A HYPEROPERATION DEFINED ON A GROUPOID EQUIPPED WITH A MAP

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ABSTRACT

The H_v*-structures* are hyperstructures where the equality is replaced by the non-empty intersection. The fact that this class of the hyperstructures is very large, one can use it in order to define several objects that they are not possible to be defined in the classical hypergroup theory. In the present paper we introduce a kind of hyperoperations which are defined on a set equipped with an operation or a hyperoperation and a map on itself.

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1. Introduction

The object of this paper is the hyperstructures called H_v*-structures* introduced in 1990 [5], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Recall some basic definitions:

Definitions 1. In a set H equipped with a hyperoperation $\cdot:H\times H\to \boldsymbol{P}(H)$, we abbreviate by *WASS* the *weak associativity*: $(xy)z\cap x(yz)\neq\emptyset,\ \forall x,y,z\in H$ and by *COW* the *weak commutativity*: $xy\cap yx\neq\emptyset,\ \forall x,y\in H$. The hyperstructure $(H,\cdot)$ is called H_v*-semigroup* if it is WASS, is called H_v*-group* if it is reproductive H_v-semigroup. The hyperstructure $(R,+,\cdot)$ is called H_v*-ring* if (+) and $(\cdot)$ are WASS, the reproduction axiom is valid for (+) and $(\cdot)$ is *weak distributive* with respect to (+): $x(y+z)\cap(xy+xz)\neq\emptyset$, $(x+y)z\cap(xz+yz)\neq\emptyset$,

$\forall x,y,z \in R$. *H_v-modulus* and *H_v-vector spaces* are also defined in a similar way.

For more definitions, results and applications on H_v-structures, see books [6,2] and on some papers such as [3-11]. A special class [6]: An H_v-structure is called *very thin* iff all its hyperoperations are operations except one, which all hyperproducts are singletons except only one, which has cardinality more than one.

The fundamental relations β^*, γ^* and ε^* are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively (see [1,6]). The way to find the fundamental classes is given by analogous theorems to the following [5,6,7]:

Theorem. Let $(H,\cdot)$ be H_v-group and let us denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset \boldsymbol{u}$ where $\boldsymbol{u}\in \boldsymbol{U}$. Then the fundamental relation β^* is the transitive closure of β.

Proof. The main point is: Take x,y such that $\{x,y\}\subset \boldsymbol{u}\in \boldsymbol{U}$ and any hyperproduct where one of the elements x,y, is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the hyperproducts of the above β-classes are products, then, they are fundamental classes. Analogous remarks for the relations γ^*, in H_v-rings, and ε^*, in H_v-vector spaces, are also applied.

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, to define the H_v-field the γ^* is used: An H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field [5], and in the sequence the general *H_v-vector space* is defined.

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an automorphism $f\in Aut(H,*)$ such that $xy\subset f(x*y)$, $\forall x,y\in H$. Then we write $\cdot\leq*$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called H_b*-structure*.

Theorem. Greater hyperoperations of the ones which are WASS or COW, are also WASS or COW, respectively.

Remark 2. The weak axioms lead to a great number of hyperoperations and these hyperoperations define hyperstructures which can be now studied in detail and, in any case, they have a substance; hence they can be considered as hyperstructures with interesting properties. These are many hyperoperations which, in the past, were unlikely to be considered because not even one property was valid in them. We can see that the hyperoperations introduced here are associative only in very special cases and before 1990 such hyperoperations could hardly be considered, even though they appeared in the research. Nevertheless, the created theory can now give results and discover new properties of the obtained hyperstructure. Thus, algebraic domains reveal constructions which seem to be chaotic. Even more so, in certain cases, some of these hyperstructures contain well known structures or hyperstructures, see also [11,12].

