Using an Inductive Approach for Definition Making: Monotonicity and Boundedness of Sequences

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The study investigated fourth-year students' construction of the definitions of monotonicity and boundedness of sequences, at the Edgewood Campus of the University of KwaZulu-Natal in South Africa. Structured worksheets based on a guided problem solving teaching model were used to help students to construct the two definitions. A group of twenty three undergraduate teacher trainees participated in the project. These students specialised in the teaching of mathematics in the Further Education and Training (FET) (Grades 10 to 12) school curriculum. This paper, specifically, reports on the investigation of students' definition constructions based on a learning theory within the context of advanced mathematical thinking and makes a contribution to an understanding of how these students constructed the two definitions. It was found that despite the intervention of a structured design, these definitions were partially or inadequately conceptualised by some students.

Currently, many Schools of Education from different South African universities (e.g., Witwatersrand, Nelson Mandela Metropolitan University, KwaZulu-Natal) are revising the presentation of differential and integral calculus modules for undergraduate pre-service mathematics FET trainee-teachers. This is to address the learner-centred approach which underpins Curriculum 2005. We report on an investigation based on the use two of worksheets on which students worked individually and later in groups to construct definitions, followed by the analysis of written responses and interviews.

Wu (2003) argued that pre-service development of teachers in Grades 6-12 require courses which consolidate, *mathematically,* those topics which do not stray far from the high school mathematics curriculum. In particular, "...they should revisit all the standard topics in high school from an advanced standpoint, and enliven them with motivation, historical background, inter-connections and above all, proofs" (Wu, 2003, p. 8-9). It was shown that many of the concepts dealt with in the Real Analysis module strengthen the ideas on which the high school mathematics is based (Brijlall, 2005). Such concepts are indicated in the next section.

The above context led us to formulate the following research question:

> *How does the implementation of a structured worksheet design influence students' construction of definitions in real analysis?*

In particular we look at the construction of the definitions for monotonicity and boundednes of infinite real sequences. The structured design used an examples and non-examples approach. In answering this question we focus on sorting, reflecting and explaining, generalising, verifying and refining, and extension of generalisation (Cangelosi, 1996).

Background

Knowledge base for teachers

One of the expectations of the Norms and Standards for Educators (DOE, 1999) is that the teacher be well grounded in the knowledge relevant to her/his occupational practice. She/he has to have a well-developed understanding of the knowledge *appropriate* to the specialism. Many mathematics teachers find themselves in a position requiring them to implement the syllabus, which includes certain topics they are unfamiliar with. According to Adler (2002), teachers with a very limited knowledge of mathematics need to develop a base of mathematical knowledge. They need to relearn mathematics so as to develop conceptual understanding. Taking this into account we attempted to make certain that trainee-teachers leave with a base of knowledge relevant to their occupational needs. Mwakapenda (2004) concurs when stating that a significant concern in school mathematics is the development of an understanding of mathematical concepts.

We show via a few examples the direct relationship of the concepts with topics from the school FET syllabus as suggested by National Curriculum Statement (DOE, 2003). The following are examples of the types of problems that are covered:

Open and closed intervals
The notion and notation of open and closed intervals play an important role throughout the FET curriculum. For example the solution to the inequality $x^2 - x < 6$ uses an open interval $x \in (-2; 3)$. The question solve for θ in $\sin\theta = \frac{1}{2}$ for $\theta \in [0^0; 360^0]$ uses a closed interval. Graduate students confront many other types of open and closed sets. The module questions the openness/closedness of $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$. This should yield a better understanding of the number system as well.

Sequences and Series
The learning outcome in the National Curriculum Statement (2003) prescribes the investigation of number patterns culminating in arithmetic and geometric sequences and series. Learners in the 2006 curriculum would at Grade 12 be expected to interpret recursive formulae (e.g., $T_{n+1} = T_n + T_{n-1}$). When proving $\sqrt{2}$ irrational an approach using recursive formulae is employed. This exposes the teacher trainee to recursive formulae, were not covered in the old syllabus. Undergraduate students therefore need to be well versed in sequences and series. The realisation that a real sequence is a function whose domain is the set of naturals is fundamental. Deeper understanding into the concepts of convergence and divergence of both sequences and series are dealt with in the Real Analysis course. This will aid student teachers when dealing with the convergence of geometric series in Grade 12. Students are also allowed the opportunity to investigate ideas and tasks to acquire a wider perspective to arithmetic, geometric, linear, quadratic and cubic sequences which are taught in Grade 11.

