta|sed(W_{\beta-1}, \beta-1)| + \phi\,|sed(W_{\beta-1}, 1)|$
$+ \phi^2|sed(W_{\beta-1}, 2)| + \phi^3|sed(W_{\beta-1}, 3)| + \phi^4|sed(W_{\beta-1}, 4)| + \ldots$
$+ \phi^{\beta-1}|sed(W_{\beta-1}, \beta-1)| + \phi^\beta|sed(W_{\beta-1}, \beta)|$
$= \phi\,[\phi\,|sed(W_{\beta-1}, 1)| + \phi^2|sed(W_{\beta-1}, 2)| + \phi^3|sed(W_{\beta-1}, 3)|$
$+ \phi^4|sed(W_{\beta-1}, 4)| + \cdots + \phi^{\beta-1}|sed(W_{\beta-1}, \beta-1)|] + \phi\,|sed(W_{\beta-1}, 1)|$
$+ \phi^2|sed(W_{\beta-1}, 2)| + \phi^3|sed(W_{\beta-1}, 3)| + \phi^4|sed(W_{\beta-1}, 4)|$
$+ \cdots + \phi^{\beta-1}|sed(W_{\beta-1}, \beta-1)|$
Since $|sed(W_{\beta-1}, 0)| = |sed(W_{\beta-1}, \beta)| = 0.$
$= \phi \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)|\phi^l + \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)|\phi^l.$
$SED(W_\beta, \phi) = \phi\ SED(W_{\beta-1}, \phi) + SED(W_{\beta-1}, \phi).$

1. By mathematical induction (MI).

It is true for $\beta = 7$.

$$\begin{aligned} SED(W_{\beta-1}, \phi) &= \phi(\phi+1)^{7-1} \\ &= \phi(\phi+1)^{6} \\ &= \phi(\phi+1)^{3}(\phi+1)^{3} \\ &= \phi^{7} + 6\phi^{6} + 15\phi^{5} + 20\phi^{4} + 15\phi^{3} + 6\phi^{2} + 1. \end{aligned}$$

Assume it is true $\forall$ $\mathbb{N}$ less than $'\beta'$.
$SED(W_\beta, \phi) = \phi(1+\phi)^{(\beta-1)-1} = \phi(1+\phi)^{\beta-2}$

$$\begin{aligned} For'\beta',\ SED(W_\beta, \phi) &= \phi\ SED(W_{\beta-1}, \phi) + SED(W_{\beta-1}, \phi) \\ using theorem - 3.1 - (2) & \\ &= \phi[\phi(\phi+1)^{\beta-2}] + \phi(\phi+1)^{\beta-2} \\ &= \phi(\phi+1)^{\beta-1} \end{aligned}$$

$\therefore$ Proved $\forall\, '\beta'$.

Table: $|sed(W_\beta, l)|$ is the number of superior eccentric dominating sets of W_β with cardinality l where $1 \leq l \leq 15$.

β \ l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0														
2	0	0													
3	0	0	0												
4	4	6	4	1											
5	5	10	10	5	1										
6	1	5	10	10	5	1									
7	1	6	15	20	15	6	1								
8	1	7	21	35	35	21	7	1							
9	1	8	28	56	70	56	28	8	1						
10	1	9	36	84	126	126	84	36	9	1					
11	1	10	45	120	210	252	210	120	45	10	1				
12	1	11	55	165	330	462	462	330	165	55	11	1			
13	1	12	66	220	495	792	924	792	495	220	66	12	1		
14	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1	
15	1	14	91	364	1001	2002	3003	3423	3003	2002	1001	364	91	14	1

Theorem 3.2. *The following properties for the co-efficients of $SED(W_\beta, \phi)$ holds.*

1. $|sed(W_\beta, 1)| = 1$ *for all* $\beta \geq 6$.

2. $|sed(W_\beta, \beta)| = 1$, $\forall\ \beta \geq 4$.

3. $|sed(W_\beta, \beta - 1)| = \beta - 1$, $\forall\ \beta \geq 6$.

4. $|sed(W_\beta, \beta-2)| = \frac{(\beta-1)(\beta-2)}{2}$, $\forall\ \beta \geq 6$.