This remark follows that constructions and hyper-constructions are needed to be enlarged or to become smaller and we can do this:

Definitions 3. Let $(H,\cdot)$ be a hypergroupoid. We say that *remove* $h\in H$, if we consider the restriction of $(\cdot)$ in $H-\{h\}$. We say that $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$. We say that $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

Most of these constructions are needed in the representation theory. Representations of H_v-groups can be considered either by generalized permutations or by H_v-matrices [6]. The representation problem by H_v-matrices is the following:

H_v*-matrix* is a matrix with entries of an H_v-ring. The hyperproduct of H_v-matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type $m\times n$ and $n\times r$, respectively, is a set of $m\times r$ H_v-matrices:

$$\mathbf{A}\cdot\mathbf{B} = (a_{ij})\cdot(b_{ij}) = \{ \mathbf{C} = (c_{ij}) \mid c_{ij} \in \oplus\boldsymbol{\Sigma} a_{ik}\cdot b_{kj} \},$$

where $\oplus$ denotes the *n-ary circle hyperoperation* on the hyperaddition.

Definition 4. Let $(H,\cdot)$ be H_v-group, take a H_v-ring $(R,+,\cdot)$ and a set $\mathbf{M}_R = \{ (a_{ij}) \mid a_{ij}\in R \}$, then any map

$$\mathbf{T}: H\to\mathbf{M}_R: h\to\mathbf{T}(h) \text{ with } \mathbf{T}(h_1h_2)\cap\mathbf{T}(h_1)\mathbf{T}(h_2)\neq\varnothing,\ \forall h_1,h_2\in H,$$

is a *H_v-matrix representation*. If $\mathbf{T}(h_1h_2)\subset\mathbf{T}(h_1)\mathbf{T}(h_2)$, then $\mathbf{T}$ is an *inclusion*, if $\mathbf{T}(h_1h_2)=\mathbf{T}(h_1)\mathbf{T}(h_2)$, then $\mathbf{T}$ is a *good* and an induced representation for the hypergroup algebra is obtained.

In the same attitude recently we defined, using hyperstructure theory, hyperoperations on any type of matrices:

Definition 5 [12]. Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ be matrix and $s,t\in N$ with $1\leq s\leq m$, $1\leq t\leq n$. Then *helix-projection* is a map $\underline{st}:\mathbf{M}_{m\times n}\to\mathbf{M}_{s\times t}: A\to A\underline{st}=(\underline{a}_{ij})$, where $A\underline{st}$ has entries

$$\underline{a}_{ij} = \{ a_{i+\kappa s, j+\lambda t} \mid 1\leq i\leq s,\ 1\leq j\leq t \text{ and } \kappa,\lambda\in N,\ i+\kappa s\leq m,\ j+\lambda t\leq n \}$$

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ be matrices, $s=\min(m,u)$, $t=\min(n,v)$. We define a hyper-addition, called *helix-addition*, as follows

$$\oplus:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{s\times t}):(A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset\mathbf{M}_{s\times t}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})=(a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ and $B=(b_{ij})\in\mathbf{M}_{u\times v}$ be matrices and $s=\min(n,u)$. We define a hyper-multiplication, called *helix-multiplication*, as follows

$$\otimes:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{m\times v}):(A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset\mathbf{M}_{m\times v}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

The helix-addition is commutative, is WASS, not associative. The helix-multiplication is WASS, not associative and it is not distributive, not even weak, to the helix-addition. If all used matrices are of the same type, then the inclusion distributivity, is valid.

From the definition of representations by H_v-matrices, we have two difficulties. The first one is to find an appropriate H_v-ring and the second one is to find an appropriate set of H_v-matrices. However, with the above hyper-multiplication we can use subsets of matrices of type $\mathbf{M}_{m\times n}$ with $m\neq n$. Thus, the representation problem is reduced, as in the classical theory, in searching appropriate sets from usual matrices. This is so, because we have now a hyperalgebra over non-square matrices.

2. New hyperoperations

We will define a hyperoperation in a groupoid equipped with a map f on it. The map plays crucial role so the hyperoperation is called *map* and it is denoted by ∂_f, because the motivation to obtain this is the property which the 'derivative' has on the product of functions. However, since there is no confusion, we will write simply *theta* ∂.