Monotonicity
A sub-skill in Learning Outcome 2 (DOE, 2003) for Grade 12 is to identify the intervals on which the function $y = a^x$, $a > 0$ and its inverse $y = \log_a x$ increase or decrease. These ideas are consolidated when studying monotonicity. A monotonic function (or monotone function) is a function which preserves the given order. This concept first arose in calculus and was later generalised to a more abstract setting. Formally, a function f is called a monotonically increasing function on $[a; b]$ if $f(x) > f(y)$ whenever $x > y$, $\forall x, y \in [a; b]$, and dually one can define a monotonically decreasing function on an interval. A function is called monotonic if it is either monotonically increasing or decreasing.

Boundedness
For this study, a bounded set is regarded as a set which has a greatest element and a least element. A bounded sequence is a sequence whose set of values is bounded. Formally, a sequence $(x_n)_{n=1}^{\infty}$ is

bounded if it is bounded above and below. A sequence $\left(x_n\right)_{n=1}^{\infty}$ is bounded above if and only if there exists a real number M such that $x_n \leq M$ for all $n \in \mathrm{N}$. A sequence $\left(x_n\right)_{n=1}^{\infty}$ is bounded below if and only if there exists a real number m such that $x_n \geq m$ for all $n \in \mathrm{N}$.

Theoretical basis

Piaget, cited in Bowie (2000), expanded and deliberated on the notion of *reflective abstraction* which has the following two components: (1) A projection of existing knowledge into a higher plane of thought, and (2) a reorganisation of existing knowledge structures. He proposed this to be a major factor for the development of mathematical cognition, and distinguished three types of abstractions: empirical abstraction, pseudo-empirical abstraction, and reflective abstraction.

Reflective abstraction refers to the construction of logico-mathematical structures by a learner during the process of cognitive development (Dubinsky, Weller, McDonald, & Brown, 2005a). The two features of this concept are: that it has no absolute beginning but appears at the very earliest ages in the coordination of sensori-motor structures, and that it continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction.

We define the following four concepts that are used in the APOS theory of conceptual understanding (Bowie, 2000):

- **A**ction: an action is a repeatable physical or mental manipulation that transforms objects
- **P**rocess: a process is an action that takes place entirely in the mind.
- **O**bject: the distinction between a process and an object is drawn by stating that a process becomes an object when it is perceived as an entity upon which actions and processes can be made.
- **S**chema: A schema is a “more or less coherent collection of cognitive objects and internal processes for manipulating these objects”

In our analysis of students’ definition making we used the following five kinds of mental constructions associated with reflective abstraction as explained by Bowie (2000) and Dubinsky et al. (2005a; 2005b):

- Interiorisation: the ability to apply symbols, language, pictures and mental images to construct internal processes as a way of making sense out of perceived phenomena. Actions on objects are interiorised into a system of operations
- Coordination: two or more processes are coordinated to form a new process
- Encapsulation: the ability to conceive a previous process as an object
- Generalisation: the ability to apply existing schema to a wider range of contexts.
- Reversal: the ability to reverse thought processes of previous interiorised processes.

This study focused on advanced mathematical thinking required for the concepts of monotonicity and boundedness of infinite real sequences. This falls under the domain of APOS theory.

Methodology

The method adopted four stages: design of worksheet, facilitation of group-work, capture of written responses, and interviews.

The first two stages were influenced by social constructivism. In the worksheets a guided structure was used to facilitate definition making in a collaborative manner. This was based on our assumption that the construction of knowledge is better facilitated in a social context which provides support. Learning in a social context is recommended by social constructivists such as Von Glasersfeld (1984), Cobb (1994), Confrey (1990) and Steffe (1992). Steffe (1992) argued that reflective ability is a major source of knowledge in all levels of mathematics. This implies it is important for students to talk about their thoughts to each other and the facilitator.

