Construction of k-Hyperideals by P-Hyperoperations

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Abstract

In this note we present a method to construction new k-hyperideals from given k-ideals of a semiring R by using of the P-hyperoperations. Then we investigate the relationship between them. In particular, we describe all k-hyperideals of the semihyperring of the nonnegative integers.

Keywords: (semi)hyperring, k-(hyper)ideal, P-hyperoperation, weak distributive [1]

1 Introduction

Hyperstructures theory was born in 1934 when Marty [12] defined hypergroups as a generalization of groups. Also Wall in 1937 defined the notion of cyclic hypergroup. This theory has been studied in the following decades and nowadays by many mathematicians. A short review of the theory of hypergroups appears in [2]. A recent books [2], [3] and [15] contain a wealth of applications. There are applications

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to the following subjects: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, groups, rational algebraic functions and etc. One of the several contexts which they arise is hyperring. First M. Krasner studied hyperrings, which is a triple $(R, +, .)$, where $(R, +)$ is a canonical hypergroup and $(R, .)$ is a semigroup, such that for all $a, b, c \in R$, $a(b + c) = ab + ac, (b + c)a = ba + ca$ ([10]).

The notion of k-ideals in ordinary semirings was introduced by D. R. Latore in 1965 ([11]). Also M. K. Sen and others worked on one-sided k-ideals and maximal k-ideals of semirings ([14], [16]).

The authors in [6] introduced the notion of k-hyperideals in the sense of Krasner and obtained some related results about this notion. We now follow [6] to introduce a method to construct new k-hyperideals from given k-ideals.

In section 2 of this paper, we gather all the preliminaries of (semi)hyperrings and k-(hyper)ideals which will be used in the next sections. In section 3, we represent some methods for construction semihyperrings from semirings by P-hyperoperations and then we investigate the relationship between their k-hyperideals and k-ideals. As an important result of this section, all k-hyperideals of the nonnegative integers $\mathbb{N}^*$ as a semihyperring, constructed by P-hyperoperations, are described. In section 4, we characterize the k-hyperideals of product of semihyperrings which are made by P-hyperoperations and a family of semirings.

2 Preliminaries

A map $\circ : H \times H \longrightarrow P_*(H)$ is called *hyperoperation* or *join operation*. A *hypergroupoid* is a set H with together a (binary) hyperoperation $\circ$. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z, \forall x, y, z \in H$ is called a *semihypergroup* .

A *hypergroup* is a semihypergroup such that $\forall x \in H$ we have $x \circ H = H = H \circ x$, which is called *reproduction axiom* (see [2]).

Let H be a hypergroup and K be a nonempty subset of H. Then K is said to be a *subhypergroup* of H if itself is a hypergroup under hyperoperation "$\circ$" restricted to K. Hence it is clear that a subset K of H is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on H.

Definition 2.1. A hyperalgebra $(R, +, .)$ is called a *semihyperring* if and only if

(i) $(R, +)$ is a semihypergroup;

(ii) $(R, .)$ is a semigroup;

(iii) $\forall a, b, c \in R,\ a.(a + b) = a.b + a.c$ and $(b + c).a = b.a + c.a.$

Remark. In Definition 2.1, if we replace (iii) by

$$\forall a, b, c \in R,\ a.(a + b) \subseteq a.b + a.c \ \text{ and } \ (b + c).a \subseteq b.c + c.a,$$

we say that R is a *weak distributive* semihyperring.

A semihyperring R is called with *zero element*, if there exists an unique element $0 \in R$ such that $0 + x = x = x + 0$ and $0x = 0 = x0$ for all $x \in R$.

A semihyperring R is called *additive commutative*, if $x + y = y + x,\ \ \forall x, y \in R$.

A semihyperring $(R, +, .)$ is called a *hyperring* provided $(R, +)$ is a canonical hypergroup.

Definition 2.2. A hyperring $(R, +, .)$ is called

(i) *commutative* if $a.b = b.a$ for all $a, b \in R$;

(ii) *with identity*, if there exists an element, say $1 \in R$, such that $1.x = x.1 = x$ for all $x \in R$.

Let $(R, +, .)$ be a hyperring, a nonempty subset S of R is called a *subhyperring* of R if $(S, +, .)$ is itself a hyperring.