Definition 6. Let $(G,\cdot)$ be a groupoid (respectively, hypergroupoid) and $f:G\to G$ be any map. We define a hyperoperation (∂), we call it *theta-operation*, on G as follows

$$x\partial y = \{f(x)\cdot y,\ x\cdot f(y)\ \} \quad \text{(respectively, } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y))$$

If $(\cdot)$ is commutative then (∂) is also commutative. If $(\cdot)$ is a COW hyperoperation, then (∂) is also COW hyperoperation.

Remark. One can use instead of single valued map f, a multivalued map as well. We will not consider this problem here.

Remark. Motivation for this definition was the map 'derivative' where only the multiplication of functions can be used. In other words, if we 'do not know' the addition of functions. Therefore, for any functions s(x), t(x), we have $s\partial t=\{s't,\ st'\}$ where $(')$ denotes the derivative.

Properties 7. If $(G,\cdot)$ is a semigroup then:

(a) For every f, the hyperoperation (∂) is WASS.

(b) If f is homomorphism then, again, (∂) is WASS.

(c) If f is homomorphism and projection, or idempotent, i.e. $f^2=f$, then (∂) is associative.

Proof.

(a) For all x,y,z in G we have

$(x\partial y)\partial z = \{f(x)\cdot y, x\cdot f(y)\}\partial z =$

$= \{ f(f(x)\cdot y)\cdot z, (f(x)\cdot y)\cdot f(z), f(x\cdot f(y))\cdot z, (x\cdot f(y))\cdot f(z) \} =$

$= \{ f(f(x)\cdot y)\cdot z, f(x)\cdot y\cdot f(z), f(x\cdot f(y))\cdot z, x\cdot f(y)\cdot f(z) \}$

$x\partial(y\partial z) = x\partial\{f(y)\cdot z, y\cdot f(z)\} =$

$= \{ f(x)\cdot(f(y)\cdot z), x\cdot f(f(y)\cdot z), f(x)\cdot(y\cdot f(z)), x\cdot f(y\cdot f(z)) \} =$

$= \{ f(x)\cdot f(y)\cdot z, x\cdot f(f(y)\cdot z), f(x)\cdot y\cdot f(z), x\cdot f(y\cdot f(z)) \}$

Therefore $(x\partial y)\partial z \cap x\partial(y\partial z) = \{ f(x)\cdot y\cdot f(z) \} \neq \emptyset$, so ($\partial$) is WASS.

(b) If f is homomorphism then we obtain

$(x\partial y)\partial z = \{ f(f(x))\cdot f(y)\cdot z, f(x)\cdot y\cdot f(z), f(x)\cdot f(f(y))\cdot z, x\cdot f(y)\cdot f(z) \}$

$x\partial(y\partial z) = \{ f(x)\cdot f(y)\cdot z, x\cdot f(f(y))\cdot f(z), f(x)\cdot y\cdot f(z), x\cdot f(y)\cdot f(f(z)) \}$

So, again $(x\partial y)\partial z \cap x\partial(y\partial z)=\{f(x)\cdot y\cdot f(z)\}\neq\emptyset$ and (∂) is WASS.

(c) If f is homomorphism and projection then we have

$(x\partial y)\partial z = \{ f(x)\cdot f(y)\cdot z, f(x)\cdot y\cdot f(z), x\cdot f(y)\cdot f(z) \}=x\partial(y\partial z)$.

Therefore, (∂) is an associative hyperoperation.

Notice that only projection without homomorpthism does not give the associativity. Commutativity does not improve the results.

3. Properties and characteristic elements.

We will discuss now some properties in the general case where $(G,\cdot)$ be a groupoid and $f:G\to G$ be a map.

Properties 8.

