Dynamic geometry: an agent for the reunification of algebra and geometry

Meira Hockman

School of Mathematics, University of the Witwatersrand
Email: meira@maths.wits.ac.za

This paper investigates the degree of separation or unity between the algebraic and geometric modes of thought of students in tertiary education. Case studies indicate that as a student is inducted into the use of algebra the insightful and visual components of geometrical and graphical modes of thought are sidelined. Based on Vygotsky's taxonomy of the psychogenesis of cultural forms of behaviour, I suggest that this separation occurs because the algebraic methods remain fixed at a naïve or algorithmic stage. The algebraic concepts may fail to be internalised because the stage of instrumental functioning of algebra as a 'tool' or 'method' of geometry is not successfully transitioned. I suggest that this stage of instrumental functioning may be stimulated by using dynamic geometry programs to promote the formation of images in conjunction with algebraic representations in problem solving. In this way the modes of thought in algebra and geometry in mathematics may be reunified.

Introduction

Algebra plays a fundamental instrumental role as a language in the practice of mathematics. It is used to reduce the complexity of problems and hence promote a method of solution through the application of an algebraic algorithm (Grabiner, 1995; Katz, 2004). While students may effectively use algebraic algorithms and routines, the purpose and/or usefulness of these 'tools' is often lost or misinterpreted by novice users. To these novice users, the algorithms are seen as an end in themselves. The signs and symbols, however, may not be used effectively as a means of reasoning. The descriptive and analytic powers of the algorithms in the initial problem or in other mathematical disciplines are often lost. Furthermore, even if the learner has assimilated the algebraic tool, the geometrical or graphical insights relevant to algebra are often sidelined and neglected. Thus the dynamic relationship between algebra and geometry (including graphs and visualisation) is broken. I believe that this break in the natural connection between these mathematical disciplines (tools for reasoning) prevents the novice learner from gaining a mature understanding of either.

The functional unity of the component processes of geometry (through visualisation) and algebra of mathematics is the focus of the paper. This research follows experiences of others in the field, in particular, visualisation in algebra and analysis (Yerushalmy *et al*., 1999; Kawski, 2002; Katz, 2004); extracting and giving meaning in formal theories through concept images (Pinto & Tall, 1999); cognitive units and algebra (Crowley & Tall, 1999) and relating to multi-representation (Sierpinska *et al*., 1999).

In order to understand the dynamics of change in algebra from its beginnings as a 'method' in geometry to its development as a legitimate branch of mathematics that sidelines geometric thinking, I examine a Vygotskian perspective on the education of cultural forms of behaviour (Vygotsky, 1994). Vygotsky proposed that a tool (such as an algebraic algorithm) and the object it acts on (such as a complex mathematical problem) have a dynamic relationship that benefits both. While it may be understood that algebra is a 'method' of solving complex problems in geometry, it should also be understood that geometry is a 'method' of understanding the formal concepts in algebra.

Vygotsky mapped the functional use of tools and signs from their primitive instrumental beginnings to final sophisticated mental processes through four stages. These stages are detailed as: *natural or primitive psychological*; *naïve psychology*; *instrumental function*; *ingrown or internalised*.

In this paper I propose that tools such as algebra develop and mature in a way similar to that of language, signs and symbols as described by Vygotsky. I distinguish between the tools' instrumental effects of *cultural amplification* (Pea, 1987; Bruner, 1966) that speed up or accelerate processes and *cognitive functioning* (Pea, 1987) that illuminate or give insight into the processes. I suggest that as the use of the tool becomes more sophisticated its attributes are internalised and the tool's relationship to its objects of action fades. Not only do the tools' roles of *amplification* and *cognitive functioning* disappear but the dynamic feedback of subject and object may be severed.

In what follows, I consider the pedagogical implications of reuniting geometry with algebra using dynamic geometry programs as the agent of reunification. I explore this separation/unification of algebra and geometry in two case studies. The first study shows how the separation of algebra from geometry has a detrimental result on problem solving within a class of university students in their second year of study. These students, majoring in mathematics, were participants in a first course in abstract algebra. I highlight the separation that occurs between existing graphical knowledge and newly acquired algebraic routines. The second study involves a mathematics education honours class in geometry. The class consisted of in-service senior school mathematics teachers. Here I highlight the changed perspective of the students after they incorporate geometry into an algebraic problem solving activity. I review the status of the students in both case studies with respect to the described Vygotskian taxonomy of the psychogenesis of cultural forms of behaviour.