Definition 2.3. A subhyperring I of a hyperring R is said to be a (resp. *right) left hyperideal* of R provided that (resp. $x.r \in I$) $r.x \in I$ for all $r \in R$ and for all $x \in I$. We say that I is a *hyperideal* if I is both a left and right hyperideal.

Definition 2.4.[11] Let $(R, +, .)$ be a semiring. A nonempty subset I of R is called a *left k-ideal* of R, if I is a left ideal of R and for $a \in I$ and $x \in R$ we have

$$a + x \in I \ \text{ or } \ x + a \in I \implies x \in I.$$

Similarly a right k-ideal is defined. A two sided k-ideal or simply a k-ideal is both a left and right k-ideal. We denote I as k-ideal (resp. ideal) of R by $I \lhd_k R$ (resp. $I \lhd R$).

In the sequel, by R we mean a semihyperring, unless otherwise specified.

Definition 2.5.[6] Let $(R, +, .)$ be a (weak distributive) semihyperring. A nonempty subset I of R is called

(i) a *left* (resp. *right) hyperideal* of R if and only if

(a) $(I, +)$ is a semihypergroup of $(R, +)$; and

(b) $rx \in I$ (resp. $xr \in I$), for all $r \in R$ and for all $x \in I$.

(ii) a *hyperideal* of R if it is both left and right hyperideal of R. The hyperideal I of R is denoted by $I \lhd_h R$.

(iii) a *left k-hyperideal* of R, if I is a left hyperideal of R and for $a \in I$ and $x \in R$ we have

$$a + x \approx I \quad \text{or} \quad x + a \approx I \quad \Longrightarrow \quad x \in I,$$

where by $A \approx B$ we mean $A \cap B \neq \emptyset$.

(iv) Similarly a right k-hyperideal is defined. A two sided k-hyperideal or simply a k-hyperideal is both a left and right k-hyperideal. We denote I as k-hyperideal of R by $I \lhd_{k.h} R$.

3 Construction of k-hyperideals by P-hyperoperations

In this section we apply three kinds of P-hyperoperations (which were introduced for H_v-structures in [15]) to construct semihyperrings from semirings. Then we investigate the relationship between their k-hyperideals and k-ideals .

Definition 3.1. Let $(R, +, .)$ be semiring and $\emptyset \neq P \subseteq R$. We define two hyperoperations as follows

$$x \oplus_c y = \{x + t + y \ \mid \ t \in P\},$$

$$x \odot y = x.y = xy,$$

which $\oplus_c$ is called *centre P-hyperoperation.*

Proposition 3.2. Let $(R, +, .)$ be semiring and $P \subseteq R$ be a nonempty such that $PR \subseteq P$ and $RP \subseteq P$, then $(R, \oplus_c, \odot)$ is a weak distributive semihyperring.

Proof . First, we show $(R, \oplus_c)$ is a semihypergroup. For this we prove that

$$(x \oplus_c y) \oplus_c z = x \oplus_c (y \oplus_c z).$$

For $x, y, z \in R$ we have

$$\begin{aligned} a \in (x \oplus_c y) \oplus_c z &\Longrightarrow \exists a_1 \in x \oplus_c y,\ a \in a_1 \oplus_c z \\ &\Longrightarrow \exists t_1, t_2 \in P,\ a = a_1 + t_1 + z,\ a_1 = x + t_2 + y \\ &\Longrightarrow a = x + t_2 + y + t_1 + z \\ &\Longrightarrow a = x + t_2 + b,\ b = y + t_1 + z \in y \oplus_c z \\ &\Longrightarrow a \in x \oplus_c b,\ b \in y \oplus_c z \\ &\Longrightarrow a \in x \oplus_c (y \oplus_c z) \\ &\Longrightarrow (x \oplus_c y) \oplus_c z \subseteq x \oplus_c (y \oplus_c z). \end{aligned}$$

Similarly, we obtain that

$$(x \oplus_c y) \oplus_c z \supseteq x \oplus_c (y \oplus_c z).$$

Clearly $(R, \odot)$ is a semigroup, since $(R, .)$ is a semigroup and $x \odot y = xy$.