5. $|sed(W_\beta, \beta-3)| = \frac{(\beta-1)(\beta-2)(\beta-3)}{6}$, $\forall\ \beta \geq 6$.

6. $|sed(W_\beta, \beta-4)| = \frac{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}{24}$, $\forall\ \beta \geq 6$.

7. $|sed(W_\beta, l)| = |sed(W_\beta, \beta-l+1)|$, $\forall\ \beta \geq 6$.

8. *If* $SED_\beta = \sum_{l=1}^{\beta} |sed(W_\beta, l)|$, $\forall\ \beta \geq 6$, *then* $SED_\beta = 2(SED_{\beta-1})$, $\forall\ \beta \geq 7$.

9. SED_β =*Total number of SED-sets in* $W_\beta = 2^{\beta-1}$, $\forall\ \beta \geq 6$.

Proof:

1. Let $V(W_\beta) = \{\wp_1, \wp_2, \wp_3, \ldots \wp_\beta\}$. In a wheel graph W_β all the vertices form a superior neighbour of central vertex $\wp_1$ except itself. Therefore the only set with single cardinality $D = \{\wp_1\}$ forms the superior eccentric dominating set of wheel graph W_β where $\beta \geq 6$.

2. The vertex set $V(W_\beta)$ forms the superior eccentric dominating set $|sed(W_\beta, \beta)| = 1$ for all $\beta \geq 4$.

3. By MI on $'\beta'$.
For $\beta = 6$, $|sed(W_6, 6-1)| = |sed(W_6, 5)| = 5$.
Assume it is true $\forall\ \mathbb{N}$ less than $'\beta'$.
For $'\beta'$, By theorem-3.1-(2) and 3.2-(2)

$$\begin{aligned}
|sed(W_\beta, \beta-1)| &= |sed(W_{\beta-1}, \beta-2)| + |sed(W_{\beta-1}, \beta-1)| \\
&= (\beta-2) + 1 \\
&= \beta - 1
\end{aligned}$$

$\therefore$ Proved $\forall\ '\beta'$.

4. By MI on $'\beta'$.
For $\beta = 6$, $|sed(W_6, 6-2)| = |sed(W_6, 4)| = 10$.
For $\beta = 7$, $|sed(W_7, 7-2)| = |sed(W_7, 5)| = 15$.
Assume it is true $\forall\ \mathbb{N}$ less than $'\beta'$.

For $'\beta'$, By theorem-3.1 and 3.2-(3)

$$\begin{aligned}
|sed(W_\beta, \beta-2)| &= |sed(W_{\beta-1}, \beta-3)| + |sed(W_{\beta-1}, \beta-2)| \\
&= \frac{(\beta-2)(\beta-3)}{2} + (\beta-2) \\
&= \frac{(\beta-2)(\beta-3)+2(\beta-2)}{2} \\
&= \frac{(\beta-2)(\beta-3+2)}{2} \\
&= \frac{(\beta-2)(\beta-1)}{2}
\end{aligned}$$

$\therefore$ Proved $\forall$ $'\beta'$.

5. By MI on $'\beta'$.
   For $\beta = 6$, $|sed(W_6, 6-3)| = |sed(W_6, 3)| = 10$.
   For $\beta = 7$, $|sed(W_7, 7-3)| = |sed(W_7, 4)| = 20$.
   Assume it is true $\forall$ $\mathbb{N}$ less than $'\beta'$.
   For $'\beta'$, By theorem-3.1 and 3.2-(4)

$$\begin{aligned}
|sed(W_\beta, \beta-3)| &= |sed(W_{\beta-1}, \beta-4)| + |sed(W_{\beta-1}, \beta-3)| \\
&= \frac{(\beta-2)(\beta-3)(\beta-4)}{6} + \frac{(\beta-2)(\beta-3)}{2} \\
&= \frac{(\beta-2)(\beta-3)(\beta-4+3)}{6} \\
&= \frac{(\beta-1)(\beta-2)(\beta-3)}{6}
\end{aligned}$$

$\therefore$ Proved $\forall$ $'\beta'$.