The students involved were undergraduate teacher trainees at the Edgewood Campus of the University of KwaZulu-Natal. They pursue a module on Real Analysis in their final year. This module, which included elementary topology of the real line, involves the learning of concepts in set theory, relations and functions, cardinality, countability, denseness, convergence and other related ideas on elementary topology.

A group of twenty three fourth-year undergraduate teacher trainees participated in the project. These students specialised in the teaching of mathematics in the FET school curriculum.

The research instruments used were worksheets, observation of classroom activity, and interviews.

Instrumentation

Design of worksheet

Worksheets were designed in accordance with ideas postulated by a guided problem solving model suggested by the work of Cangelosi (1996). His work modeled how meaningful mathematics teaching could be planned with the aim of simultaneously addressing the cognitive and affective domains. The model, which is illustrated in Figure 1, has the following three levels or phases:

- Inductive reasoning (conceptual level)
- Inductive and deductive reasoning (simple knowledge and knowledge of a process level), and
- Deductive reasoning (application level).

However, note that there is always interplay between inductive and deductive reasoning. They are continuously present and constantly following each other in mathematical thinking. For example, in an inductive process very often a preliminary 'generalising' step is reached. A conclusion or the finalisation of an inductive part is the beginning of the deductive part. Therefore generalising at each of the different levels implies that the deductive mode of reasoning comes into play. In creating constraints for the examples and non-examples in the guided worksheets implemented in this research we kept in mind the characteristics of boundaries as suggested by Mason and Watson (2004). Boundary examples are those examples that "distinguish having and not having a specified property" (Mason & Watson, 2004, p.9). They suggested that if students are only offered well-behaved examples, or examples which have additional, but irrelevant features, then the reason for careful statements in a definition might pass them by. In choosing items for the examples and non-examples the conditions in the definitions of monotonocity and boundedness were considered. To illustrate this, the sequence $\left((-1)^n\right)_{n=1}^{\infty}$ was given as an example of a sequence that is bounded above. However, it is a bounded sequence since it is also bounded below.

For the design of the worksheets inductive learning activities were used to construct the concepts of monotonicity and boundedness of real sequences. These activities had the following stages:

- *Sorting* (examples and non-examples) and categorising, which target action in APOS Theory.
- *Reflecting and explaining* the rationale for categorising, which target process in APOS Theory.
- *Generalising* by describing the concept in terms of attributes (that is, what sets examples of the concept apart from non-examples).
- *Verifying and refining* (the description or definition is tested and refined if necessary), which both target object in APOS Theory. To determine the monotonicity of a sequence require constructing knowledge of a process. Here an algorithm had to be formulated which required the calculation of successive terms of a sequence, and then comparing them.

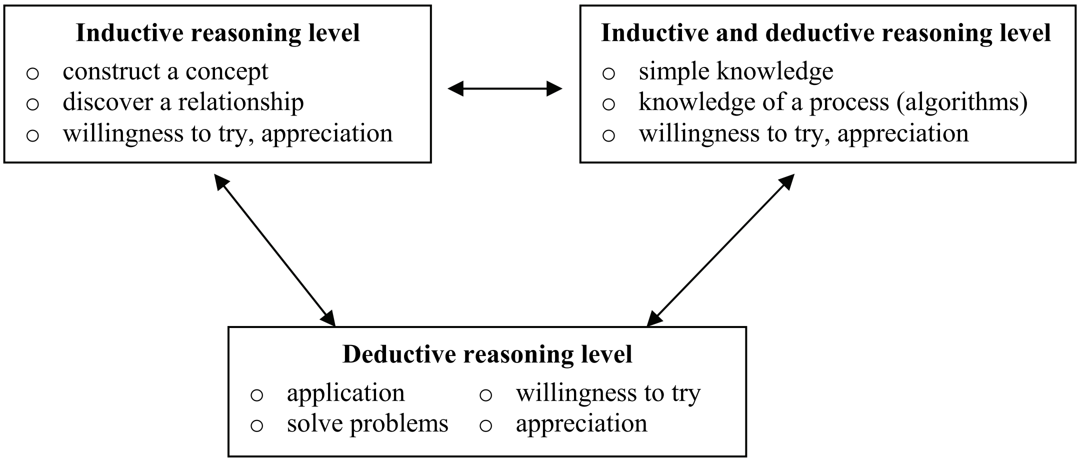

Figure 1: *A guided problem solving teaching model*

Data collection procedures

Facilitation of group-work

The fourth year class of 23 students were presented the worksheets and engaged with the activities individually for approximately fifteen minutes. We believed that this would prepare the students to make constructive contributions in the group-work context which was to follow. When constructing the concept of montonicity they worked in seven groups (five comprising of three members and two with four). Each group, after discussing and reaching a collective decision, documented their thoughts for presentation to the class.