Reproductivity. For the reproductivity we must have

$$x\partial G =\bigcup_{g\in G}\{f(x)\cdot g, x\cdot f(g)\}=G \text{ and } G\partial x =\bigcup_{g\in G}\{f(g)\cdot x, g\cdot f(x)\}=G.$$

Thus, if ($\cdot$) is reproductive then (∂) is also reproductive, because

$$\bigcup_{g\in G}\{f(x)\cdot g\}=G \quad\text{and}\quad \bigcup_{g\in G}\{g\cdot f(x)\}=G.$$

Commutativity. If $(\cdot)$ is commutative then $x\partial y=\{f(x)\cdot y,\ x\cdot f(y)\}=y\partial x$, so (∂) is commutative. If f is into $\boldsymbol{Z}_G=\{z\in G \mid z\cdot g=g\cdot z, \forall g\in G\}$, the centre of G, then (∂) is a commutative hyperoperation. If $(G,\cdot)$ is a COW hypergroupoid then, obviously (∂) is a COW hypergroupoid.

Unit elements. In order to have a right unit element u we must have $x\partial u=\{f(x)\cdot u,\ x\cdot f(u)\}\ni x$. But, the unit must not depend on the f(x), so we must have f(u)=e, where e be unit in $(G,\cdot)$ which must be a monoid. The same it is obtained for the left units. Therefore, the elements of the kernel of f, i.e. u for which f(u)=e, are the units of (G,∂).

Inverse elements. Let u be a unit in (G,∂), then $(G,\cdot)$ is a monoid with unit e and f(u)=e. For given x in order to have an inverse element x' with respect to u, we must have

$$x\partial x'=\{f(x)\cdot x',\ x\cdot f(x')\}\ni u \quad \text{and} \quad x'\partial x=\{f(x')\cdot x,\ x'\cdot f(x)\}\ni u.$$

So the only cases, which do not depend on the image $f(x')$, are

$$x' = (f(x))^{-1}u \quad \text{and} \quad x' = u(f(x))^{-1}$$

the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$. For example, if u belongs to the centre of G. In some cases, some elements may have a second inverse.

Proposition 9. Let $(G,\cdot)$ be a group and f(x)=a, a constant map on G. Then $(G,\partial)/\beta^*$ is a singleton.

Proof. For all x in G we can take the hyperproduct of the elements, a^{-1} and $a^{-1}x$

$$a^{-1}\partial(a^{-1}\cdot x) = \{f(a^{-1})\cdot a^{-1}\cdot x,\ a^{-1}\cdot f(a^{-1}\cdot x)\} = \{x,a\}.$$

thus $x\beta a$, $\forall x\in G$, so $\beta^*(x)=\beta^*(a)$ and $(G,\partial)/\beta^*$ is singleton. q.e.d.

Remark. If $(G,\cdot)$ be a group and f(x)=e, then we obtain $x\partial y=\{x,y\}$ which is the smallest incidence hyperoperation.

Remark. Every $f:G\to G$ defines a partition of G by setting two elements x,y in the same class iff f(x)=f(y), we shall call this partition f-partition and we will denote the class of x by f[x]. So, in the above Proposition, we have $f[x] = G = \beta^*(x)$ for all x in G.

Proposition 10. Let (G,·) be a group, e the unit, and f homomorphism, then for (G,∂), we have $x\beta f(x)$.

Proof. Indeed $e\partial x = \{f(e)\cdot x, e\cdot f(x)\} = \{x, f(x)\}$. q.e.d.

Obviously we have $x \beta f(x) \beta f(f(x)) \beta \ldots$

Theorem 11. Let (G,·) be a group and f be an homomorphism, then

$$f[x] \subset \beta^*(x) \text{ for all } x \text{ in } G.$$

Proof. Let $y \in f[x]$, then $f(y)=f(x)$ but from Proposition10, we have

$$x\beta f(x) = f(y)\beta y, \quad \text{so} \quad x\beta^* y. \qquad \text{q.e.d.}$$

4. Special cases and applications

In this paragraph we present some applications and we give some examples in order to see that a large field of research is open.

