Ratio Mathematica Volume 46, 2023

Some new odd prime graphs

S. Meena[*]
G. Gajalakshmi[†]

Abstract

For a graph G, a bijection f is called an odd prime labeling , if f from V to $\{1, 3, 5, ..., 2|V| - 1\}$ for each edge uv in G the greatest common divisor of the labels of end vertices $(f(u), f(v))$ is one. In this paper we investigate the existence of odd prime labeling of some new classes of graphs and we prove that the graphs such as the $Z - P_n$ graph, Fish graph, Umbrella graph, Cocount tree , F-tree, Y-tree and Double Sunflower graph are odd prime graphs.
Keywords: Odd prime graph, $Z - P_n$ graph, Fish graph, Umbrella graph,Cocount tree , F-tree, Y-tree, Double Sunflower graph.
2020 AMS subject classifications: 05C78 [1]

[*]Department Of Mathematics, Govt, Arts & Science College, Chidambaram 608 102, India; meenasaravanan14@gmail.com.
[†]Department Of Mathematics, Govt, Arts & Science College, Chidambaram; gaja61904@gmail.com.
[1]Received on September 15, 2022. Accepted on December 15, 2022. Published on March 20, 2023. DOI: 10.23755/rm.v46i0.1056. ISSN: 1592-7415. eISSN: 2282-8214. ©S. Meena et al. This paper is published under the CC-BY licence agreement.

1 Introduction

In this paper by a graph $G = \langle V(G), E(G) \rangle$ we mean a simple graph. For graph theoretical terminologies we refer J .A.Bondy and U .S. R.Murthy [1976].

A graph labeling is an assignment of integers to the vertices or edges or both subject to some constraints. For entire survey of graph labeling we refer Gallian [2009]. The concept of prime labeling was introduced by Roger Etringer and was studied further by many researchers Deretsky et al. [1991], Tout et al. [1982].Meena and Vaithilingam [2013] proved that crown related graphs are prime graphs. Meena and Naveen [2018] investigated about the prime labeling of graphs related to bicyclic graphs. Prime labeling in the concept of graph operation was discussed by Meena and Kavitha [2015]. The Existence of odd prime labeling for some new classes of graph was discussed by Meena et al. [2021].

The notion of odd prime labeling was introduced by Prajapati and Shah [2018] and they have proved many researchers. We give some families of odd prime graphs and some necessary condition for a graph to be odd prime graph.

Motivated by this study, further studied by in this paper we investigate the existence of odd prime labeling of some new classes of odd prime graphs.

Definition 1.1. *Let $G = \langle V(G), E(G) \rangle$ be a graph. A bijection $f : V(G) \to O_{|V|}$ is called an odd prime labeling if for each edge $uv \in E$, greatest common divisor $\langle f(u), f(v) \rangle is 1$. A graph is called an odd prime graph if its admits odd prime labeling.*
Here $O_{|V|}$ = $\{1, 3, 5, ...2|V| - n\}$

Definition 1.2. *$Z - P_n$ is a graph obtained from a pair of P_1 and P_2 of path of length n in which the i^{th} vertex of a path P_1 is joined with $(i + 1)^{th}$ vertex of a path P_2.*

Definition 1.3. *Fish graph is a graph obtained by attaching one of the vertex of K_3 to any one of the vertex of C_n.It is denoted by $C_n@K_3$.*

Definition 1.4. *For any integers $m > 2$, $n > 1$.An umbrella graph $U(m, n)$ is the graph obtained by identifying the end vertex of path P_n with the central vertex of a Fan graph F_m.*

Definition 1.5. *Coconut tree graph is obtained by identifying the central vertex of $K_{1,m}$ with a pendant vertex of the path P_n.*

Definition 1.6. *F- tree on $n + 2$ vertices, denoted by F_n is obtained from a path P_n by attaching exactly two pendant vertices of the $n - 1$ and n^{th} vertex of P_n.*

