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Strong forms of b-continuous multifunctions

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Abstract

In this paper we have introduced strong forms of b-continuous multifunctions namely $b^{\#}$-multicontinuity and *b-multicontinuity and studied their properties and characterizations. Also investigate the relationship with other type of functions with suitable examples.

Keywords: b-open, multi-function, u.$b^{\#}$-c, u.*b-c

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1. Introduction

Recently topologists concentrate their research in several types of continuous multi functions. A weak form of b-continuous multifunctions was studied in [4]. The variations of multi continuity were discussed in [5]. The weak and strong forms of continuity of multi functions were introduced in [6]. Certain properties of topological spaces preserved under multivalued continuous mappings were investigated in [7]. Certain strong forms of mixed continuous multi functions were characterized in [8] and the upper and lower β-continuous multi functions were studied in [11]. The notions of $b^{\#}$-continuity and *b-continuity were respectively discussed and studied in [9] and [3].

In this paper we have introduced strong forms of b-continuous multifunctions namely $b^{\#}$-multicontinuity and *b-multicontinuity and also studied their properties and characterizations with suitable examples.

2. Preliminaries

Throughout this paper it is assumed that X and Y are non-empty sets and τ and σ are topologies on X and Y respectively and τ' and σ' denote the collections of closed sets in X and Y respectively. The notation P: X⇉Y is used for a multivalued function. For the notations in multifunction theory**,** the reader may consult (Thangavelu, Premakumari, 2015). We use the following abbreviations and notations.

"continuous" ="c", "upper continuous" = "u.c" and "lower continuous" = "*l*.c**"**. Further

$V \in(\sigma, x, P(x),\subseteq) \Rightarrow V\in\sigma$, $x\in X$ and $P(x)\subseteq V$.

$U\in[\tau, x, P, V,\subseteq] \Rightarrow U\in\tau$, $x\in U$ and $P(U)\subseteq V$.

$V \in(\sigma, x, P(x), \varnothing) \Rightarrow V\in\sigma$, $x\in X$ and $P(x)\cap V\neq\varnothing$.

$U \in[\tau, x, P, V, \varnothing] \Rightarrow U\in\tau$, $x\in U$ and $P(u)\cap V\neq\varnothing \forall u\in U$.

$\chi\in\{b^{\#}, *b\}$.

Definition 2.1. The set A is called β(resp. b, *b)-open[1] (resp.[2], resp.[3]) if $A\subseteq Cl(Int(Cl(A)))$ (resp. $Cl(Int(A))\cup Int(Cl(A))$, $Cl(Int(A))\cap Int(Cl(A))$ and $b^{\#}$-open [9,10] if $A=Cl(Int(A))\cup Int(Cl(A))$. The complements of β(resp. b,*b, $b^{\#}$)-open sets are β(resp. b,*b, $b^{\#}$)-closed sets.

Lemma 2.2. The set B is

(i) χ-open ⇒ b-open
(ii) open ⇒ b-open
(iii) b-open⇒β-open

Definition 2.3. The multifunction P is u.c [5,6,7] if $\forall$ $V\in(\sigma, x, P(x), \subseteq)$, $\exists$ $U\in[\tau, x, P,V, \subseteq]$ and is *l*.c if $\forall$ $V\in(\sigma, x, P(x), \varnothing)$, $\exists$ $U\in[\tau, x, P, V, \varnothing]$.

Analogously u.b-c [4] and u.β-c [11] may be defined by replacing "τ" in [τ, *x*, P, V,⊆] respectively by "bO(X,τ)" and "βO(X,τ)". Also *l*.b-c [4] and *l*.β-c [11] may be defined by replacing "τ" in [τ, *x*, P, V, ∅] by "bO(X,τ)" and "βO(X,τ)" respectively.

Definition 2.4. The multifunction P is c if P is u.c and *l*.c. The notions b-c and β-c can be similarly defined.

3.χ-multi continuity where χ∈{*b, $b^{\#}$}

Definition 3.1. The multivalued function P is u.$b^{\#}$-c (resp. u.*b-c) if $P^{+}(V)$ is $b^{\#}$-open (resp. *b-open) ∀ V∈σ.

Proposition 3.2. Consider the following statements.
(i)P is u.χ-c.
(ii)$P^{-}(B)$ is χ-closed ∀ B∈σ′.
(iii)P^{-}(*Cl* (B)) is χ-closed ∀ B⊆Y.
(iv)P^{+} (*Int* (B)) is χ-open ∀ B⊆Y.
The implications (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) always hold.
Proof: Suppose (i) holds. Let B∈σ′ that implies, P^{+}(Y\B) is χ-open so that X\ $P^{-}(B)$ = P^{+}(Y\B) is χ-open that further shows that $P^{-}(B)$ is χ-closed. This proves (i) ⇒(ii).

Now we assume (ii). Let V∈σ that implies by (ii), P^{-}(Y\V) is χ-closed so that X\ $P^{+}(V)$ is χ-closed that further shows that $P^{+}(V)$ is χ-open. This proves (ii) ⇒(i). Other implications follow easily.