Based on this evidence I suggest that dynamic geometry programs such as *Geometer's Sketchpad* can be used to reunite the mathematical disciplines of algebra and geometry at school and at undergraduate level. Such programs help produce a multi-perspective understanding of the mathematical content and highlight the illumination that the processes of algebra and geometry bring to each other. With such understanding the students may become aware of the instrumental function of algebra with respect to other branches of mathematics.

Algebra as A tool in Mathematics

Tools used to accelerate and to illuminate

Algebra as a 'method', a 'tool', a 'technology' or 'language' in mathematics is widely accepted (Applebaum, 1999; Noss & Hoyles, 1996). Since algebra helps the user mediate or transcend the limitations of geometric thinking, learning and problem solving it may be classified as a cognitive technology (Pea, 1987: 91). Algebra as a tool may be used in mathematics in strategically different ways. That is as a tool of *cultural amplification* and of *cognitive functioning* (Pea, 1987; Berger, 1998: 15).

In the case of cultural amplification, the tool provides the techniques for calculations, approximations or constructions that are otherwise laborious, complicated or simply tedious. The mathematics student/teacher/researcher benefits from using the tool by gaining insight into the nature of solutions to problems, through saved time and through maintaining interest that might otherwise dwindle. This attribute of the tool has been referred to by Pea as a means of empowering human cognitive capacities.

In the case of cognitive functioning, the tool provides a functional role of revealing changes and/or invariance that may be present. In particular, using the method of algebraic representation, geometric concepts are seen from an algebraic perspective. This approach adds rigour and depth of understanding. Pea (1987: 96) refers to such technological attributes as the *reorganisation of mental or cognitive functioning*. He suggests that this technological feedback externalises thought processes, keeping a record of results and allowing patterns to be observed (1987: 97). These records are available for inspection, correction and reflection. This reflection leads to reformed action or to hypotheses that can be tested.

In the case of algebra, I propose that the distinct attributes of *cultural amplification* and *cognitive functioning* emerge as algebra becomes part of the culture of mathematical practice of students of mathematics and mathematicians alike. As the use of algebraic techniques mature there is a shift in the dynamic relationship between algebra (the tool) and the geometric problem (the object or subject matter it was designed to serve). The induction of the learner into the use of algebra as a mathematical language is similar to the education of cultural forms and behaviours (signs, symbols and language) as explained by Vygotsky (1994). Vygotsky indicates that a tool develops from a naïve (external) to a sophisticated (internal) means of support to learning. I believe that the algebraic tool develops similarly. In what follows, I will make this path explicit.

The Education of cultural forms of behaviour

Vygotsky (1978), Bruner (1986) and Pea (1987) suggest that there is a dynamic link between the tool and the object it acts on. Vygotsky notes that if one changes the available tools of thinking, the mind will develop a radically different structure (1978: 126). Bruner (1986: 72) suggests that tools and technologies provide a means for turning around one's thoughts and seeing the technologies in a new light.

This changing dynamic is what Vygotsky calls psychogenesis of cultural forms of behaviour. A person may master his/her external behaviour by a culturally acquired technique of signs. This method of using signs is "*not only a key to the understanding of higher forms of a child's behaviour which originate in the process of cultural development, but also a means to the practical mastering of them in the matter of education and school instruction*" (Vygotsky, 1994: 70). The method consists of natural psychological processes, and yet unites these processes in a complicated functional and structural way (1994: 61). Vygotsky notes that remembering with signs and tools, such as maps and plans, may be an example of the cultural development of memory. The process of 'remembering' will be determined by the character of the signs or tools that are selected as an aid. That is, the cultural development consists of mastery methods of behaviour based on the use of signs. Culture thus transforms nature to suit the ends of the child.

I suggest that Vygotsky's mapping of the changing roles of the signs, symbols and language in learning throws light on how algebra, as a systems of signs, symbols and language, may become integrated into mathematical practice. Algebra may be learned in a mechanical way, but may not be used as a means of reasoning in much the same way as a child may be able to speak but not reason in his/her native language. Visualisation of the algebraic symbols, through graphs or geometry, reunites the tool with the object of action, to enhance the cultural development of algebra. This unification may thus determine the character of the system of signs, symbols and images that comprise algebra and geometry. The algebra and geometry should fuse into a functional unit of processes.

