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Bounds on fuzzy dominator chromatic number of fuzzy soft bipartite graphs

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Abstract

An FSG G^S(T,V) fuzzy's soft dominator colouring (FSDC) is a suitable Fuzzy Soft Colouring (FSC) where every node of a colour group is dominated by a vertex of G^S(T,V). In the current work, we characterize the sharp bounds for the Fuzzy Dominator Chromatic Number (FDCN) of fuzzy soft bipartite graphs and we present limits on the FDCN of fuzzy soft bipartite graph in terms of the $\gamma^e(G^S(T,V))$. Furthermore, we classify fuzzy soft bipartite graphs into three classes based on FDCN.

Keywords:Fuzzy Soft Bipartite Graph, Fuzzy dominator chromatic number, Fuzzy soft path, Fuzzy soft cycle, Strong arcs.
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1 Introduction

Fuzzy soft graphs are a useful mathematical tool for simulating the ambiguity of the actual world in view of parameters. Fuzzy soft graphs, which are often utilised in many different disciplines, including decision-making issues, combine fuzzy soft sets and the graph model. Scheduling problems are just one of many practical issues involving the allocation of scarce resources for which graph colouring is used as a model. Graph colouring also has an important place in discrete mathematics and combinatorial optimization.

To address ambiguous issues in the fields of engineering, social science, economics, medical research, and environment, Molodstov [1999] created the idea of soft set theory. Fuzzy soft sets, a blend of a fuzzy set and a soft set, were first introduced by P.K. Maji and Biswas [2001]. Thumkara and George introduced the idea of a soft graph in 2014. Rosenfeld first proposed the idea of fuzzy graph theory in 1975. Fuzzy soft graphs were independently introduced in 2015 by Mohinta and Samanta as well. Akram and Nawaz [2015] presented fuzzy soft graphs and examined their operations as well as several other graph theoretical ideas.

Domination is a fast growing area of graph theory study, and the numerous ways it is used to networks, distributed computers, social networks, and online graphs helps to explain why there is more interest in this subject. The dominator colouring problem in graphs was first described by Gera [2007]. An appropriate colouring of a graph G with the extra characteristic that each vertex in the graph dominates an entire class is known as a dominator colouring of G. The smallest number of colour classes in a graph's dominator colouring is known as the dominator chromatic number. We want to reduce the number of colour categories. In 2007,Gera [2007] explored the dominator chromatic number for the hypercube and more broadly for bipartite graphs. He also presented dominator colorings in bipartite graphs. Chellali and Maffary [2012] studied dominator colorings in some classes of graphs.

The idea of fuzzy dominator colouring in fuzzy graphs was created by Hussain and Fathima [2015]. They investigated the fuzzy dominator chromatic number for a number of fuzzy graphs and explored its boundaries. The FDCN of bipartite, middle, and subdivision fuzzy graphs was created by Hussain and Fathima [2015], and established its limits. Domination in fuzzy soft graphs was the subject of study done by Hussain and Hussain [2017].

Fuzzy dominator coloring applied to fuzzy soft graph yields fuzzy soft dominator coloring which concentrates on minimizing the number of color classes of FSG. A Soft, Fuzzy Dominator FSG's colouring should be done in such a way that every node of a colour group is dominated by a vertex of $G^S(T, V)$. The proposed method concentrates on strength of connectedness, strong arc and strong neighbor of fuzzy soft bipartite graph in view of parameters from the existing method.

In this paper, we introduced Bounds of Fuzzy Soft Bipartite Graphs with Fuzzy Dominator Chromatic Number.

2 Preliminaries

Definition 2.1. *Let R be a parameter set and T is a subset of R, Let V =* $\{x_1, x_2, x_3, \cdots, x_n\}$ *is a non-empty set. As well*
(i) α*: T* $\rightarrow F(V)$ *(V's collection of all fuzzy subsets)*
e $\dashv$ $\alpha(e) = \alpha_e(say)$
α_e *: V* $\rightarrow [0, 1]$
(T,α*) : Fuzzy Soft Vertex*

(ii) β*: T* $\rightarrow$ *F(V*$\times$ *V) (V*$\times$ *V's collection of all fuzzy subsets)*
e $\dashv$ $\beta(e) = \beta_e$ *(say)*
β_e *: V*$\times$ *V* $\rightarrow [0, 1]$
(T,β*) : Fuzzy Soft Edge*
followed by ((T,α*),(T,*β*)) iff* $\beta_e(x, y) \leq \alpha_e(x) \wedge \alpha_e(y)$ *for all e in T and this FSGs are written by* $G^S(T, V)$*, it is referred to as a Fuzzy Soft Graph (FSG). Additionally, a FSG is a parametrized family unit of fuzzy graphs.*