We now prove weak distributivity, that is

$$\begin{aligned} x \odot (y \oplus_c z) &\subseteq (x \odot y) \oplus_c (x \odot z) \\ &= xy \oplus_c xz. \end{aligned}$$

For this we have

$$\begin{aligned} a \in x \odot (y \oplus_c z) &\Longrightarrow \exists a_1 \in y \oplus_c z,\ a = x \odot a_1 = xa_1 \\ &\Longrightarrow \exists t \in P,\ a = xa_1,\ a_1 = y + t + z \\ &\Longrightarrow a = x(y + t + z) \\ &= xy + xt + xz \in xy \oplus_c xz \quad (\ RP \subseteq P\ ) \\ &\Longrightarrow x \odot (y \oplus_c z) \subseteq xy \oplus_c xz. \end{aligned}$$

Similarly we conclude that $(y \oplus_c z) \odot x \subseteq yx \oplus_c zx.\square$

Definition 3.3. Let $(R, +, .)$ be a semiring and $\emptyset \neq P \subseteq R$. We define the following hyperoperations

$$x \oplus_r y = \{x + y + t \ \mid \ t \in P\}, \qquad x \oplus_l y = \{t + x + y \ \mid \ t \in P\},$$

$$x \odot y = xy,$$

which $\oplus_r$ and $\oplus_l$ are called *right P-hyperoperation* and *left P-hyperoperation* respectively.

Proposition 3.4. Let $(R, +, .)$ be a semiring and $P \subseteq R$ be a nonempty such that $PR \subseteq P$ and $RP \subseteq P$ and $x + P = P + x$, for all $x \in R$. Then $(R, \oplus_r, \odot)$ and $(R, \oplus_l, \odot)$ are weak distributive semihyperrings.

Proof. First, we prove that

$$(x \oplus_r y) \oplus_r z = x \oplus_r (y \oplus_r z).$$

For this we have

$$\begin{aligned}
a \in (x \oplus_r y) \oplus_r z &\Longrightarrow \exists a_1 \in x \oplus_r y, \ a \in a_1 \oplus_r z \\
&\Longrightarrow \exists t_1, t_2 \in P, \ a_1 = x + y + t_1, \ a = a_1 + z + t_2 \\
&\Longrightarrow \exists t_1, t_2 \in P, \ a = x + y + t_1 + z + t_2 \qquad (1)
\end{aligned}$$

also we have

$$\begin{aligned}
b \in x \oplus_r (y \oplus_r z) &\Longrightarrow \exists b_1 \in y \oplus_r z, \ b \in x \oplus_r b_1 \\
&\Longrightarrow \exists w_1, w_2 \in P, \ b_1 = y + z + w_1, \ b = x + b_1 + w_2 \\
&\Longrightarrow \exists w_1, w_2 \in P, \ b = x + y + z + w_1 + w_2 \qquad (2)
\end{aligned}$$

From (1) we have

$$\begin{aligned}
a = x + y + t_1 + z + t_2 &= x + y + z + w_1 + t_2, \ \exists w_1 \in P \qquad (z + P = P + z) \\
&\Longrightarrow a \in x \oplus_r (y \oplus_r z) \qquad \text{(by (2))} \\
&\Longrightarrow (x \oplus_r y) \oplus_r z \subseteq x \oplus_r (y \oplus_r z).
\end{aligned}$$

Similarly we can prove that

$$(x \oplus_r y) \oplus_r z \supseteq x \oplus_r (y \oplus_r z).$$

Clearly $(R, \odot)$ is semigroup, since $(R, .)$ is a semigroup. In a similar way to the Proposition 3.2 we can prove weak distributivity. Therefore $(R, \oplus_r, \odot)$ is a weak distributive semihyperring. Analogously we can prove that $(R, \oplus_l, \odot)$ is a weak distributive semihyperring. $\square$

Remark. In Propositions 3.2 and 3.4, if we replace the conditions $RP \subseteq P$ and $PR \subseteq P$ by $rP = P = Pr$ for all $r \in R$, then $(R, \oplus_c, \odot)$ and $(R, \oplus_r, \odot)$ and $(R, \oplus_l, \odot)$ become semihyperring.