Vygotsky proposes that the cultural development of the child passes through four phases or stages that follow consecutively one after another (1994: 64). These stages form a complete cycle of cultural development of any one psychological function. Following Vygotsky, I list the four stages and elaborate to the mathematical domain:

Stage 1. In this stage *natural or primitive psychological* means are used to resolve a problem. That is, the task is not above the natural abilities of the child and he/she will master it using present memory and intellectual development. For example, the child may operate with quantities even though he cannot count. In the case of language, a student may be able to communicate in the language but does not understand how to reason or use words to draw conclusions.

Referring to algebra, this stage is epitomised by learners representing unknown quantities as "x" but not making use of this representation in the solution to a problem. For example, representing a parabola by ax^2+bx+c with no understanding of the significance of the constants a, b, and c and variable x.

Stage 2. *Naïve psychology* is the stage in which the sign or tools are adequately used in an algorithmic way. The task is above the natural capabilities of the child and the child is initiated into the use of new tools or technology. In this stage Vygotsky explains that children may learn to use a mnemonic aid (the tool) to enhance memory. The solution of the problem requires the application of this mnemonic aid. The child solves an inner problem by means of the exterior object (mnemonic aid). The external object (say fingers or counters) takes on the functional importance of a sign (replacing the object). However, the child is unaware of how the object helped him solve the problem. The external connection between the method and the problem is not forged. For example, a child may imitate counting and repeat sequences of words but does not know for what purpose counting is used.

In a similar way mathematics students may learn to build equations and solve problems using variables and fixed constants. For example, students may use the quadratic formula (the tool) to solve for the roots of the equation, but have no concept of what a root or quadratic equation actually signify. In this case the formula (the tool) plays a definite and functional role. However, while the student may mechanically repeat the methods on the same type of problem with success, he may not be aware of the significance or deeper meanings of the method. This is the stage that Pea (1987) refers to as *cultural amplification*.

Stage 3. In this stage the *instrumental function* of the tool is established and the tool is used appropriately as an intellectual tool. The child learns how the method works and how to make proper use of techniques. The processes forming part of the method form a complicated functional and structural unity. The unity is effected by both the task that must be completed and by the means by which the method can be followed. The structure of the problem solving activity is moulded by available means (e.g. using fingers as a tool to help in the counting process by placing them in a one-to-one correspondence with the numbers).

This is the stage of cognitive functioning (Pea, 1987). In algebra, for example, the student may recognise that curves bear a definite relationship to points on a straight line (the X-axis) and that this relationship may be expressed by a single equation. Here, there is recognition that the quadratic formula (tool) relates to the shape and position of a parabola with respect to coordinate axes and that the roots are intersections with the X-axis. In this role the respective tools, used repeatedly and in diverse ways, illuminate and extend knowledge.

Stage 4. In this stage the use of the tool becomes ingrown or internalised. The external activity of using an object as a means of finding solutions passes on into internal activity. This assimilation indicates a maturation of the tools as a working strategy. A problem once solved leads to a correct solution in all analogous situations even when external conditions have changed radically (e.g. counting in the mind is an illustration of 'complete in-growing' of the technique of placing objects in one-to-one correspondence with the natural numbers).

In the case of algebra, polynomial equations replace the graphs of the polynomials and their properties are now established algebraically with no reference to their physical reality. The algebraic representations replace the geometric objects and a geometric problem is solved algebraically. Mental images or rough sketches of the changes produce insights. The algebraic method and the visual changes have been internalised. This internalisation promotes generalisation, where different examples are represented by the same or similar algebraic relationships. This stage marks the maturation of the role of the tool from technology to science.

Instrumental function

While Vygotsky points out that language as a tool of speech and a tool of reasoning have entirely different roots, he stresses that at a certain moment the two lines of development cross each other. At that moment a child discovers the 'instrumental function' of a word (1994: 68). Prior to this stage the intellectual behaviour of the child indicates an independence of intellect from speech. Following the stage of discovery of the functional importance of a word is the stage of transition from external to internal speech. Vygotsky observes that these three main stages in the development of speech and reasoning correspond to the three main stages of cultural development of tools and signs as discussed above. The pre-speech reasoning corresponds to primitive and naïve behaviour (Stages 1 and 2), while the instrumental function of a word, which Vygotsky describes as the "greatest discovery of the life of a child" (1994: 69), corresponds to the third stage of the scheme. In this stage the tool is used appropriately as an intellectual resource. Finally, the transition from external speech to internal speech corresponds to the transformation of external activity to internal activity as described in Stage 4.