Definition 2.2. *A Path is intended to be a set of different points* $x_1, x_2, \cdots x_n$ *in an FSG such that for all e in T and* β_e $(x_{i-1}, x_i) > 0$*, for all i = 1 to n.*

Definition 2.3. *If an FSG* $G^S(T, V)$ *contains more than 1 smallest arc,* $\forall\ e \in T$ *it is referred to as a Fuzzy Soft Cycle.*

Definition 2.4. *A FSG* $G^S(T, V)$ *is supposed to be a fuzzy soft bipartite if the node set V can be divided into 2 non-empty sets* V_1^e *and* V_2^e *such that* V_1^e *and* V_2^e *are fuzzy independent sets. These sets are called fuzzy bipartition of V, thus each efficient arc of FSG has one end in* V_1^e *and other end in* V_2^e*,* $\forall\ e \in T$*.*

Definition 2.5. *A fuzzy soft bipartite graph is Complete if for an individual node* V_1^e*, each single node of* V_2^e *is an efficient neighbour,* $e \in T$*.*

3 Fuzzy soft graphs with fuzzy dominator colouring

Definition 3.1. *If* $\beta_e(x, y) = \beta_e^{\infty}$ *(x,y),* $e \in T$ *iff the arc (x,y) in FSG is said to be a strong arc, where* β_e^{∞} *(x,y),* $e \in T$ *is the maximum strength of all pathways among x and y.*

Definition 3.2. *A Soft, Fuzzy Dominator a FSG's colouring should be done in such a way that every node of a colour group is dominated by a vertex of* G^S*(T,V).*

Definition 3.3. *In a FSDC of FSG, an FDCN is the least number of colour groups, and it is denoted as* χ^e_{fd} *(*G^S*(T,V)), e* $\in$ *T.*

4 Bounds on fuzzy soft bipartite graphs with fuzzy dominator chromatic number

Theorem 4.1. *Let* G^S*(T,V) be a connected fuzzy soft bipartite graph. Soon after* $2 \leq \chi^e_{fd}(G^S(T,V)) \leq \lfloor \frac{n}{2} \rfloor$*+1, e* $\in$ *T and these limits are precise.*

Proof. Take into account a connected fuzzy soft bipartite graph G^S(T,V). Every acceptable fuzzy soft colour must also be present in every fuzzy soft dominator colour, $\chi^e_f(G^S(T,V))$ = 2, Hussain and Farhana [2020], as a result $\chi^e_{fd}(G^S(T,V))$ $\geq$ 2, e $\in$ T.
To acquire the upper bound, let V^e_1 and V^e_2 be the 2 bipartite sets of G^S(T,V) with the condition $|V^e_1| \leq |V^e_2|$. After that, allocate colours 1,2,...,$|V^e_1|$ to the nodes of V^e_1 and colour $|V^e_1|$ +1 to the nodes of V^e_2, is a least FSDC. Hence $\chi^e_{fd}(G^S(T,V))$ $\leq |V^e_1|$ + 1,since $|V^e_1| \leq \lfloor \frac{n}{2} \rfloor$, which implies $\chi^e_{fd}(G^S(T,V)) \leq \lfloor \frac{n}{2} \rfloor$+ 1, e $\in$ T.□

Definition 4.1. *Consider a FSG* G^S*(T,V) with n nodes. Connect each node of* G^S*(T,V) onto any one of the n isolated nodes, where n is the total number of nodes in the FSG.The resultant graph is called Corona of* G^S*(T,V) and is symbolized as Cor(*G^S*(T,V))= ((T,*α_1*),(T,*β_1*))where*
$\alpha^e_1(u) = \alpha_e(u)$*,u* $\in$*V, e*$\in$*T and*
$\alpha_e(u) = \alpha^e_1(u) \in$ *(0,1], if u is isolated, e*$\in$*T.*
$\beta^e_1(u,v) = \beta_e(u,v) \in$ *E, e*$\in$*T and*
$\beta^e_1(u,v) = \alpha^e_1(u) \wedge \alpha_e(v)$*, if u* $\in$ *V and v is isolated, e*$\in$*T.*

Remark:
1. Obviously Cor(G^S(T,V)) is a FSG if G^S(T,V) be a FSG.
2. If G^S(T,V) has 'n' nodes and 'e' edges then Cor(G^S(T,V)) has 2n nodes and e+n edges.
3. Cor(G^S(T,V)) has k+n strong arcs if the no. of strong nodes in G^S(T,V) is 'k'.
Now we discuss about the sharpness of lower bound and upper bound of Theorem:4.1. Since the lower bound is sharp, it may be inferred that the entire fuzzy soft bipartite graph's FDCN is 2. The subsequent theorem proves the sharpness of the upper bound.