Theorem 3.5. Let $(R, +, .)$ be a semiring with zero and P be the same as Proposition 3.2 such that $0 \in P$. Then there is a one-to-one correspondence between the k-ideals of $(R, +, .)$ containing P and k-hyperideals of $(R, \oplus_c, \odot)$.

Proof. Let I be a k-ideal of $(R, +, .)$ containing P. First we prove that $I \lhd_h (R, \oplus_c, \odot)$. Suppose that $x, y \in I$, we prove $x \oplus_c y \subseteq I$. For this we have

$$\begin{aligned} z \in x \oplus_c y &\Longrightarrow \exists t \in P \subseteq I,\ z = x + t + y \\ &\Longrightarrow z = x + t + y \in I \quad (\text{ since } x, t, y \in I\ ) \\ &\Longrightarrow x \oplus_c y \subseteq I. \end{aligned}$$

Also if $r \in R$ and $x \in I$, then $r \odot x = rx \in I$, since $I \lhd (R, +, .)$. Thus I is a hyperideal of $(R, \oplus_c, \odot)$. We now prove that $I \lhd_{k.h} (R, \oplus_c, \odot)$. For $r \in R$ and $x \in I$ we have

$$\begin{aligned} r \oplus_c x \approx I &\Longrightarrow \exists z \in r \oplus_c x \approx I \\ &\Longrightarrow \exists t \in P,\ z = r + t + x,\ z \in I \\ &\Longrightarrow r + t + x \in I,\ t + x \in I \\ &\Longrightarrow r \in I \qquad (\text{ since } I \lhd_k (R, +, .)\ ) \\ &\Longrightarrow I \lhd_{k.h} (R, \oplus_c, \odot). \end{aligned}$$

Conversely, suppose that $I \lhd_{k.h} (R, \oplus_c, \odot)$. We prove that I is a k-ideal of $(R, +, .)$ containing P. For this we have

$$\begin{aligned} x, y \in I &\Longrightarrow x \oplus_c y \subseteq I \qquad (\ I \lhd_h (R, \oplus_c, \odot)\ ) \\ &\Longrightarrow \forall t \in P,\ x + t + y \in I \\ &\Longrightarrow x + y \in I \qquad (\ 0 \in P\ )\ . \end{aligned}$$

On the other hand

$$\begin{aligned} r \in R, x \in I &\Longrightarrow r \odot x \in I \qquad (\ I \lhd_h (R, \oplus_c, \odot)\ ) \\ &\Longrightarrow rx \in I. \end{aligned}$$

Also we have

$$\begin{aligned} r + x \in I, x \in I &\Longrightarrow r + 0 + x \in I,\ x \in I \quad (0 \in P) \\ &\Longrightarrow r \oplus_c x \approx I,\ x \in I \\ &\Longrightarrow r \in I \qquad\qquad (\ I \lhd_{k.h} (R, \oplus_c, \odot) \\ &\Longrightarrow I \lhd_k (R, +, .). \end{aligned}$$

We have $0 \oplus_c 0 \subseteq I$, then $\{0 + t + 0 \mid t \in P\} \subseteq I$, therefore $P \subseteq I$. $\square$

Theorem 3.6. Let $(R, +, .)$ be a semiring with zero and P be the same as Proposition 3.4 such that $0 \in P$. Then there is a one-to-one correspondence between k-ideals of $(R, +, .)$ containing P and k-hyperideals of ($(R, \oplus_l, \odot)$) $(R, \oplus_r, \odot)$.

Proof. The proof is similar to the proof of Theorem 3.5 by some manipulation. $\square$

Examples. (i) Let $\mathbb{N}$ be the set of natural numbers and $2\mathbb{N} = \{2, 4, 6, 8, ...\}$. Clearly $(\mathbb{N}, +, .)$ is a semiring and $2\mathbb{N}$ is a k-ideal of $(\mathbb{N}, +, .)$. Now if $P = \{4, 8, 12, 16, ...\} \subseteq 2\mathbb{N}$, then it is easy to verify that $(\mathbb{N}, \oplus_c, \odot)$ is a weak distributive semihyperring, where for all $m, n \in \mathbb{N}$ we have

$$m \oplus_c n = \{m + k + n \mid k \in P\} \quad \text{and} \quad m \odot n = mn.$$

Thus $2\mathbb{N}$ is a k-hyperideal of $(\mathbb{N}, \oplus_c, \odot)$.