Proposition 3.3. If P is u.χ-c then ∀ V∈(σ, *x*, P(*x*), ⊆),∃ U∈[χO(X,τ), *x*, P, V, ⊆].

Proof: Let P be u.χ-c and V∈(σ, *x*, P(*x*), ⊆). Since P(*x*)⊆V, *x*∈$P^{+}(V)$. Since $P^{+}(V)$ is χ-open ∃ a χ-open set U with *x*∈U⊆ $P^{+}(V)$. Clearly U∈[χO(X,τ), *x*, P, V, ⊆].

Proposition 3.4. P is u.χ-c ⇒ it is u.b-c and u.β-c.

Definition 3.5. The multifunction P is *l*.$b^{\#}$-c(resp. *l*.*b-c) if $P^{-}(V)$ is $b^{\#}$-(resp.*b)-open ∀ V∈σ.

Proposition 3.6. Consider the following statements.
(i)P is *l*.χ-c.
(ii) $P^{+}(B)$ is χ-closed ∀ B∈σ′.
(iii) P^{+}(*Cl* (B)) is χ-closed ∀ B ⊆Y.
(iv) P^{-}(*Int* (B)) is χ-open ∀ B ⊆Y.
The implications (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) always hold.

Proof: Suppose (i) holds. Let $B \in\sigma'$ that implies $P^{-}(Y\backslash B)$ is χ-open so that $X\backslash P^{+}(B)$ is χ-open that further shows that $P^{+}(B)$ is χ-closed. This proves (i) $\Rightarrow$(ii).
Now we assume (ii). Let $V\in\sigma$ that implies by (ii)), $P^{+}(Y\backslash V)$ is χ-closed so that $X\backslash P^{-}(V)$ is χ-closed that further shows that $P^{-}(V)$ is χ-open. This proves (ii) $\Rightarrow$(i). The rest follows easily.

Proposition 3.7. If P is *l*.χ-c then$\forall$ $V \in(\sigma, x, F(x), \emptyset)$, $\exists$ $U\in[\chi O(X,\tau), x, P, V, \emptyset]$.

Proof: Analogous to Proposition 3.3.

Proposition 3.8. P is *l*.χ-c$\Rightarrow$ it is *l*.b-c and *l*.β-c

Definition 3.9. P is $b^{\#}$-c (resp.*b-c) if it is u.$b^{\#}$-c (resp.u.*b-c) and *l*.$b^{\#}$-c(resp. *l*.*b-c).

The next proposition follows from previous definition, Proposition 3.2 and Proposition 3.6.

Proposition 3.10. Consider the following statements.
(i) P is χ-continuous.
(ii) $P^{+}(V)$ and $P^{-}(V)$ are χ-open $\forall$ $V\in\sigma$.
(iii) $P^{+}(B)$ and $P^{-}(B)$ are χ-closed $\forall$ $B\in\sigma'$.
(iv) $P^{+}(Int\,(B))$ and $P^{-}(Int\,(B))$ are χ-open $\forall B \subseteq Y$.
(v) $P^{+}(Cl\,(B))$ and $P^{-}(Cl\,(B))$ are χ-closed $\forall B \subseteq Y$.
The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) always hold.
The following diagrams always hold.

Diagram 3.11. Let t=u or *l*.
(i) t.$b^{\#}$-c $\Rightarrow$ t.b-c $\Leftarrow$ t.*b-c.
(ii) t.c $\Rightarrow$ t.b-c $\Rightarrow$ t.β-c.

Examples 3.12. In this section some examples are given to illustrate certain results in the third section.
Let $X = \{p, q\ r, s\}$, $Y = \{1, 2, 3\}$, $\sigma = \{\phi, \{1\}, Y\}$,
$\tau = \{\emptyset, \{r\}, \{q\}, \{q, r\}, \{p, q\}, \{p, q, r\}, \{q, r, s\}, X\}$.

(i) $F_1(p) = \{1, 2\}$, $F_1(q)=\{1, 3\}$ $F_1(r) = \{1\}$ and $F_1(s) = \{1\}$ then $F_1^{+}(\{1\})=\{r, s\}$ is $b^{\#}$-open so that F_1 is u.$b^{\#}$-c.

(ii) If $F_2(p) = \{1, 2\}$, $F_2(q)=\{1\}$, $F_2(r) = \{1, 3\}$ and $F_2(s) = \{3\}$ then $F_2^{+}(\{1\})=\{r\}$ is *b-open that implies F_2 is u.*b-c.

(iii) If $F_3(p) = \{1\}$, $F_3(q)=\{1\}$, $F_3(r) =\{1,2\}$ and $F_3(s) = \{1\}$ then $F_3^+(\{1\})=\{p, q, s\}$ is b-open and β-open and hence F_3 is u.b-c and u.β-c. However F_3 is not u.χ-c as $F_3^+(\{1\})= \{p, q, s\}$ is not χ-open.