Similarly, I argue that the movement from algebra as a tool of amplification in Stage 2 to algebra as a tool of cognition in Stage 3 is reached when the student understands the instrumental function of algebra in solving geometric problems. This progress in roles from amplification to cognition provokes a change in the algebra itself and effects the relationship between algebra and geometry. In this case, the conscious reflections on the tool (algebraic method) causes the associated skills to mature into a branch of knowledge or a science in their own right (Stage 4). That is the 'method of algebra' changes into the 'science of algebra' through a process of conscious reflection.

The historical development of algebra (Boyer, 1968; Eves, 1953) mirrors this argument. While this fact is not used as an argument of the present thesis, it adds depth to understanding the way algebra matures as a discipline in a mathematical mind. Historically we see the initial role of algebra as a 'method' in the service of giving rigor to geometry metamorphosise into its present role as a 'central subject area or discipline' in mathematics as its nature and attributes are internalised by mathematicians. As a discipline today modern algebra rarely shows its geometrical beginnings explicitly or draws on the visual images or intuition that geometry provides (Atiyah, 1982; Grabiner, 1995; Hilton, 1990). I suggest that it is in Vygotsky's Stage 3, when the method of algebra is used externally to support geometry, that the student may discover the true usefulness of algebra. Conversely as algebra develops and matures into a discipline in its own right, geometry or geometrical intuition can be recalled to support new algebraic ideas. Pedagogically it is in this stage that the links between algebra and geometry can be forged to produce mutual feedback and illumination.

Case studies of separated and unified practice

The theory presented here is examined in two different situations. The first case study is undertaken in a regular second year abstract algebra class at university. The second case study focuses on a class of mathematics education honours students participating in a course in geometry. The two case studies were carried out for different purposes. In the first case study the aim was to support or contradict a hypothesis that students of abstract algebra do not integrate their new algebraic skills with existing techniques from other mathematical perspectives (graphical or geometrical). The second case study aimed at examining a change in attitude and approach by a cohort of students to a problem that appears to be purely algebraic, when a geometric approach is encouraged. In this case the students first attempted the problem with no interventions, and then redid the problem after visualisation was actively promoted.

The two cohorts of students were substantially different. The first was a group of students who were studying full time at a South African university and majoring in mathematics. The second was a group of in-service mathematics senior school teachers who were taking a mathematics education honours degree part-time at the same university. In both cases the author was the course designer and lecturer. It must be noted that in both these studies, formal consent was received from the students to use their work, their responses and questionnaires as research data.

Case Study One: An example of the separation in modes of thought

MATH 204 Abstract Algebra is a first abstract algebra course offered to selected students in their second year of study at university. The students are selected on the grounds of their good achievement in an analysis course. The cohort was made up of 53 students, who completed three problems about bijective (one-to-one and onto) mappings that were selected from tutorial exercises. Question 1(a) and (b) appeared to be algebraic but had geometric or graphical solutions. Question 2 was theoretical with a solution that makes use of the formal definitions of one-to-one and onto mappings. The students had been introduced to algebraic methods of establishing whether given mappings were one-to-one and onto. These new algebraic techniques were introduced to support and complement the geometric methods that students had used at school. It was emphasised in the lectures preceding the case study that the geometric methods learned at school were still useful when the mapping could be represented geometrically or by a graph. The students were encouraged to use any method of solution to the problems.

The exercise given to the students comprised the following (see Equation 1):

Q1: State whether the given mapping is onto, one-to-one, bijective.

(a) $\alpha : \mathrm{N} \to \mathrm{N}$ defined by $\alpha(n) = \begin{cases} \frac{n+1}{2} \text{ if } n \text{ is odd} \\ \frac{n}{2} \text{ if } n \text{ is even} \end{cases}$

(b) $\alpha: \mathrm{P} \to \mathbf{P} \times \mathbf{P}$ defined by $\alpha(x) = (x+1, x-1)$

Q 2: If $\beta\alpha$ is one-to-one and α is onto, show that β is one-to-one.

Equation 1: Exercise for Case 1 study.