(ii) Let $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\mathbb{N}^*[x] = \{f(x) = \sum_{i=1}^{n} a_i x^i \mid a_i \in \mathbb{N}^*\}$. Clearly $(\mathbb{N}^*[x], +, .)$ is a semiring and $< x >= \{f(x) \in \mathbb{N}^*[x] \mid a_0 = 0\}$ is a k-ideal of $(\mathbb{N}^*[x], +, .)$ generated by x. Set $P =< x^m >$ for $m \in \mathbb{N}$. Obviously, $0 \in P \subseteq < x >$. Then by Propositions 3.2 and 3.5, $(\mathbb{N}^*[x], \oplus_c, \odot)$ is a weak distributive semihyperring and $< x >$ is a k-hyperideal of $(\mathbb{N}^*[x], \oplus_c, \odot)$.

In the next theorem we describe all k-hyperideals of semihyperring of the natural numbers constructed by P-hyperoperation. For this we consider the semiring $(\mathbb{N}, +, .)$ of natural numbers by usual ordinary operations.

Theorem 3.7. Let $0 \in P \subseteq \mathbb{N}^*$ and $P\mathbb{N}^* \subseteq P$ and $\mathbb{N}^*P \subseteq P$ and $P \subseteq I$. Then I is a k-hyperideal of $(\mathbb{N}^*, \oplus_c, \odot)$ if and only if there exists $a \in \mathbb{N}^*$ such that $I = \{na \mid n \in \mathbb{N}^*\}$.

Proof. By Theorem 3.5, $I \lhd_{k.h} (\mathbb{N}^*, \oplus_c, \odot)$ if and only if $I \lhd_k (\mathbb{N}^*, +, .)$. Also by Proposition 4.1 [14], $I \lhd_k (\mathbb{N}^*, +, .)$ if and only if there exists $a \in \mathbb{N}^*$ such that $I = \{na \mid n \in \mathbb{N}^*\}$. $\square$

4 Product of k-hyperideals

In the sequel by $\prod_{i \in I} R_i$, we mean the *cartesian product* of the family $\{R_i\}_{i \in I}$. It means

$$\prod_{i \in I} R_i = \{(x_i)_{i \in I} \mid x_i \in R_i\}.$$

Proposition 4.1. Let $\{R_i\}_{i \in I}$ be a family of semirings and $P_i \subseteq R_i$ be nonempty such that $R_i P_i \subseteq P_i$ and $P_i R_i \subseteq P_i$, for all $i \in I$. For $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} R_i$. Define

$$(x_i)_{i \in I} \oplus_c (y_i)_{i \in I} = \{(x_i + t_i + y_i)_{i \in I} \mid t_i \in P_i\},$$

$$(x_i)_{i \in I} \odot (y_i)_{i \in I} = (x_i y_i)_{i \in I}.$$

Then $(\prod_{i \in I} R_i, \oplus_c, \odot)$ is a weak distributive semihyperring .

Proof. First we show that $(\prod_{i\in I} R_i, \oplus_c)$ is a semihypergroup. For this we prove that

$$(x_i)_{i\in I} \oplus_c [(y_i)_{i\in I} \oplus_c (z_i)_{i\in I}] = [(x_i)_{i\in I} \oplus_c (y_i)_{i\in I}] \oplus_c (z_i)_{i\in I}.$$

We have $\quad A \in (x_i)_{i\in I} \oplus_c [(y_i)_{i\in I} \oplus_c (z_i)_{i\in I}]$

$$\begin{aligned}
&\Longrightarrow \ \exists t_i \in P_i,\ A \in (x_i)_{i\in I} \oplus_c (y_i + t_i + z_i)_{i\in I}\\
&\Longrightarrow \ \exists t'_i \in P_i,\ A = (x_i + t'_i + y_i + t_i + z_i)_{i\in I}\\
&\Longrightarrow \ A \in (x_i + t'_i + y_i)_{i\in I} \oplus_c (z_i)_{i\in I}\\
&\Longrightarrow \ A \in [(x_i)_{i\in I} \oplus_c (y_i)_{i\in I}] \oplus_c (z_i)_{i\in I}\\
&\Longrightarrow \ (x_i)_{i\in I} \oplus_c [(y_i)_{i\in I} \oplus_c (z_i)_{i\in I}] \subseteq [(x_i)_{i\in I} \oplus_c (y_i)_{i\in I}] \oplus_c (z_i)_{i\in I}.
\end{aligned}$$