(iv) If $F_4(p) =\{2\}$, $F_4(q)=\{3\}$, $F_4(r) =\{1, 2\}$ and $F_4(s) = \{1, 3\}$ then $F_4^-(\{1\})= \{r, s\}$ is $b^\#$-open that implies F_4 is $l.b^\#$-c.

(v) If $F_5(p) = \{1, 2\}$, $F_5(q)=\{1, 3\}$, $F_5(r) = \{2\}$ and $F_5(s) = \{3\}$ then $F_5^-(\{1\})=\{p, q\}$ is *b-open and hence F_5 is *l*.*b-c.

(vi) If $F_6(p) =\{2\}$, $F_6(q)=\{1, 2\}$, $F_6(r) = \{3\}$ and $F_6(s) = \{1, 3\}$ then $F_6^-(\{1\})=\{q, s\}$is b-open and β-open so that F_6 is *l*.b-c and *l*.β-c. However F_6 is not *l*.χ-c as $F_6^-(\{1\})= \{q, s\}$ is not χ-open.

(vii) If $F_7(p) = Y$, $F_7(q)=\{1, 3\}$, $F_7(r) = \{1\}$ and $F_7(s)=\{1\}$ then $F_7^+(\{1\})=\{r, s\}$ and $F_7^-(\{1\})= X$ are $b^\#$-open we see that F_7 is u.$b^\#$-c and $l.b^\#$-c and hence $b^\#$-c.

(viii) If $G_1(p) = \{2\}$, $G_1(q)=\{1\}$, $G_1(r) = \{1, 2\}$ and $G_1(s) = \{3\}$ then $G_1^+(\{1\})= \{q\}$ and $G_1^-(\{1\})= \{q, r\}$ are *b-open so that G_1 is u.*b-c and *l*.*b-c and hence *b-c.

(ix) If $G_2(p) = \{1, 3\}$, $G_2(q)=\{1\}$, $G_2(r) = \{2\}$ and $G_2(s) = \{1\}$ then $G_2^+(\{1\})= \{q, s\}$ and $G_2^-(\{1\})= \{p, q, s\}$ are b-open and β-open we see that G_2 is b-c and β-c. However G_2 is not χ-c as $G_2^+(\{1\})=\{q, s\}$ and $G_2^-(\{1\})=\{p, q, s\}$ are not χ-open .

(x) If $G_3(p) = \{2, 3\}$, $G_3(q)=\{1\}$, $G_3(r) =\{1\}$ and $G_3(s) = \{1\}$ then $G_3(\{1\})= \{q, r, s\}$is open that implies G_3 is u.c. However G_3 is not u.$b^\#$-c as $G_3^+(\{1\})= \{q, r, s\}$ is not $b^\#$-open. If $G_4(p)=\{2, 3\}$, $G_4(q)=\{1, 2\}$, $G_4(r)=\{1\}$ and $G_4(s) =\{1\}$ then $G_4^+(\{1\})=\{r, s\}$ is $b^\#$-open so that G_4 is u.$b^\#$.c. However G_4 is not u.c as $G_4^+(\{1\})=\{r, s\}$ is not open.

(xi) If $G_5(p) = \{2, 3\}$, $G_5(q)=\{1, 2\}$, $G_5(r) = \{1, 3\}$ and $G_5(s) = \{1\}$ then $G_5^-(\{1\})= \{q, r, s\}$ is open we see that G_5 is *l*.c . However G_5 is not $l.b^\#$-c as $G_5^+ (\{1\}) = \{q, r, s\}$ is not $b^\#$-open. If $G_6 (p) = \{3\}$, $G_6 (q) = \{2\}$, $G_6(r) = Y$ and $G_6(s) = \{1, 3\}$ then $G_6^-(\{1\})=\{r, s\}$ is $b^\#$-open we see that G_6 is $l.b^\#$-c. However G_6 is not *l*.c as $G_6^- (\{1\}) =\{r, s\}$ is not open.

(xii) If $F(p) = \{2, 3\}$, $F(q)=\{1\}$, $F(r) = \{1, 3\}$ and $F(s) = \{1, 2\}$ then $F^+(\{1\})=\{q\}$ and $F^-(\{1\})=\{q, r, s\}$ are open we see that F is u.c and *l*.c so that it is c. However F is neither u.$b^\#$-c nor $l.b^\#$-c as $F^+ (\{1\}) = \{q\}$ and $F^- (\{1\}) = \{q, r, s\}$ are not $b^\#$-open. If $G(p) = \{3\}$, $G(q)=\{2, 3\}$, $G(r) =\{1\}$ and $G(s) = \{1\}$then $G^+(\{1\})=\{r, s\}= G^-(\{1\})$ is $b^\#$-open we see that G is u.$b^\#$-c and $l.b^\#$-c so that it is $b^\#$-c. However G is not c as $G^+ (\{1\}) = \{r, s\} = G^- (\{1\})$ is not open.

4. Conclusions

The concepts of strong forms of b-continuous multifunctions namely $b^{\#}$-multicontinuous and *b-multicontinuous functions are suitable for future extension research.

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