6. By MI on $'\beta'$.
   The result is true for $\beta = 6$, $|sed(W_6, 6-4)| = |sed(W_6, 2)| = 5$.
   For $\beta = 7$, $|sed(W_7, 7-4)| = |sed(W_7, 3)| = 15$.
   Assume it is true $\forall$, $\mathbb{N} < \beta$.

For $'\beta'$, By theorem-3.1 and 3.2-(5)

$$
\begin{aligned}
|sed(W_\beta, n-4)| &= |sed(W_{\beta-1}, \beta-5)| + |sed(W_{\beta-1}, \beta-4)| \\
&= \frac{(\beta-2)(\beta-3)(\beta-4)(\beta-5)}{24} \\
&+ \frac{(\beta-2)(\beta-3)(\beta-4)}{6} \\
&= \frac{(\beta-2)(\beta-3)(\beta-4)(\beta-5+4)}{24} \\
&= \frac{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}{24}
\end{aligned}
$$

$\therefore$ Proved $\forall\ '\beta'$.

7. By MI on $'\beta'$. The result is true for $\beta = 6$.
   $|sed(W_6, 2)| = |sed(W_6, 6-2+1)| = |sed(W_6, 5)| = 5$
   $|sed(W_7, 3)| = |sed(W_7, 7-3+1)| = |sed(W_7, 4)| = 20$.
   Assume it is true $\forall\ \mathbb{N}$ less than $'\beta'$. For $'\beta'$, by theorem-3.1 we have

$$
\begin{aligned}
|sed(W_\beta, l)| &= |sed(W_{\beta-l}, l-1)| + |sed(W_{\beta-1}, l)| \\
&= |sed(W_{\beta-1}, (\beta-1-(l-1)+1))| \\
&+ |sed(W_{\beta-1}, (\beta-1-(l)+1))| \\
&= |sed(W_{\beta-1}, (\beta-1-l+1+1))| \\
&+ |sed(W_{\beta-1}, (\beta-1-l+1))| \\
&= |sed(W_{\beta-1}, (\beta-l+1))| \\
&+ |sed(W_{\beta-1}, (\beta-l))| \\
&= |sed(W_\beta, (\beta-l+1))|
\end{aligned}
$$

$\therefore$ Proved $\forall\ '\beta'$.

8. $SED_\beta = \sum_{l=1}^{\beta} |sed(W_\beta, l)|$
   By theorem-3.1 we have

$$
\begin{aligned}
SED_\beta &= \sum_{l=1}^{\beta} [|sed(W_{\beta-1}, l-1)| + |sed(W_{\beta-1}, l)|] \\
&= \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)| + \sum_{l=1}^{\beta-1} |sed(W_{\beta-1}, l)| \\
&= SED_{\beta-1} + SED_{\beta-1} \\
SED_\beta &= 2(SED_{\beta-1})
\end{aligned}
$$

9. By MI on $'\beta'$. When $\beta = 6$,
   $SED_6 = 2^{6-1} = 2^5 = 32$.
   $SED_7 = 2^{7-1} = 2^6 = 64$.
   Assume it is true $\forall$ $\mathbb{N}$ less than $'\beta'$.
   $SED_{\beta-1} = 2^{\beta-1-1} = 2^{\beta-2}$
   For $'\beta'$,

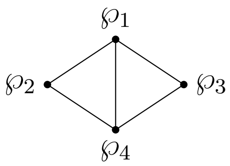
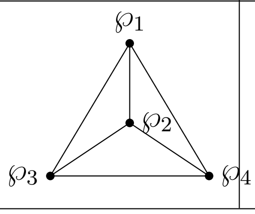
$$\begin{aligned} SED_\beta &= 2[SED_{\beta-1}] \quad \text{from theorem} - 3.2 - (8) \\ &= 2[2^{\beta-2}] \\ &= 2^{\beta-1} \end{aligned}$$

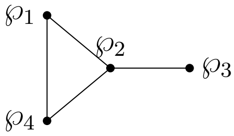
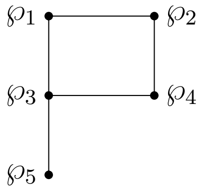
   $\therefore$ Proved $\forall$ $'\beta'$.
   Hence the theorem.