Theorem 4.2. *Having a 'n'-node fuzzy soft path* G^S*(T,V) and n' is the number of nodes in* $cor(G^S(T,V))$*. Then* $\chi^e_{fd}(cor(G^S(T,V))$ = $\lfloor \frac{n'}{2} \rfloor$*+1, e* $\in$ T*.*

Proof. Consider a fuzzy soft path having v_1,v_2,...,v_n as nodes. Noticeably G^S(T,V) is a fuzzy soft bipartite graph, Hussain and Farhana [2020] and so

$Cor(G^S(T,V))$ is also a fuzzy soft bipartite graph. This implies every arc is a strong arc in $Cor(G^S(T,V))$. The number of nodes in G^S(T,V) is n, then n' = 2n nodes in $Cor(G^S(T,V))$.

At this instant, we have to colour the nodes of $Cor(G^S(T,V))$. To obtain the least fuzzy soft dominator colouring, every node in minimum dominating set be full of distinctive colour. The nodes in G^S(T,V)is strong adjoining to just one node of $Cor(G^S(T,V))$. This implies that every node of G^S(T,V) is in a minimum dominating set. Therefore $\gamma^e(Cor(G^S(T,V)))$=n, e∈T.

Color 1,2,...,n is distributed to the nodes of G^S(T,V), and colour n + 1 is distributed to the left over nodes. This is the least FSDC of $Cor(G^S(T,V))$. Hence we prove $\chi^e_{fd}(Cor(G^S(T,V)))$ = $\lfloor \frac{n'}{2} \rfloor$+1, e ∈ T which is the upper bound and it is sharp.□

Now we prove that the values in the middle of 2 and $\lfloor \frac{n'}{2} \rfloor$ + 1 can be achieved as the FDCN of some fuzzy soft bipartite graphs of order n.

Theorem 4.3. *Let k be an integer with 2 ≤ k ≤ $\lfloor \frac{n}{2} \rfloor$+1, If so, a connected fuzzy soft bipartite graph G^S(T,V) with n- nodes and its FDCN is k.*

Proof. Consider a fuzzy soft path P^e_k =((T,α_1),(T,β_1)) with k ≥ 2 nodes and it is assumed as v_1,v_2,...v_k. Now put up a FSG from fuzzy soft path by extending k nodes u_1,u_2,...,u_k in order that u_iv_i ∈ E,(1≤i≤k) and by adding n-2k nodes x_j to v_k (1≤j≤n-2k). We have n-2k ≥ 1 and k ≥2, it follows that n ≥5, where

$\alpha^e(v_1)$ =$\alpha^e_1(v_i)$,e ∈ T

β_e(u,v) = β^e_1(u,v), (u,v)∈ E, e ∈ T

$\beta_e(u_i$,$v_j)$= $\alpha_e(u_i) \wedge \alpha_e(v_j)$, for every i, e ∈ T.

$\beta_e(v_k$, $x_j)$= $\alpha_e(v_k) \wedge \alpha_e(x_j)$, for every j, e ∈ T.

Clearly, G^S(T,V) is a fuzzy soft tree because there are no fuzzy soft cycles of odd length, implying that it is a fuzzy soft bipartite graph, Hussain and Farhana [2020]. At the moment, the sets $\{v_1, v_2, ..., v_k\}$ and $\{u_1, u_2, ...u_{k-1}, v_k\}$ have the lowest dominance. As a result,$\gamma^e(G^S$(T,V)) = k, e ∈ T.

Since $\gamma^e(G^S$(T,V)) = k, we require no less than k distinctive colours for least FSDC. Allocate colours 1,2,...,k to v_1,v_2,...,v_k respectively and k+1 more colour to u_1,u_2,...,u_k,x_1, x_2,...,x_{n-2k}. So each node has k+1 colours offered in G^S(T,V) dominate no less than one colour group. At last, the colouring is least FSDC.

Alternatively, we allocate k unique colours to the nodes of the set $\{u_1, u_2, ...u_{k-1}, v_k\}$ then we have a FSDC of 'k' colours, since we can colour the

remaining nodes using k colours. Thus $\chi^e_{fd}(G^S(T,V))$ = k e $\in$ T.□
We have a necessary condition for the upper bound.

Theorem 4.4. *Define G^S(T,V) a soft fuzzy bipartite graph with double-edged (U,V). Following this, corona of G^S(T,V) is a fuzzy soft bipartite graph whose χ^e_{fd} is $\lfloor\frac{|cor(G^S(T,V))|}{2}\rfloor$+1, e$\in T$ where $|cor(G^S(T,V))|$ is the number of nodes in $cor(G^S(T,V))$.*

Proof. Assume a soft fuzzy bipartite graph with bipartition (U,V) and $|U|$ + $|V|$ = number of nodes in G^S(T,V). Label G^S(T,V)' be the corona of G^S(T,V). Undoubtedly G^S(T,V)' is a fuzzy soft bipartite graph.