Capture of written responses

Each student was given a guided activity sheet. When they were in groups they were provided a separate worksheet which required the collective group response to the activities. The following five instructions appeared on the worksheets: (a) Complete each worksheet on an individual basis. (b) You are now required to form groups of three or four. (c) Now discuss your findings within the group to reach consensus. (d) Write down a collective response and elect a leader to discuss with class. (e) Finally conclude findings as a class with tutor. These worksheets were then collected by the lecturer for analysis of student thinking.

Interviews with group leaders

During the analysis of the written responses it became necessary to interview the group leaders. Based on the written responses, questions were formulated to clarify the written responses of certain groups. Interviews with the groups were audio recorded and transcribed. This was used for the analysis and discussion section of this paper.

Instruments for learning and tools for data collection

Monotonic sequences

The following extract from the worksheet, modeled on the construction of a concept at the inductive reasoning level (Cangelosi, 1996), indicates the task based on examples and non-examples which the students engaged with:

Sorting
The following infinite real sequences are called ***monotonically increasing***:

1. $\left(2^n\right)_{n=1}^{\infty}$ 2. $\left(2n+1\right)_{n=1}^{\infty}$ 3. $\left(\dfrac{n+1}{2}\right)_{n=1}^{\infty}$ 4. $\left(2n-1\right)_{n=2}^{\infty}$

The following infinite real sequences are ***not monotonically increasing:***

1. $\left(2^{-n}\right)_{n=1}^{\infty}$ 2. $\left(\dfrac{2}{2n+1}\right)_{n=1}^{\infty}$ 3. $\left(\dfrac{1}{2n}\right)_{n=1}^{\infty}$ 4. $\left((-1)^{n}\right)_{n=1}^{\infty}$

Reflecting and explaining
After interrogating the above examples and non-examples of monotonic increasing infinite real sequences, explain why one would categorise them as such.

Generalising the description of ***monotonically increasing*** *infinite real sequences*
Now write out a statement which you would adopt to describe (define) an arbitrary infinite real sequence $\left(x_n\right)_{n=1}^{\infty}$:

Bounded sequences

The following extract from the worksheet, modeled on the construction of a concept at the inductive reasoning level (Cangelosi, 1996), indicates the task based on examples and non-examples which the students engaged with:

Sorting
The following are examples of infinite real sequences which are ***bounded below:***

1. $\left(2^{n}\right)_{n=1}^{\infty}$ 2. $\left(2n+1\right)_{n=1}^{\infty}$ 3. $\left(\dfrac{n+1}{2}\right)_{n=1}^{\infty}$ 4. $\left(2n-1\right)_{n=1}^{\infty}$

The following are examples of infinite real sequences are ***not bounded below:***

1. $\left(-\dfrac{n+1}{2}\right)_{n=1}^{\infty}$ 2. $\left(1-2n\right)_{n=1}^{\infty}$

The following are examples of infinite real sequences which are ***bounded above.***

1. $\left(2^{-n}\right)_{n=1}^{\infty}$ 2. $\left(\dfrac{2}{2n+1}\right)_{n=1}^{\infty}$ 3. $\left(\dfrac{1}{2n}\right)_{n=1}^{\infty}$ 4. $\left((-1)^{n}\right)_{n=1}^{\infty}$

The following are examples of infinite real sequences are ***not bounded above***

1. $\left(2^{n}\right)_{n=1}^{\infty}$ 2. $\left(2n-1\right)_{n=1}^{\infty}$

Reflecting and explaining
After interrogating the above examples and non-examples of bounded (above or below) infinite real sequences, explain why one would categorise them as such.