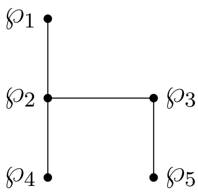
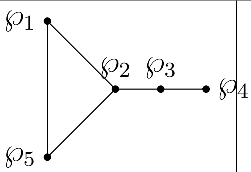
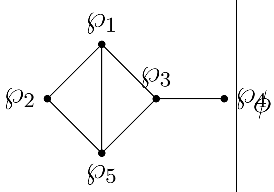
Remark 3.1. .

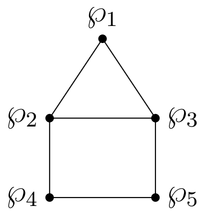
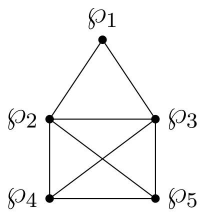
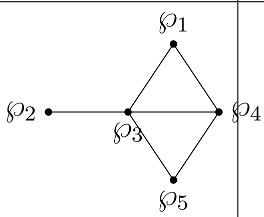
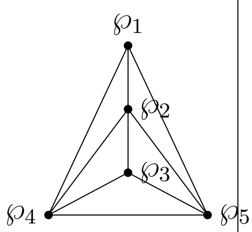
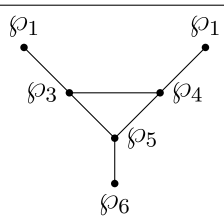
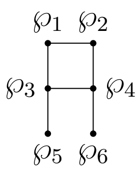
1. *For any graph* G 0 *is one of the root of every* $SED(G, \phi)$.
2. *A graph with more than 3 pendant vertices has at least 2 real roots.*

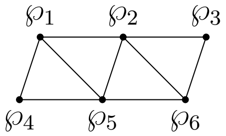
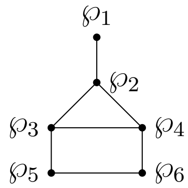
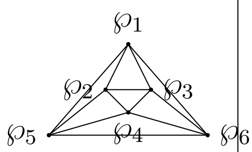
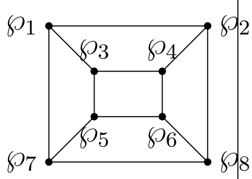
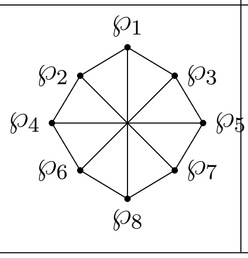
$SED(G, \phi)$ **of different standard graphs and their roots are tabulated below:**

Graph	Figure	Superior eccentric domination polynomial $SED(G, \phi)$	Roots
Diamond graph		$\phi^4 + 4\phi^3 + 5\phi^2$	$\phi_1 = 0$, $\phi_2 = -0.4563$, $\phi_3 = -1.7718 + 1.1151i$, $\phi_4 = -1.7718 - 1.1151i$.
Tetrahedral graph		$\phi^4 + 4\phi^3 + 6\phi^2 + 4\phi$	$\phi_1 = 0$, $\phi_2 = -2$, $\phi_3 = -1 + i$, $\phi_4 = -1 - i$.

Graph	Figure	Superior eccentric domination polynomial $SED(G,\phi)$	Roots
Paw graph		$\phi^4 + 3\phi^3 + 3\phi^2 + \phi$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -1$, $\phi_4 = -1$.
Banner graph		$\phi^5 + 4\phi^4 + 5\phi^3 + 3\phi^2$	$\phi_1 = 0$, $\phi_2 = -2.4656$, $\phi_3 = -0.7672 + 0.7926i$, $\phi_4 = -0.7672 - 0.7926i$.

Graph	Figure	Superior eccentric domination polynomial $SED(G,\phi)$	Roots
Fork graph		$\phi^5 + 4\phi^4 + 5\phi^3 + 2\phi^2$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -1$, $\phi_4 = -2$.
(3,2)-Tadpole graph		$\phi^5 + 4\phi^4 + 3\phi^3 + \phi^2$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -1$, $\phi_4 = -2$.
Kite graph		$\phi^5 + 5\phi^4 + 6\phi^3 + 2\phi^2$	$\phi_1 = 0$, $\phi_2 = -0.378 +0.1877i$, $\phi_3 = -0.378 -0.1877i$, $\phi_4 = -2.122 +1.0538i$, $\phi_5 = -2.122 -1.053i$.