In a similar way, we can prove the reverse inclusion. Therefore, $(\prod_{i\in I} R_i, \oplus_c)$ is a semihypergroup. Clearly $(\prod_{i\in I} R_i, \odot)$ is a semigroup. It is enough we prove weak distributivity. For this we should prove that

$$(x_i)_{i\in I} \odot [(y_i)_{i\in I} \oplus_c (z_i)_{i\in I}] \subseteq (x_iy_i)_{i\in I} \oplus_c (x_iz_i)_{i\in I}.$$

We have $\quad A \in (x_i)_{i\in I} \odot [(y_i)_{i\in I} \oplus_c (z_i)_{i\in I}]$

$$\begin{aligned}
&\Longrightarrow && \exists t_i \in P_i,\ A \in (x_i)_{i\in I} \odot (y_i + t_i + z_i)_{i\in I}\\
&\Longrightarrow && A = (x_i(y_i + t_i + z_i))_{i\in I}\\
&= && (x_iy_i + x_it_i + x_iz_i)_{i\in I}\\
&\in && (x_iy_i)_{i\in I} \oplus_c (x_iz_i)_{i\in I} \qquad (\ R_iP_i \subseteq P_i\ ).
\end{aligned}$$

This completes the proof. □

Proposition 4.2. If $\{R_i\}_{i\in I}$ is a family of semirings and for all $i \in I$, $P_i \subseteq R_i$ is nonempty such that $R_iP_i \subseteq P_i$ and $P_iR_i \subseteq P_i$ and $x_i + P_i = P_i + x_i$, for all $x_i \in R_i$, then $(\prod_{i\in I} R_i, \oplus_r, \odot)$ and $(\prod_{i\in I} R_i, \oplus_l, \odot)$ are weak distributive semihyperring where

$$(x_i)_{i\in I} \oplus_r (y_i)_{i\in I} = \{(x_i + y_i + t_i)_{i\in I} \ \mid\ t_i \in P_i\},$$

$$(x_i)_{i\in I} \oplus_l (y_i)_{i\in I} = \{(t_i + x_i + y_i)_{i\in I} \ \mid\ t_i \in P_i\},$$

$$(x_i)_{i\in I} \odot (y_i)_{i\in I} = (x_i y_i)_{i\in I}.$$

Proof. First we prove that $(\prod_{i\in I} R_i, \oplus_r)$ is a semihypergroup. For this we prove that

$$(x_i)_{i\in I} \oplus_r [(y_i)_{i\in I} \oplus_r (z_i)_{i\in I}] = [(x_i)_{i\in I} \oplus_r (y_i)_{i\in I}] \oplus_r (z_i)_{i\in I}.$$

We have $\quad A \in (x_i)_{i\in I} \oplus_r [(y_i)_{i\in I} \oplus_r (z_i)_{i\in I}]$

$$\begin{aligned}
&\Longrightarrow && \exists t_i \in P_i,\ A \in (x_i)_{i\in I} \oplus_r (y_i + z_i + t_i)_{i\in I} \\
&\Longrightarrow && \exists t_i' \in P_i,\ A = (x_i + y_i + z_i + t_i + t_i')_{i\in I} \\
&\Longrightarrow && \exists w_i \in P_i,\ A \\
&= && (x_i + y_i + w_i + z_i + t_i')_{i\in I} \quad (\text{ since } z_i + P_i = P_i + z_i\ ) \\
&\in && (x_i + y_i + w_i)_{i\in I} \oplus_r (z_i)_{i\in I} \\
&\subseteq && [(x_i)_{i\in I} \oplus_r (y_i)_{i\in I}] \oplus_r (z_i)_{i\in I} \\
&\Longrightarrow && (x_i)_{i\in I} \oplus_r [(y_i)_{i\in I} \oplus_r (z_i)_{i\in I}] \subseteq [(x_i)_{i\in I} \oplus_r (y_i)_{i\in I}] \oplus_r (z_i)_{i\in I}.
\end{aligned}$$

Similarly, we can prove that the reverse inclusion.