Generalising the description of ***bounded above/below*** *infinite real sequences*

1. Now write out a statement which you would adopt to describe (define) an arbitrary infinite real sequence $\left(x_n\right)_{n=1}^{\infty}$ as bounded above:
2. Now write out a statement which you would adopt to describe (define) an arbitrary infinite real sequence $\left(x_n\right)_{n=1}^{\infty}$ as bounded below:

Verifying and refining
Check whether the following are ***bounded above/below*** infinite real sequences by applying the above definition.

1. $\left(\log_2 n\right)_{n=1}^{\infty}$ 2. $\left(2^n\right)_{n=1}^{\infty}$ 3. $\left(\log_{\frac{1}{2}} n\right)_{n=1}^{\infty}$

Pre-knowledge

At this stage in the course students covered fundamental concepts on set theory, methods of proof and logic, relations, and basic ideas on topology of the real line. Hereafter, they were engaged in issues relating to real infinite sequences. They covered the definition of a real sequence and its convergence as:

- A real infinite sequence $\left(x_n\right)_{n=1}^{\infty}$ is a function $f: \mathbb{N} \rightarrow \mathbb{R}$ defined as $f(n) = x_n$
- A real infinite sequence $\left(x_n\right)_{n=1}^{\infty}$ converges to L if for every $\in > 0$, there exist a natural number N, such that $n > N$ implies $\left|x_n - L\right| < \in$.

Post-knowledge

During the guided problem solving activity they developed the following definitions:

- A real infinite sequence $\left(x_n\right)_{n=1}^{\infty}$ is called monotonically increasing (decreasing) if $x_{n+1} > x_n$ ($x_{n+1} < x_n$) for all $n \in \mathbb{N}$.
- A real infinite sequence $\left(x_n\right)_{n=1}^{\infty}$ is bounded above (below) if there exists a real number M (m) such that $x_n \leq M$ ($x_n \geq m$) for all $n \in \mathbb{N}$.

A real infinite sequence $\left(x_n\right)_{n=1}^{\infty}$ is monotone or monotonic if it is either increasing *or* decreasing and bounded if it is bounded above *and* below. The aim hereafter is to prove the theorem "Every monotonic bounded real infinite sequence is convergent". It is much easier to show that convergence implies boundedness. That the converse is not necessarily true can be demonstrated by the bounded sequence $\left((-1)^n\right)_{n=1}^{\infty}$ which is not covergent. The need for monotonicity allows for the truth of the converse. This theorem then will be treated as a *final object* (Dubinsky et al., 2005a; 2005b), since the construction of the two concepts discussed in this paper applied finite processes and the resulting generalisations culminated in the formation of objects. We end at this theorem as a last step and thus envisage it as a final object.

Findings and discussion

Table 1 summarises the seven group responses in reflecting and explaining, and generalising the concept monotonically increasing sequences. Characterisation of coded categories is as follows:

- *none* was used for no response
- *inadequate* codes an incorrect or unclear response
- *partial* codes gaps in description
- *complete* codes a mathematically correct response.

Table 1: *Results on group constructions for monotonically increasing (N = 7)*

	Number of group responses			
Stages	**None**	**Inadequate**	**Partial**	**Complete**
Reflecting and Explaining	1	0	5	1
Generalising	1	2	3	1

It was observed that five of the seven groups when reflecting and explaining constructed a partial understanding of this concept. Examples of such constructions stated by two of the groups were as follows:

Group E: The sequence always increase, which imply that term (T_{n+1}) minus T_n will give a positive value and in monotonic increasing infinite real sequences there will be no variable in the denominator.

The correct use of the symbols T_{n+1} and T_n to explain an increasing sequence implies that group E viewed such sequences as an object. However, the exclusion of variables in the denominator indicates an incomplete understanding. This misconception arose because the four illustrative examples excluded such cases. During the interview the group leader was asked why the group referred to "no variable in the denominator".

Group E leader: Because I think from our observations this monotonical don't have a variable denominator but in the nonmonotonical there is … that's why.

It was pointed out to the respondent that the sequence $\left((-1)^n\right)_{n=1}^{\infty}$, which is a non-example, had no variable in the denominator. To this he responded: "The other thing is, if I can divide this thing by n, it means that the value of the term will decrease." It seemed that this group, after working with only the examples generalised their definition for monotonicity. In the case of the non-example they forced a variable in the denominator to satisfy their definition. Here, the mathematical reasoning was faulty. Note that in the above sequence if each term is divided by the replacement value for *n*, the sequence does not decrease.