Definition 1.7. *Y - tree on $n+1$ vertices, denoted by Y_n is obtained from a path P_n by attaching a pendant vertex of the n^{th} vertex of P_n.*

Definition 1.8. *A double sunflower graph order n, denoted by DSF_n, is a graph obtained from the graph SF_n by intserting a new vertex C_i on each edges a_ia_{i+1} and adding edges for each i.*

2 Main results

Theorem 2.1. *$Z-(P_n)$ is an odd prime graph for all integers $n \geq 3$.*

Proof. Let G = $Z-(P_n)$ be the graph
$V(G) = \{u_i, v_i/1 \leq i \leq n\}$
$E(G) = \{(u_iu_{i+1}), (v_iv_{i+1})/1 \leq i \leq n-1\} \cup \{(v_iu_{i+1})/1 \leq i \leq n-1\}$
Now $|V(G)|$ = 2n and $|E(G)|$ = 3(n-1)
Define a labeling $f : V \to O_{2n}$ as follows
$f(u_i)$ = 4i-5 for $1 \leq i \leq n$
$f(v_i)$ = 4i+1 for $1 \leq i \leq n$
Clearly vertex labels are distinct.
For each $e = uv \in E$, if $gcd(f(u), f(v)) = 1$
(i) $e = u_1u_2$, $gcd(f(u_1), f(u_2))$ = $gcd(1, 3) = 1$
(ii) $e = u_iu_{i+1}$, $gcd(f(u_i), f(u_{i+1}))$ = $gcd(4i-5, 4i-1) = 1$ for $1 \leq i \leq n$
(iii)$e = v_iv_{i+1}$, $gcd(f(v_i), f(v_{i+1}))$ = $gcd(4i+1, 4i+5) = 1$ for $1 \leq i \leq n$
(iv) $e = v_iu_{i+1}$, $gcd(f(v_i), f(u_{i+1}))$ = $gcd(4i+1, 4i-5) = 1$ for $1 \leq i \leq n$
(v) $e = v_{n-1}v_n$, $gcd(f(v_{n-1})), f(v_n)$ = $gcd(4n-3, 4n-1) = 1$ for $1 \leq i \leq n$
Thus f admits odd prime labeling on $Z-(P_n)$ and hence $Z-(P_n)$ is an odd prime graph. ☐

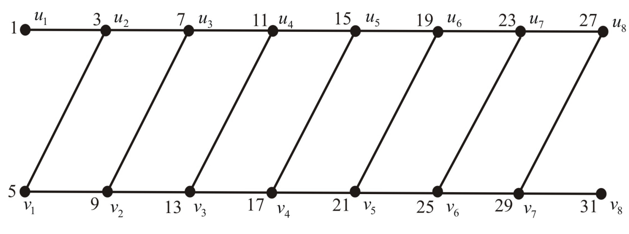

Figure 1: $Z-(P_n)$ and its odd prime labeling

Theorem 2.2. *Fish graph is an odd prime graph for $n \geq 3$.*

Proof. Let G = $C_n@K_3$ be the graph
$V(G)$ = $\{u_i/1 \leq i \leq n\} \cup \{(v_1v_2)\}$
$E(G)$ = $\{(u_iu_{i+1})/1 \leq i \leq n-1\} \cup \{(u_1u_n)\} \cup \{u_1v_i/1 \leq i \leq 2\} \cup \{(v_1, v_2)\}$