The exercise was chosen in order to examine the methods of solution, the accuracy of answers and to indicate whether or not the students, when in an algebraic environment, felt comfortable or inclined to use geometric representations to support their arguments or findings.

The mapping in Q 1(a) is **not one-to-one** and using the graphical representation a counter example can be found (Figure 1). The range of the mapping in Q1(b) can be drawn to produce support to the fact that the mapping is not a bijection (Figure 2) since it is **not onto $\mathbf{P} \times \mathbf{P}$**.

The solution of Question 2 uses the algebraic technique for proving one-to-oneness both as a given property and a required property. It also calls for the use of the definition of onto mappings. As a result the algebraic manipulations are quite sophisticated. The problem is abstract and needs the application of theory (see Equation 2).

Analysis of Data

In Q1(a) out of the 53 students only six students used graphical representation to solve the problem. The remaining students all used a combination of lengthy algebraic techniques and counter examples to establish (correctly and incorrectly) the results. In Q1(b) none of the students used any form of graphical representation to support their arguments. Only two students who used graphical representation in part 1(a) gave correct answers in Q1(b) (without using a graphical representation). It

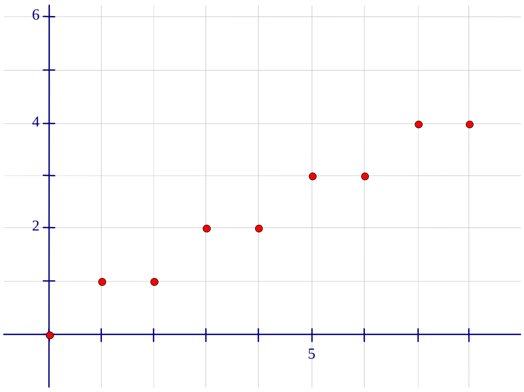

Figure 1: The graph of α in Q1(a)

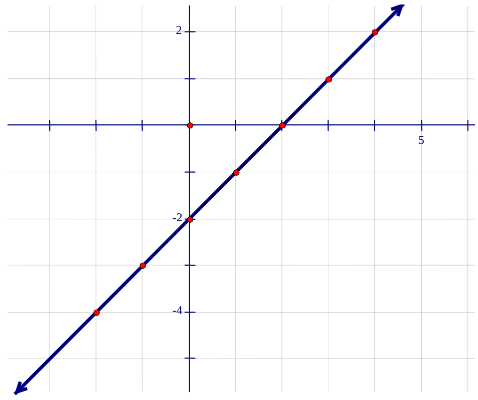

Figure 2: The range of α in Q1(b)

is noted that all 53 students correctly proved the mapping in Q1(b) is one-to-one, 48 correctly proved the mapping in Q1(a) to be onto, and 38 of the 53 managed to answer Q2. Yet in all, only 11 students answered the three questions correctly.

This study supports my contention that while graphical representation is available to students, Q1(a) was one-to-one. The algebraic routines and not the mappings appeared to be the focus of their attention.

This case study indicates that when a mapping satisfies the bijective property the students are able to solve the problems using the algebraic techniques they have mastered. The problems

$b_1, b_2 \in \boldsymbol{B} \Rightarrow \exists\, a_1, a_2 \in A$ with $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$, since α is onto *B*

$\beta(b_1) = \beta(b_2) \Rightarrow \beta(\alpha(a_1)) = \beta(\alpha(a_2)) \Rightarrow \beta\alpha(a_1) = \beta\alpha(a_2) \Rightarrow a_1 = a_2$, since $\beta\alpha$ is one-to-one.

Therefore β is a one-to-one mapping.

Equation 2: Solution to Q2.

this means of dealing with a problem is largely ignored in an ‘algebra’ course, once an algorithmic routine is learned and assimilated by the students. This study also highlights the fact that while algebraic techniques may be mastered they are not always the best method of solving a problem. In Q2(b) the students were successful in showing that the mapping was one-to-one but half the class also, erroneously, ‘proved’ it was onto. Similarly many students, following the technique for proving a mapping one-to-one ‘proved’ that the mapping in involved the concepts of one-to-one and onto for which they had learned an algorithm. The mapping in Q1(a) was onto and the mapping in Question 1(b) was one-to-one and so both could be correctly solved using the algorithm. The majority of the students performed this task. In terms of Vygotsky’s taxonomy they were at Stage 2, indicating that they could use the algorithms. However, the responses to the mapping in Q1(a) being one-to-one and the mapping in Q2(b) being onto show that most of the students are not at Stage

	Q 1(a) “*Proved*” Map 1-1	**Q 1(a)** *Proved* **Map onto**	**Q 1(b)** *Proved* **Map 1-1**	**Q 1(b) “*Proved*” Map onto**	**Q 2** *Proved*	**All proved correctly**
No. of Correct answers	17	48	53	24	38	11

Table 1: The distribution of answers in the class.