Group F: A monotically increasing sequence is a sequence that approaches ∞ from the negative side as $n \to \infty$. In other words, as $n \to \infty$, x_n increases in value to ∞.

Note that a monotonically increasing sequence could converge to a finite limit. The response of group F was also coded as such due to the phrase *from the negative side*. To clarify this, the following question was posed during the interview:

Interviewer: What do you mean by "from the negative side?"

Group F leader: What we meant was the value was becoming more and more positive as *n* approaches infinitely. Even if the sequence started with the first term as say -5 or something like that … it er … if you had to analyse it as a graph and unfortunately in maths I do that all the time, I analyse everything as a graph and then I look at sort of limits and the limit of that as you move along … from the left or from the negative side per say … that's were we got it from.

The striking feature of this response is the visual aid adopted to respond to the question. It seems that the replacement values of n were used as the x-coordinates of a point on the Cartesian plane, while the resulting values for the terms were used for the corresponding y-coordinates. In the context of APOS theory the conceptual understanding is at the level of a schema. The cognitive objects are the ordered pairs and the visualisation of these graphically were the result of internal processes for manipulating these objects.

Table 2 summarises the seven group responses in reflecting and explaining, and generalising the concept boundedness of sequences. Characterisation of coded categories is as for Table 1.

Table 2: *Results on group constructions for bounded below/above (N = 7)*

		Number of group responses			
Stages		**None**	**Inadequate**	**Partial**	**Complete**
Reflecting and Explaining	Bounded above	0	5	2	0
	Bounded below	0	5	2	0
Generalising	Bounded above	0	4	1	2
	Bounded below	0	4	1	2

A striking observation is that the majority of the responses were categorised as inadequate. Such responses for bounded sequences under reflecting and explaining led to inadequate generalisations. This indicates that during the process of constructions of concepts, inadequate reflecting and explaining is likely to produce inadequate generalisations. Group B was the exception to this assertion. Their response for reflecting and explaining is:

> A sequence X_n is bounded if $\{X_n / n \in N\}$ is a bounded set. For a sequence to be bounded below it has X_1 as it's minimum value where it is a lower limit. For a sequence to be bounded above it has X_1 as it's maximum value where it is an upper limit.

The response regarded the first term of a sequence as a determining factor in concluding the type of boundedness of a sequence. The following question was posed at the interview:

Interviewer: Why did you all include the first term of the sequence as being important in concluding the type of boundedness of the sequence?

Group B leader: They figured if they chose the first term of the sequence that must be the lowest one … the first value of the sequence will be highest or lowest, and the sequence will either go up/down, increase/decrease … . If your sequence is increasing then, then your first number will be your lower bound and if decreasing the first number will be your upper bound.

Note that this group used monotonocity to reflect and explain the concept of boundedness. Their explanation is limited since it does not apply to the sequence $\left((-1)^n\right)_{n=1}^{\infty}$.

In generalising the description of bounded above this group's response was:

> If $x_n \leq M$ for $n \in$ N, M is a constant (independent of n) we say that the sequence $\left(x_n\right)_{n=1}^{\infty}$ bounded above and M is called an upper bound.

This group correctly generalised the dual of this statement for the concept bounded below. These responses were in accordance with the definitions of bounded above/below as discussed in the section on post knowledge. Further the use of terminology such as upper/lower bound and upper/lower limit was spontaneously introduced by the students in group B. Despite inadequately framing their responses (for the reflecting and explaining processes) in the construction of the concept of boundedness, their generalisation displayed that they conceptualised sequences as objects. This supports the claim of an interplay between inductive and deductive reasoning within the framework of the guided problem solving teaching model (see Figure 1).

Group E gave a partial response in reflecting and explaining the concept of boundedness, yet were able to correctly generalise. This is more likely to occur than the exception which was illustrated above during the analysis and discussion of the response by group B. The written response by Group E for reflecting and explaining the concept of boundedness was:

Group B: A sequence $\left(x_n\right)_{n=1}^{\infty}$ is bounded above $\Leftrightarrow x_n < x_1, \forall\ n \in \mathrm{N}$.
It is then bounded below $\Leftrightarrow x_n > x_1, \forall\ n \in \mathrm{N}$.