Now $|V(G)|$= $n+2$ and $|E(G)|$ = $n+3$
Define a labeling $f : V \rightarrow O_{n+2}$ as follows.
$f(u_1) = 1$
$f(v_1) = 3$
$f(v_2) = 5$
$f(u_i) = 2i + 3$ for $2 \leq i \leq n$
Clearly vertex labels are distinct.
For each $e = uv \in E$, if $gcd(f(u), f(v)) = 1$
(i) $e = v_1v_2, gcd(f(v_1), f(v_2)) = gcd(3, 5) = 1$
(ii) $e = u_1v_1, gcd(f(u_1), f(v_1)) = gcd(1, 3) = 1$ for $1 \leq i \leq n$
(iii)$e = u_1v_2, gcd(f(u_1), f(v_2)) = gcd(1, 5) = 1$ for $1 \leq i \leq n$
(iv) $e = u_iu_{i+1}, gcd(f(u_i), f(u_{i+1}) = gcd(2i + 3, 2i + 5) = 1$ for $2 \leq i \leq n - 1$
(v) $e = u_1u_2, gcd(f(u_1), f(u_2) = gcd(1, f(u_2) = 1$
(vi) $e = u_1u_n, gcd(f(u_1), f(u_n) = gcd(1, f(u_n) = 1$
Thus f admits odd prime labeling on $C_n@K_3$ and hence Fish graph is an odd prime graph. □

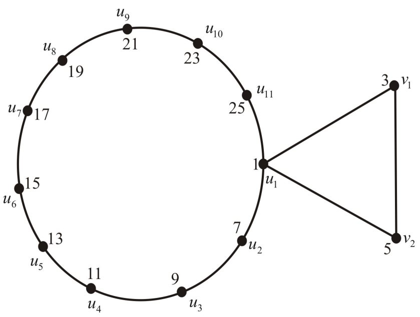

Figure 2: Fish graph $C_n@K_3$ and its odd prime labeling

Theorem 2.3. *The Umbrella graph $U(m, n)$ is an odd prime graph.*

Proof. Consider the umbrella graph $U(m, n)$ with vertex set.
$V(U(m, n))$ = $\{x_i, y_i/1 \leq i \leq m, 1 \leq i \leq n\}$
$E(U(m, n))$ = $\{x_ix_{i+1}/1 \leq i \leq m - 1\}$
$\cup \{x_iy_1/1 \leq i \leq m\} \cup \{y_iy_{i+1}/1 \leq i \leq n - 1\}$
Now $|V(U(m, n))|$ = $m + n$ and $|E(U(m, n))|$ = $2m + n - 2$
Define $f : V \rightarrow O_{m+n}$ as follows.
$f(x_i)$ = $2i + 1$ for $1 \leq i \leq m$
$f(y_1)$ = 1
$f(y_i)$ = $2(i + m) - 1$ for $2 \leq i \leq n$
Clearly vertex labels are distinct. With this labeling for each $e = uv \in E$, if $gcd(f(u), f(v)) = 1$.

(i) $e = x_i x_{i+1}, gcd(f(x_i), f(x_{i+1})) = gcd(2i{+}1, 2i{+}3) = 1$ for $1 \leq i \leq m{-}1$
(ii) $e = x_i y_1, gcd(f(x_i), f(y_1)) = gcd(2i + 1, 1) = 1$ for $1 \leq i \leq m$
(iii)$e = y_1 y_2, gcd(f(y_1), f(y_2)) = gcd(1, 2m + 3) = 1$
(iv) $e = y_i y_{i+1}, gcd(f(y_i), f(y_{i+1}) = gcd(2(i + m) - 1, 2(i + m) + 1) = 1$ for $2 \leq i \leq n - 1$
as they are consecutive odd integers.
This f is a odd prime labeling on $U(m, n)$ and hence it is an odd prime graph. □