Clearly $(\prod_{i\in I} R_i, \odot)$ is a semigroup. Also the weak distributivity is obtained similar to the proof of Proposition 4.1. Therefore $(\prod_{i\in I} R_i, \oplus_r, \odot)$ is a semihyperring. Analogously we can prove that $(\prod_{i\in I} R_i, \oplus_l, \odot)$ is a weak distributive semihyperring. This completes the proof. $\square$

Remark. In Propositions 4.1 and 4.2, if we replace the conditions $R_iP_i \subseteq P_i$ and $P_iR_i \subseteq P_i$ by the condition $r_iP_i = P_i = P_ir_i$, for all $r_i \in R_i$ and for all $i \in I$, then $(\prod_{i\in I} R_i, \oplus_c, \odot)$, $(\prod_{i\in I} R_i, \oplus_r, \odot)$ and $(\prod_{i\in I} R_i, \oplus_l, \odot)$ will be semihyperrings.

Proposition 4.3. If $\{R_j\}_{j\in J}$ is a family of semirings and for all $j \in J$, $P_j \subseteq R_j$ is nonempty such that $R_jP_j \subseteq P_j$ and $P_jR_j \subseteq P_j$. Then I is a k-hyperideal of $(\prod_{j\in J} R_j, \oplus_c, \odot)$ if and only if $I = \prod_{j\in J} I_j$ such that $I_j \lhd_{k.h} (R_j, \oplus_{c_j}, \odot_j)$, where

$$x_j \oplus_{c_j} y_j = \{x_j + t_j + y_j \ \mid\ t_j \in P_j\},$$

$$x_j \odot_j y_j = x_j y_j.$$

Proof. ($\Longrightarrow$) For all $j \in J$ define

$$I_j = \{x \in R_j \ \mid \ (x_i)_{i\in J} \in I, \ \exists x_i \in R_i, \ x = x_j\}.$$

We have

$$\begin{aligned}
x, y \in I \ &\Longrightarrow \ \exists x_i, y_i \in R_i, \ (x_i)_{i\in J}, (y_i)_{i\in J} \in I, \ x = x_j, y = y_j \\
&\Longrightarrow \ (x_i)_{i\in J} \oplus_c (y_i)_{i\in J} \subseteq I \qquad (I \lhd_h (\prod_{j\in J} R_j, \oplus_c, \odot)) \\
&\Longrightarrow \ \forall t_i \in P_i, \ (x_i + t_i + y_i)_{i\in J} \in I \qquad (\forall i \in J) \\
&\Longrightarrow \ \forall t_j \in P_j, \ x + t_j + y \in I_j \\
&\Longrightarrow \ x \oplus_{c_j} y \subseteq I_j.
\end{aligned}$$

Now suppose that

$$\begin{aligned}
r_j \in R_j, x \in I_j \ &\Longrightarrow \ \exists r_i \in R_i, \ (r_i)_{i\in J} \in \prod_{i\in J} R_i \text{ and } \exists x_i \in R_i, \ (x_i)_{i\in J} \in I, x = x_j \\
&\Longrightarrow \ (r_i)_{i\in J} \odot (x_i)_{i\in J} \in I \qquad (\ I \lhd_h (\prod_{i\in J} R_i, \oplus_c, \odot) \ ) \\
&\Longrightarrow \ (r_i x_i)_{i\in J} \in I \\
&\Longrightarrow \ r_j x_j \in I_j \qquad (\text{ by definition of } I_j \ ) \ .
\end{aligned}$$

Therefore $I_j \lhd_h R_j$.