Application 12. Taking the application on the derivative, consider all polynomials of first degree $g_i(x) = a_i x+b_i$. We have

$$g_1\partial g_2 = \{a_1a_2x+ a_1b_1, a_1a_2x+b_1b_2\},$$

so it is a hyperoperation inside the set of first degree polynomials. Moreover all polynomials x+c, where c be a constant, are units.

Application 13. If R^+ be the set of positive reals and $a\in R^+$, then we take the exponential map $x\to x^a$. The theta-operation takes the form $x\partial y = \{ x^a y, xy^a\}$ for all x,y in R^+. The only one unit is the 1. In order to find the inverses x', of the element $x\in R^+$, we must have $x\partial x'=\{x^a x', x(x')^a\}\ni 1$. From which we obtain that for every element x, there are two inverses, the x^{-a} and $x^{-1/a}$.

Example 14. In the group ($\boldsymbol{Z}_5$-{0},·) we consider the map f: $\underline{1}\to\underline{1}$, $\underline{2}\to\underline{2}$, $\underline{3}\to\underline{3}$, $\underline{4}\to\underline{2}$. Then we obtain the multiplicative table

∂	1	2	3	4
1	1	2	3	{4, 2}

2	2	4	1	{3, 4}
3	3	1	4	{2, 1}
4	{4, 2}	{3, 4}	{2, 1}	3

We remark that there exists only one fundamental class. The map-hyperoperation is not associative but it is WASS, because, for example, $\underline{2}\partial(\underline{4}\partial\underline{4}) = \{\underline{1}\}$ and $(\underline{2}\partial\underline{4})\partial\underline{4} = \{\underline{1}, \underline{2}, \underline{3}\}$.

Example 15. Consider the group $(\boldsymbol{Z}_6,+)$ and the map $f:\boldsymbol{Z}_6\to\boldsymbol{Z}_6: x\to x^{-1}$. Then the map-operation is given from the table

∂	0	1	2	3	4	5
0	0	{1, 5}	{2, 4}	3	{2, 4}	{1, 5}
1	{1, 5}	0	{1, 5}	{2, 4}	3	{2, 4}
2	{2, 4}	{1, 5}	0	{1, 5}	{2, 4}	3
3	3	{2, 4}	{1, 5}	0	{1, 5}	{2, 4}
4	{2, 4}	3	{2, 4}	{1, 5}	0	{1, 5}
5	{1, 5}	{2, 4}	3	{2, 4}	{1, 5}	0

This is a commutative hyperoperation, it is WASS, because, for example, $\underline{1}\partial(\underline{1}\partial\underline{2}) = \{\underline{2}, \underline{4}\}$ and $(\underline{1}\partial\underline{1})\partial\underline{2} = \{\underline{0}, \underline{2}, \underline{4}\}$, so $(\boldsymbol{Z}_6,\partial)$ is a commutative H_v-group. One can obtain that

$$(\boldsymbol{Z}_6,\partial)/\beta^* = \{\{\underline{0}, \underline{2}, \underline{4}\},\{\underline{1}, \underline{3}, \underline{5}\}\} \cong \boldsymbol{Z}_2.$$

This is not cyclic since $x\partial x = \{\underline{0}\}$ for all x in $\boldsymbol{Z}_6$, i.e. every element has itself as the only one inverse element.

Example 16. Consider the group $(\boldsymbol{Z}_6,+)$ and the map

$$f: \underline{0}\to\underline{0}, \underline{1}\to\underline{1}, \underline{2}\to\underline{2}, \underline{3}\to\underline{3}, \underline{4}\to\underline{4}, \underline{5}\to\underline{2}.$$