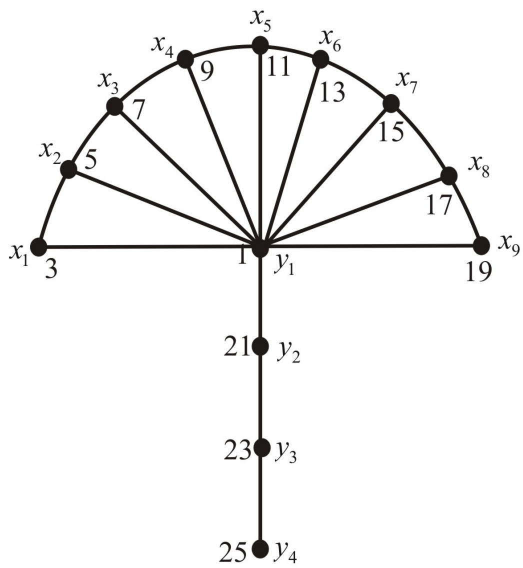

Figure 3: $U(m,n)$ and its odd prime labeling

Theorem 2.4. *Cocount tree $CT(m, n)$ is an odd prime graph.*

Proof. $V(G) = \{u_i, v_i/1 \leq i \leq m, 1 \leq i \leq n\}$ and edge set
$E(G) = \{u_i v_1/1 \leq i \leq m\} \cup \{v_i v_{i+1}/1 \leq i \leq n - 1\}$
Now $|V(CT(m, n))| = m + n$ and $|E(CT(m, n))| = m + n - 1$
Define a labeling $f : V(G) \to \{1, 3, 5, ...2m + 2n - 1\}$ as follows
$f(u_i) = 2(n + i) - 1$ for $1 \leq i \leq m$
$f(v_i) = 2i - 1$ for $1 \leq i \leq n$
Clearly vertex labels are distinct.
For each $e = uv \in E$, if $gcd(f(u), f(v)) = 1$
(i) $e = u_i v_1, gcd(f(u_i), f(v_1)) = gcd(2(n + 1) - 1, 1) = 1$ for $1 \leq i \leq n$;
(ii) $e = v_i v_{i+1}, gcd(f(v_i), f(v_{i+1})) = gcd(2i - 1, 2i + 1)$=1 for $1 \leq j \leq m - 1$.
Thus f admits odd prime labeling on $CT(m, n)$ and hence $CT(m, n)$ is an odd prime graph. □

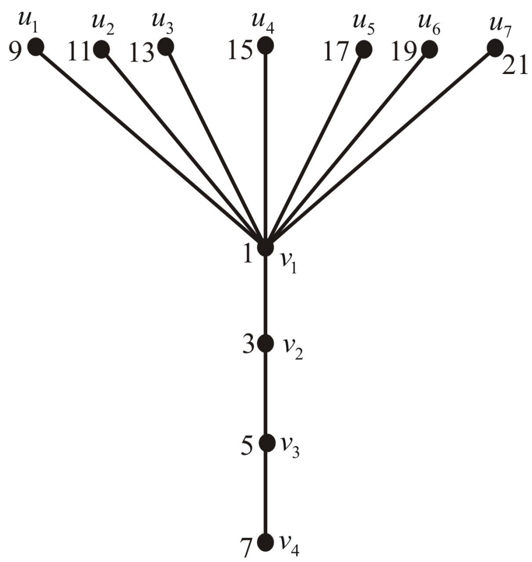

Figure 4: $CT(m, n)$ and its odd prime labeling

Theorem 2.5. *Let G be the graph obtained by identifying a pendant vertex of P_m with a leaf of $K_{1,n}$ then G is an odd prime graph for all m and n.*