We now show that $I_j \lhd_{k.h} R_j$ for all $j \in J$. We have

$$\begin{aligned}
r_j \in R_j, \ x_j \in I_j, \ r_j \oplus_{c_j} x_j \approx I_j \ &\Longrightarrow \ \exists t_j \in P_j, \ r_j + t_j + x_j \in I_j \\
&\Longrightarrow \ (r_j)_{j\in J} \oplus_c (x_j)_{j\in J} \approx I,
\end{aligned}$$

where $(r_j)_{j\in J} \in \prod_{j\in J} R_j, \ (x_j)_{j\in J} \in \prod_{j\in J} I_j$. Then since $I \lhd_{k.h} (\prod_{j\in J} R_j, \oplus_c, \odot)$ we have

$$\begin{aligned}
(r_j)_{j\in J} \in I \ &\Longrightarrow \ r_j \in I_j, \ \ \forall j \in J \\
&\Longrightarrow \ I_j \lhd_{k.h} R_j.
\end{aligned}$$

($\Longleftarrow$) Suppose that $I = \prod_{j\in J} I_j$ such that $I_j \lhd_{k.h} (R_j, \oplus_{c_j}, \odot_j)$. First we prove $I \lhd_h$ $(\prod_{j\in J} R_j, \oplus_c, \odot)$. Let $(x_j)_{j\in J}, (y_j)_{j\in J} \in I$, then

$$(x_j)_{j\in J} \oplus_c (y_j)_{j\in J} = \{(x_j + t_j + y_j)_{j\in J} \ \mid \ t_j \in P_j\} \subseteq \prod_{j\in J} I_j;$$

also we have

$$\begin{aligned} I_j \lhd_h (R_j, \oplus_{c_j}, \odot_j) &\Longrightarrow \forall t_j \in P_j,\ x_j + t_j + y_j \in I_j \\ &\Longrightarrow (x_j)_{j\in J} \oplus_c (y_j)_{j\in J} \subseteq I. \end{aligned}$$

Now if $(r_j)_{j\in J} \in \prod_{j\in J} R_j$ and $(x_j)_{j\in J} \in I$, then $(r_j)_{j\in J} \odot (x_j)_{j\in J} = (r_j x_j)_{j\in J} \in \prod_{j\in J} I_j$, since $r_j x_j \in I_j$ by hypothesis. We now prove that $I \lhd_{k.h} (\prod_{j\in J} R_j, \oplus_c, \odot)$. For this we have

$$(r_j)_{j\in J} \in \prod_{j\in J} R_j,\ (x_j)_{j\in J} \in I,\ (r_j)_{j\in J} \oplus_c (x_1, x_2) \approx I$$

$$\begin{aligned} &\Longrightarrow \exists t_j \in P_j,\ (r_j + t_j + x_j)_{j\in J} \in I = \prod_{j\in J} I_j \\ &\Longrightarrow \exists t_j \in P_j,\ r_j + t_j + x_j \in I_j,\quad \forall j \in J \\ &\Longrightarrow r_j \oplus_{c_j} x_j \approx I_j,\ r_j \in R_j,\ x_j \in I_j \\ &\Longrightarrow r_j \in I_j \qquad\qquad (\ I_j \lhd_{k.h} (R_j, \oplus_{c_j}, \odot_j)\ ) \\ &\Longrightarrow (r_j)_{j\in J} \in \prod_{j\in J} I_j. \ \square \end{aligned}$$

Proposition 4.4. Let $\{R_j\}_{j\in J}$ be a family of semirings. Suppose that $P_j \subseteq R_j$ be nonempty such that $R_j P_j \subseteq P_j$ and $P_j R_j \subseteq P_j$ and $x_j + P_j = P_j + x_j$, for all $x_j \in R_j$ and for all $j \in J$. Then I is a k-hyperideal of $(\prod_{j\in J} R_j, \oplus_r, \odot)$ (resp. $(\prod_{j\in J} R_j, \oplus_l, \odot)$) if and only if $I = \prod_{j\in J} I_j$ such that for all $j \in J$, $I_j \lhd_{k.h} (R_j, \oplus_{r_j}, \odot_j)$, (resp. $I_j \lhd_{k.h} (R_j, \oplus_{l_j}, \odot_j)$), where

$$x_j \oplus_{r_j} y_j = \{x_j + y_j + t_j \ \mid\ t_j \in P_j\},$$

$$x_j \oplus_{l_j} y_j = \{t_j + x_j + y_j \ \mid\ t_j \in P_j\},$$

$$x_j \odot_j y_j = x_j y_j.$$

Proof. The proof is similar to the proof of Proposition 4.3. $\square$

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