To gain insight into this response we asked the following during the interview:

Interviewer: Do you consider your written responses as complete?

Group E leader: I don't think so … I think what we were trying to do there … we were just trying to abbreviate it … but by sort of taking the bounded above if and only if the nth term is less than a particular limit for all sort of natural number values or for any term basically.

To gain further clarity we then asked:

Interviewer: What you are saying … that x_1 is the first term all the time that is involved in your definition … so if for instance the sequence $\left((-1)^n\right)_{n=1}^{\infty}$, … , now in this case it is bounded above. Is it?

Group E leader: Yes. That one … is 1, -1, 1, -1, … and we kind of could not explain that in terms of our definition.

During the interview it became clear to this group leader that the group's definition for that stage was incomplete. The use of an appropriate counter-example, in this case $\left((-1)^n\right)_{n=1}^{\infty}$, assisted the group leader to realise the flaw in the definition that they constructed. Such counter-examples support the processes associated with *reversal* construction during reflective abstraction in APOS theory.

Table 3 summarises the seven group responses during the verifying and refining stages leading to further abstraction in constructing the concept of boundedness.

Table 3: *Results on verifying and refining concepts bounded below/above (N = 7)*

		Number of group responses		
Sequence		**None**	**Incorrect**	**Correct**
$\left(\log_2 n\right)_{n=1}^{\infty}$	Bounded above	5	1	1
	Bounded below	2	1	4
$\left(2^n\right)_{n=1}^{\infty}$	Bounded above	5	1	1
	Bounded below	2	1	4
$\left(\log_{\frac{1}{2}} n\right)_{n=1}^{\infty}$	Bounded above	3	0	4
	Bounded below	4	2	1

It was observed that six out of the seven groups when identifying the boundedness of a sequence believed that it is bounded either above or below, and not both. For example, Group E when discussing the second sequence in the table wrote:

Group E:

$$n. > 1$$
$$\therefore 2^n > 2^1$$
$$\therefore xn. > x.1$$
$$\therefore \left(2^n\right)_{n=1}^{\infty} \textit{ is bounded below.}$$

We note the misuse of notation in line 3, despite the correct conclusion of the sequence being bounded below. No investigation into the sequence being bounded above was done. During the interview the following was posed:

Interviewer: The question asked that you look at both bounded above/below. You looked at bounded below for the sequence $\left(\log_2 n\right)_{n=1}^{\infty}$. Why did you all not look at bounded above?

Group E leader: I think we kind of assumed that for each particular function unless it is periodic it is not going to be bounded below and above.

We observe that this group did not find it necessary to investigate the other case of boundedness, if the sequence was not periodic.

Group B was the only group that investigated both the types of boundedness for each sequence. Their response for the second sequence in Table 3 was:

Group B: It is bounded below by 2 and not above.

This was a concise and apt response. Note that group B displayed understanding when generalising the concept of boundedness and also conceptualised sequences as an object (see the analysis and discussion for Table 2). In Dubinsky's stages of reflective abstraction the processes of interiorisation, coordination, encapsulation and generalisation were demonstrated by group B. This seems to intimate that these processes are pre-requisites for successful construction of concepts.

Conclusion

The structured worksheets, based on the examples and non-examples approach for constructing the concepts of monotonocity and boundedness of sequences, had a positive impact. This was so because it encouraged group-work, which fostered an environment conducive to social constructivism and reflective abstraction.

The findings (as demonstrated by the analysis and discussion of the responses of group B) showed that students demonstrated the ability to apply symbols, language, and mental images to construct internal processes as a way of making sense of the concepts of monotonocity and boundedness of sequences. On perceiving sequences as objects students could apply actions on these objects which were interiorised into a system of operations. The verifying and refining stages in the construction of the two concepts required a conceptualisation of these concepts as objects. This conceptualisation enabled the formulation of new schema which we expect to be applied to a wider range of contexts. The concept of boundedness of sequences should now lead to the construction of the definitions of *supremum* (least upper bound) and *infimum* (greatest lower bound) of sequences.

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