Proof. $V(G)$ =$\{u, u_i, v_j/1 \leq i \leq n, 2 \leq i \leq m\}$ and the edge set
$E(G)$ = $\{uv_i/1 \leq i \leq n\}$ $\cup\{v_jv_{j+1}/2 \leq j \leq m-1\}$ $\cup\{uv_2\}$
Here $u = v_1$
Now $|V(G)|$ =m+n and $|E(G)| = m + n - 1$
Define a labeling $f : V(G) \to \{1, 3, 5, ...2m + 2n - 1\}$ as follows
$f(u)$ = 1
$f(u_i) = 2i + 1$ for $1 \leq i \leq n$
$f(v_i) = 2(n + i) - 1$ for $2 \leq i \leq n$
Clearly the vertex labels are distinct.
For each $e = uv \in E$ if $gcd(f(u), f(v)) = 1$
(i) $e = uu_i$, $gcd(f(u), f(u_i))$ = $gcd(1, 2i + 1) = 1$ for $1 \leq i \leq n$;
(ii) $e = uv_2$, $gcd(f(u), f(v_2))$ = $gcd(1, 2n + 3) = 1$ for $1 \leq i \leq n$;
(iii)$e = v_iv_{i+1}$, $gcd(f(v_i), f(v_{i+1}))$= $gcd(2(n + i) - 1, 2(i + n) + 1)$ =1 for $2 \leq i \leq n$;
Thus f admits odd prime labeling on G and hence G is an odd prime graph. □

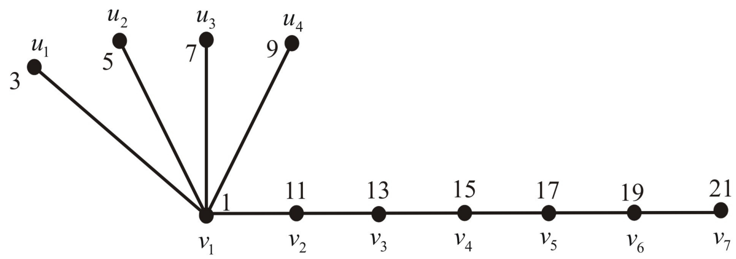

Figure 5: G and its odd prime labeling

Theorem 2.6. *F- tree FP_n $n \geq 3$ is an odd prime graph.*

Proof. Let $V(G) = \{u, v, v_i, /1 \leq i \leq n-1\}$
$E(G) = \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{uv_2, vv_1\}$
be the vertex set and edge set of FP_n
Now $|V(FP_n)| = n+2$ and $|E(FP_n)| = n+1$
Define a labeling $f : V(G) \to \{1, 3, 5, ...2n+3\}$ as follows
$f(u) = 3$
$f(v) = 1$
$f(v_i)$ = 2i+3 for $1 \leq i \leq n$
Clearly vertex labels are distinct.
For each $e = uv \in E$ if $gcd(f(u), f(v)) = 1$
(i) $e = vv_1, gcd(f(v), f(v_1)) = gcd(1, 5) = 1$
(ii)$e = uv_2, gcd(f(u), f(v_2)) = gcd(3, 7) = 1$
(iii)$e = v_iv_{i+1}, gcd(f(v_i), f(v_{i+1})) = gcd(2i+3, 2i+5) = 1$
Thus f admits odd prime labeling on FP_n and hence F- tree FP_n is an odd prime graph. □

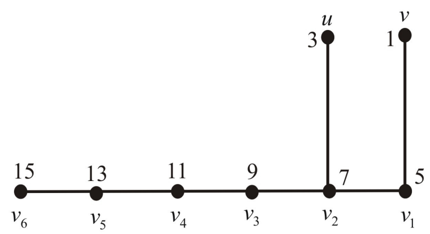

Figure 6: FP_n and its odd prime labeling

Theorem 2.7. *Y- tree is an odd prime graph.*

Proof. Let $V(G) = \{uv_i, /1 \leq i \leq n\}$
$E(G) = \{v_iv_{i+1}, v_{n-1}u \ /1 \leq i \leq n-1\}$ be the vertex set and edge set of y-tree
Now $|V(G)| = n+1$ and $|E(G)| = n$

Define a labeling $f : V(G) \to \{1, 3, 5, ...2n + 1\}$ as follows
$f(u)$ = 2n+1
$f(v_i)$ = 2i-1 for $1 \leq i \leq n$
Clearly vertex labels are distinct.
For each $e = uv \in E$ if $gcd(f(u), f(v)) = 1$
(i) $e = uv_{n-1}, gcd(f(u), f(v_{n-1}))$ = $gcd(2n + 1, 2n - 1) = 1$
(iii) $e = v_iv_{i+1}, gcd(f(v_i), f(v_{i+1}))$ = $gcd(2i - 1, , 2i + 1) = 1$ for $1 \leq i \leq n - 1$
Thus f admits odd prime labeling on Y-tree and hence Y- tree is an odd prime graph. □