Then the map-operation is given from the table

∂	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\{\underline{2}, \underline{5}\}$
$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\{\underline{0}, \underline{3}\}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\{\underline{1}, \underline{4}\}$
$\underline{3}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\{\underline{2}, \underline{5}\}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\{\underline{0}, \underline{3}\}$
$\underline{5}$	$\{\underline{2}, \underline{5}\}$	$\{\underline{0}, \underline{3}\}$	$\{\underline{1}, \underline{4}\}$	$\{\underline{2}, \underline{5}\}$	$\{\underline{0}, \underline{3}\}$	$\underline{1}$

One can obtain that

$$(\boldsymbol{Z}_6,\partial)/\beta^* = \{\{\underline{0}, \underline{3}\},\{\underline{1}, \underline{4}\},\{\underline{2}, \underline{5}\}\} \cong \boldsymbol{Z}_3.$$

$(\boldsymbol{Z}_6,\partial)$ is a cyclic H_v-group where $\underline{1}$ and $\underline{5}$ are generators of period 5.

Example 17. Consider the group $(\boldsymbol{Z}_6,+)$ and the map

$$f:\ \underline{0}\to\underline{0},\ \underline{1}\to\underline{1},\ \underline{2}\to\underline{2},\ \underline{3}\to\underline{3},\ \underline{4}\to\underline{2},\ \underline{5}\to\underline{5}.$$

Then the map-operation is given from the table

∂	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\{\underline{2}, \underline{4}\}$	$\underline{5}$
$\underline{1}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\{\underline{3}, \underline{5}\}$	$\underline{0}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\{\underline{0}, \underline{4}\}$	$\underline{1}$
$\underline{3}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\{\underline{1}, \underline{5}\}$	$\underline{2}$
$\underline{4}$	$\{\underline{2}, \underline{4}\}$	$\{\underline{3}, \underline{5}\}$	$\{\underline{0}, \underline{4}\}$	$\{\underline{1}, \underline{5}\}$	$\underline{0}$	$\{\underline{1}, \underline{3}\}$
$\underline{5}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\{\underline{1}, \underline{3}\}$	$\underline{4}$

One obtains that

$$(\boldsymbol{Z}_6,\partial)/\beta^* = \{\{\underline{0}, \underline{2}, \underline{4}\},\{\underline{1}, \underline{3}, \underline{5}\}\} \cong \boldsymbol{Z}_2.$$

For the reproductivity, the element $\underline{4}+\underline{4}$ which does not appeared in the normal position in the result it appears, in the general case, as follows:

$$x+x\in x\partial(x+x-f(x))=\{f(x)+x+x-f(x),\ x+f(x+x-f(x))\},\ \forall x\in \boldsymbol{Z}_6,$$

so the reproductivity is clear.

We conclude with a theorem on this field.

Theorem 18. Consider the commutative group of integers $(\boldsymbol{Z},+)$ and let $n\neq 0$ be a natural number. Take the map f such that $f(n)=0$ and $f(x)=x$ for all x in $\boldsymbol{Z}$-{n}. Then

$$(\boldsymbol{Z},\partial)/\beta^* \cong \boldsymbol{Z}_n.$$

Proof. First, for all x,y in $\boldsymbol{Z}$-{n} we have, for the theta-operation,

$$x\partial y = \{f(x)+y,\ x+f(y)\} = \{x+y\},$$

so the hypersum is a singleton and coincides with the usual sum in $\boldsymbol{Z}$.

For all x in $\boldsymbol{Z}$-{n} we have

$$x\partial n = n\partial x = \{f(x)+n,\ x+f(n)\} = \{x+n,\ x\}.$$

Finally $n\partial n = \{f(n)+n,\ n+f(n)\} = \{n\}$.

Therefore $x\beta(x+n)$. Moreover, from the above, we obtain that for all x,y in $\boldsymbol{Z}$, the hypersum $\{x,\ x+n\}\partial\{y,\ y+n\}$ belongs to the same class modn. Thus, the fundamental classes are the classes modn.

Therefore $(\boldsymbol{Z},\partial)/\beta^* \cong \boldsymbol{Z}_n$. q.e.d.

Remark that this construction is an analogous case to the case of the uniting the elements 0 and n, see [6].

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