Graph	Figure	Superior eccentric domination polynomial $SED(G, \phi)$	Roots
House graph		$\phi^5 + 5\phi^4 + 6\phi^3$	$\phi_1 = 0$, $\phi_2 = -0.3076 + 0.3182i$, $\phi_3 = -0.3076 - 0.3182i$, $\phi_4 = -2.1924 + 0.5479i$, $\phi_5 = -2.1924 - 0.5479i$.
House X graph		$\phi^5 + 5\phi^4 + 6\phi^3$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -1$, $\phi_4 = -0.382$, $\phi_5 = -2.618$.
Dart graph		$\phi^5 + 4\phi^4 + 6\phi^3 + 3\phi^2$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -\frac{3}{2} + \frac{\sqrt{3}}{2}i$, $\phi_4 = -\frac{3}{2} - \frac{\sqrt{3}}{2}i$.
Johnson solid skeleton 12 graph		$\phi^5 + 5\phi^4 + 10\phi^3 + 10\phi^2 + 5\phi$	$\phi_1 = 0$, $\phi_2 = -0.691 + 0.9511i$, $\phi_3 = -0.691 - 0.9511i$, $\phi_4 = -1.809 + 0.5878i$, $\phi_5 = -1.809 - 0.5878i$.
Net graph		$\phi^6 + 3\phi^5 + 3\phi^4 + \phi^3$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -1$, $\phi_4 = -1$.
A graph		$\phi^6 + 4\phi^5 + 6\phi^4 + 4\phi^3 + \phi^2$	$\phi_1 = 0$, $\phi_2 = -1$, $\phi_3 = -1$, $\phi_4 = -1$, $\phi_5 = -1$.

Graph	Figure	Superior eccentric domination polynomial $SED(G, \phi)$	Roots
4-polynomial graph 1		$\phi^6 + 4\phi^5 + 4\phi^4$	$\phi_1 = 0,$ $\phi_2 = -2.$
Antenna graph		$\phi^6 + 4\phi^5 + 3\phi^4 + \phi^3$	$\phi_1 = 0,$ $\phi_2 = -1,$ $\phi_3 = -2,$ $\phi_4 = -1.$
Octahedral graph		$\phi^6 + 6\phi^5 + 15\phi^4 + 20\phi^3$ $+15\phi^2 + 6\phi$	$\phi_1 = 0,$ $\phi_2 = -2,$ $\phi_3 = -0.5$ $+0.866i,$ $\phi_4 = -0.5$ $-0.866i,$ $\phi_5 = -1.5$ $+0.866i,$ $\phi_6 = -1.5$ $-0.866i.$
Cubical graph		$\phi^8 + 8\phi^7 + 28\phi^6 + 56\phi^5$ $+68\phi^4 + 48\phi^3 + 16\phi^2$	$\phi_1 = 0,$ $\phi_2 = -0.6714$ $+0.5756i,$ $\phi_3 = -0.6714$ $-0.5756i,$ $\phi_4 = -0.8352$ $+1.4854i,$ $\phi_5 = -0.8352$ $-1.4854i,$ $\phi_6 = -2.4934$ $+0.9097i,$ $\phi_7 = -2.4934$ $-0.9097i.$
Wagner graph		$\phi^8 + 8\phi^7 + 24\phi^6$ $+32\phi^5 + 16\phi^4$	$\phi_1 = 0,$ $\phi_2 = -2,$ $\phi_3 = -2,$ $\phi_4 = -2,$ $\phi_5 = -2.$

4 Conclusions

In this paper SED polynomial for a graph was defined. Formula to find the SED polynomials of family of wheel graphs were stated and proved. Corelation between the coefficients of SED polynomials were discussed. The SED polynomial and its roots for different standard graphs are tabulated. In the course of future work the graphs can be classified based on the roots of SED polynomials. The similarproperties of coefficients based on similar roots for different standard graphs can be discussed. The same concept can be extended to other domination parameter.

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