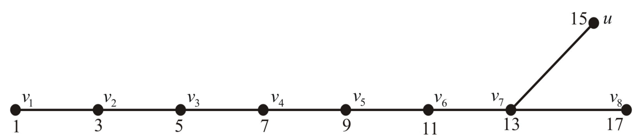

Figure 7: Y-tree and its odd prime labeling

Theorem 2.8. *For any natural numbers $k \geq 3$ graph DSF_n is an odd prime graph.*

Proof. The vertex set and edge set of DSF_n of order K respectively are
$V(DSF_n)$ = $\{l_i, m_i, n_i/1 \leq i \leq k\}$
$E(DSF_n)$ = $\{l_im_i, l_in_i, m_in_i, m_il_{i+1}, l_km_k, l_kn_n, l_1n_k/1 \leq i \leq n - 1\}$
Now $|V(DSF_n)| = 3k$ and $|E(DSF_n)| = 5k$
Define a labeling $f : V(DSF_n) \to \{1, 3, 5, ...6n - 1\}$ as follows
$f(l_i) = 6i - 5$ for $1 \leq i \leq n$
$f(m_i) = 6i - 3$ for $1 \leq i \leq n$
$f(n_i) = 6i - 1$ for $1 \leq i \leq n$
Clearly all the vertex labels are distinct. With this labeling for each $e = uv \in E$ if $gcd(f(u), f(v)) = 1$
(i) $e = l_im_i, gcd(f(l_i), f(m_i)) = gcd(6i - 5, 6i - 3) = 1$ for $1 \leq i \leq n$
(ii) $e = l_in_i, gcd(f(l_i), f(n_i)) = gcd(6i - 5, 6i - 1) = 1$ for $1 \leq i \leq n$
(iii) $e = m_in_i, gcd(f(m_i), f(n_i) = gcd(6i - 3, 6i - 1) = 1$ for $1 \leq i \leq n$
(iv) $e = m_il_{i+1}, gcd(f(m_i), f(l_{i+1})) = gcd(6i - 3, 6i - 5) = 1$ for $1 \leq i \leq n - 1$
(v) $e = m_kl_1, gcd(f(m_k), f(l_1)) = gcd(f(m_k), f(l_1)) = 1$
Thus f is a odd prime labeling on DSF_n and hence DSF_n is an odd prime graph. □

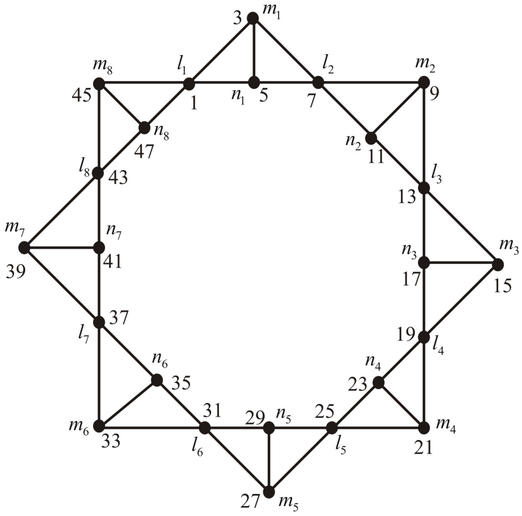

Figure 8: DSF_n and its odd prime labeling

3 Conclusions

Odd Prime labelings of various classes of graphs such as $Z - P_n$ graph, Fish graph, Umbrella graph, Cocount tree, F-tree, Y-tree and Double Sunflower graph are investigated. To derive similar results for other graph families is an open area of research.

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