3. They ‘forced’ the algorithm to ‘work’ in order to ‘prove’ the results. I would suggest that the 11 students who achieved correct answers to all the questions were at Stage 3. They recognised that the algorithm failed to give results in these cases and used alternative methods to find a solution.

It is my suggestion that with the help of the geometric representation many of the students would have successfully answered all the questions. The graphical representations would help the students to focus on the actual mappings and to use the algorithms or set routines where appropriate. This support would help the students achieve the functional cognition of Stage 3. Evidence from Case 2 indicates that as students broaden their perspectives to incorporate a graphical or geometric representation, their understanding of the algebraic concepts is enhanced.

The Problem: For which values of *k* will $\frac{x^2-x+1}{x^2+x+1}=k$ have real roots?
What are the maximum and minimum values of *k* for *x* real?

Equation 3: Problem for Case 2 study.

Case Study Two: An example of the reunification of modes of thought

SCED 400 is a two-year, part-time, postgraduate course in Mathematics Education. 22 students (two students did not respond to questionnaire), all of whom are practicing teachers, participated in a geometry module as part of the course. This particular case study took place at the first lecture of this course. The study was broken into two parts: An initial investigation where a problem was solved by the students in groups in a classroom situation, and a follow up investigation where the problem was explored in a computer laboratory by students individually.

The aim of the study was to investigate:

- How students solved the problem.
- How they categorised the problem.
- Whether students understood the nature of the problem and its solution.
- Changes in student responses resulting from joint geometric and algebraic exploration of the problem.

The students’ responses to the questionnaire, their written solutions and comments were examined. This questionnaire was first completed after the initial work period, and then again after the computer laboratory session. In addition the students (working in groups) compiled a document recording their solutions, impressions and rough work of the two sessions and their reflections on the problem solving experience.

The problem can be looked at in various ways.

$$\frac{x^2-x+1}{x^2+x+1}=k \Rightarrow x^2-x+1=k(x^2+x+1)$$
$$\Rightarrow x^2\left(1-k\right)+x(-1-k)+1-k=0$$

$$\Delta=(-1-k)^2-4\left(1-k\right)(1-k)$$
$$=k^2+2k+1-4(k^2-2k+1)$$
$$=-3k^2+10k-3$$
$$=(-3k+1)(k-3)$$

$$\Delta\ge 0 \Rightarrow (-3k+1)(k-3)\ge 0$$
$$\Rightarrow (-3k+1)\ge 0 \text{ and } (k-3)\ge 0 \text{ or } (-3k+1)\le 0 \text{ and } (k-3)\le 0$$
$$\Rightarrow k\le\frac{1}{3} \text{ and } k\ge 3 \text{ or } k\ge\frac{1}{3} \text{ and } k\le 3$$
$$\Rightarrow \frac{1}{3} \le k\le 3$$

Equation 4: Algebraic solution to problem in Case 2 study.

The algebraic approach is to find a quadratic equation in x and solve $\Delta \geq 0$ (Equation 4). The method does not require any geometric representation and does not refer to the polynomials comprising the rational function in any way.

Geometrically the problem could be looked at in three different ways. The dynamic geometry system can be used to animate the various positions of the parameter k.

Case 1. Sketch intersections of the curves $f(x)=\dfrac{x^2-x+1}{x^2+x+1}$ and $g(x) = k$, allowing k to vary from $min\{f(x) \mid x \in \mathbb{R}\}$ to $max\{f(x) \mid x \in \mathbb{R}\}$ as in Figure 3.

Case 2. Sketch the quadratic function $y = x^2(1-k)+x(-1-k)+1-k$, varying k through all real numbers, showing its limiting positions for real solutions (Figure 3).

Case 3. Sketch the discriminant Δ of $y = x^2(1-k)+x(-1-k)+1-k$, and indicate where $\Delta \geq 0$ as in Figure 5.