The points of soft fuzzy bipartite graph allocated with colours 1,2,...$|V|$ and 1+$|V|$ to the left behind nodes, which is a FSDC of G^S(T,V)'. Seeing as each node of G^S(T,V) dominate itself and the left over nodes dominates its nearest nodes. This colouring is a least FSDC. This implies that each node in minimum dominating set gets an exclusive colour. Hence $\chi^e_{fd}(cor(G^S(T,V)))$ = $\lfloor\frac{|cor(G^S(T,V))|}{2}\rfloor$+1, e$\in T$.

5 Bounds on fuzzy dominator chromatic number in terms of domination number

Theorem 5.1. *Define a soft fuzzy bipartite graph G^S(T,V). In that case γ^e(G^S(T,V)) $\leq \chi^e_{fd}(G^S(T,V)) \leq$ 2 + γ^e (G^S(T,V)), e$\in T$.*

Proof. Allow G^S(T,V) to be a soft fuzzy bipartite graph and c to be the least FSDC with colours 1,2,...,$\chi^e_{fd}(G^S(T,V))$. For every colour group of G^S(T,V), allow a_n be a point in the colour group n having 1 $\leq$ n $\leq \chi^e_{fd}(G^S(T,V))$,e$\in T$. We have to prove that S = $\{a_n$:1 $\leq$ n $\leq \chi^e_{fd}(G^S(T,V))$, e$\in T\}$ is a dominating set.

By the definition of FSDC, all nodes of G^S(T,V) will dominate all nodes of some colour class. Since S contains a node of each colour group, each node of G^S(T,V) dominate some node in S. As a result, S is a fuzzy dominating set, and every least FSDC of G^S(T,V) yields a fuzzy dominating set of G^S(T,V). Hence γ^e(G^S(T,V)) $\leq \chi^e_{fd}(G^S(T,V))$, e$\in T$.

We must now validate the upper bound. Because the FCN of fuzzy soft bipartite graph is 2, Hussain and Farhana [2020], we can colour the nodes of G^S(T,V) using 2 colours 1 and 2. Allot colours 3,4,...,γ^e(G^S(T,V))+2 to the nodes of S and offer colours 1 and 2 to the lasting nodes of G^S(T,V) such that 2 strong adjoining nodes be given dissimilar colours. This colouring fall out in fuzzy soft dominator colouring since it is an appropriate fuzzy soft colouring and each node in G^S(T,V) dominate all nodes of at least one colour group. Hence $\chi^e_{fd}(G^S(T,V)) \leq 2$ + γ^e (G^S(T,V)), e$\in T$.□

Observation:
Fuzzy Soft Bipartite Graphs G^S(T,V) on the basis of the limits, may be divided into three categories on $\chi^e_{fd}(G^S(T,V))$.
1. G^S(T,V) is of CLASS 0 if $\chi^e_{fd}(G^S(T,V))$ = γ^e(G^S(T,V)), e$\in T$.
2. G^S(T,V) is of CLASS 1 if $\chi^e_{fd}(G^S(T,V))$ = γ^e(G^S(T,V)) + 1, e$\in T$.
3. G^S(T,V) is of CLASS 2 if $\chi^e_{fd}(G^S(T,V))$ = γ^e(G^S(T,V)) + 2, e$\in T$.

Example:CLASS 0
Consider a fuzzy soft path of k $\geq$ 2 nodes. Now form a FSG as in **Theorem:4.3**. Then FSG is of CLASS 0.
Example:CLASS 1
Consider a complete fuzzy soft bipartite graph $K_{1,n}$. The FDCN of $K_{1,n}$ is two and its γ^e(G^S(T,V)) is one, therefore $\chi^e_{fd}(G^S(T,V))$ = γ^e(G^S(T,V)) + 1. Hence G^S(T,V) is of CLASS 1.
Example:CLASS 2
The fuzzy soft cycle of length n$\geq$ 5 has FSDC $\lceil \frac{n}{3} \rceil$+2 and its γ^e(G^S(T,V)) is $\lceil \frac{n}{3} \rceil$, therefore $\chi^e_{fd}(G^S(T,V))$ = γ^e(G^S(T,V)) + 2. Hence G^S(T,V) is of CLASS 2.

6 Conclusions

In this work, we characterized the bounds on FDCN of fuzzy soft bipartite graphs and also presented bounds in respect of γ^e(G^S(T,V)). Additionally, based on the fuzzy dominator chromatic number (FDCN) that was found, the fuzzy soft bipartite graphs are divided into three categories as fuzzy soft path in CLASS 0, complete fuzzy soft bipartite graph in CLASS 1 and fuzzy soft cycle in CLASS 2. We suggest this study on a few distinct classes of fuzzy soft graphs.

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