In Figure 3 and Figure 4 the continuously varying k can be used to corroborate the solution $\frac{1}{3} \leq k \leq 3$, obtained either algebraically or from Case 3 (Figure 5), and that these bounds are the minimum and maximum values of k respectively.

Figure 3: Graph in Case 1.

In this example the *Geometer's Sketchpad* introduces different perspectives of the same problem. The confusing appearance of the quadratic equation $\Delta(k) = -3k^2+10k-3$ that needs to be solved for $\Delta \geq 0$ may be avoided or in turn explained, as the inequality $\frac{1}{3} \leq k \leq 3$ keeps reoccurring in each approach.

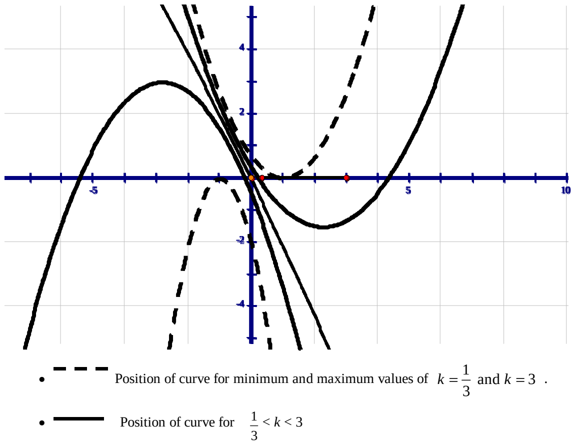

Figure 4: Graph in Case 2.

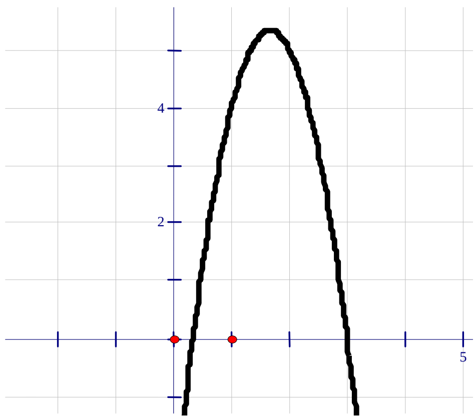

Figure 5: Graph as in Case 3.

Analysis of Data

The 22 students completed the problem in four groups, comprising two groups of five (Groups 1 and 4) and two groups of six (Groups 2 and 3). The problems were discussed on the board and the solutions submitted. It was evident that all the students understood what was expected of them and completed the problem routinely. They expressed the fact that the problem was one that they were familiar with and was a standard problem at grade 11 and 12 at school. When asked about the geometrical significance of the problem they all drew the Real line and indicated the interval where the possible solutions lay. The class remained unresponsive when pressed for a further geometrical interpretation of the problem: "*Initially … we believed the question to be only* *algebraic in nature. We could all reach the correct answer but were not sure of the meaning of this result*" (Group 4 reaction) and "*Initially we only relied on algebraic manipulation of the equation, with no idea that the equation (problem) could be dealt with geometrically*" (Group 1 reaction). The students were surprised to be asked about a geometrical significance. They felt the problem was an algebraic problem.

The class then set up the curves with *Geometer's Sketchpad*, using the various forms (see Figures 3, 4, and 5) of the equation (intersecting curves) as indicated. In each case the solution $\frac{1}{3} \le k \le 3$ reappeared, although the equation occurred in equivalent forms.

The class' responses to the problem presumably changed as a result of the intervention. The results of the questionnaire indicate a swing from thinking strictly algebraically to mixing algebra and geometry. Before the intervention 15 of the students classified the problem as being strictly algebraic and five classified it as being both algebraic and geometric (with two of the five indicating the geometric aspects of the problem lay in the solution on a number line). The work handed in concurred with this result. The only 'geometry' in their work was the representation of the solution to the problem on a number line.

The questionnaire also indicated that all the students believed that you needed knowledge of quadratic formula, equations and roots to **understand** the problem, while knowledge of graphs, curves and their intersections was not relevant (only one student saw this aspect as being highly significant to the problem) for understanding. After the intervention with dynamic geometry all the students acknowledged the geometric or graphical significance of the problem and of its solution.

The reflections of the students in the documents recording their impressions of the intervention are also illuminating. These expressions give evidence of the effectiveness in combining a geometrical and algebraic approach to this type of problem solving activity.

> "*We also realise that the geometry aspect of the problem is very important as it attempts to show how these values are true for the given equation, it also gives insight in terms of explaining why the values $x \in [0.3;3]$ will produce real roots for the equation*"(Group 1 reaction)
>
> "*We were very impressed with Sketchpad's capability to show how the graph changed from parabola concave up, to straight line, to parabola concave down depending on k…the graphical approach using Sketchpad did not provide exact answers but allowed us to view the problem from different graphical perspectives. We felt that this approach to teaching mathematics would act as a catalyst to develop conceptual understanding rather than procedural understanding. Finally, for a complete understanding of the problem, we felt that we should consider the problem from as many angles as possible*" (Group 4 reaction).
>
> "*The solution we got from this parabola confirms the solution we got from the algebraic method where $\frac{1}{3} \le k \le 3$. We discovered that both the algebraic method and the graphical method*

	Algebraic		Algebraic and Geometric	
	Before	**After**	**Before**	**After**
Classification of problem	15	2	5	18
Understanding the Problem	16	3	4	18
Solving the problem	18	13	2	7

Table 2: Analysis of Rational Function Questionnaire.

(Sketchpad) complement each other but the graphical one gives insight into the solutions though sometimes the sketchpad does not give answers but estimations… the algebraic solution of inequalities in most cases gives learners problems especially where the inequality sign has to change. Using both methods can be more helpful to learners because they will have more insight into the problems" (Group 3 reactions).

Referring to the Vygotsky taxonomy, I believe the students had all achieved the naïve understandings of Stage 2 before the intervention. I suggest that introducing the graphical representations using *Sketchpad* broadened the perspectives of the students. The intervention stimulated the students to ask questions about the meaning and interpretation of the results. The algebra could then be seen as bringing rigor to the geometric insights. In the case of Group 3 and 4 there was an indication that the third stage of Vygotsky's taxonomy was engaged. This is evidenced by their clear and insightful reflections on the *Sketchpad* intervention.

Conclusions

The two case studies examined above serve to support the hypothesis that for most students in the groups there is a separation in the thinking modes of algebra from geometry.

The first study underscored the fact that even among a mathematically talented group of students, the compartmentalising of the different disciplines was almost complete. The students were familiar with the process of checking one-to-one and onto graphically from school, but the new algorithmic procedure took precedence in their solutions, even when it proved insufficient to the task. In the second study, no more than the algebraic algorithm was needed to solve the problem. However, insight and understanding into what the algorithm was doing was missing. The introduction of the geometric approach to the problem broadened the perspective of the students allowing them to see the purpose of algebra as a tool in geometry. Here dynamic geometry played an important role. Hence in both cases the separation robbed the mathematical endeavour of its depth and relevance.

As I have noted previously, algebraic representation is used to reduce the complexity of a problem in mathematics. Once the problem has an algebraic formulation a solution may be found through an appropriate algorithm. Yet it is the resolution of the problem and not the application of the algorithm that is the centre of mathematical attention. If the attention of the student is drawn back to the given problem then the instrumental function of the algebraic tool will be established. The unity between the task that must be completed and the method used to complete it, indicates that Stage 3 of the cultural development has been engaged. It is here that visualisation and the technologies of dynamic geometry can play a vital role. The graphical representations allow reflection on the purpose and process of the algorithm.

An algebraic problem stated in a decontextualised form promotes algorithmic solution as the primary activity. Thus the activity belongs at Stage 2 of naïve amplification in Vygotsky's taxonomy. If the same algebraic polynomials are related to the curves they represent then through visualisation, with dynamic geometry or through rough sketches, a gateway to deeper understanding may be opened. This activity may encourage students to proceed into Stage 3 of cognitive functioning.

Vygotsky suggests that it is in the important third stage of 'instrumental functioning' that the child masters his external nature by means of techniques or technical means. In this stage the tool has not yet been internalised and still serves as a technology in solving problems. I suggest that this instrumental function of algebra in solving problems in geometry lies at the gateway of the internalised science of algebra. Here the geometry brings deeper understanding of the algebraic concepts. When viewing algebra as a tool we have access to the very process of formation of the higher forms of behaviour as manifest in abstract algebra. I believe that in order to understand the extent of concept development in abstract algebra we need to accentuate and observe the external process of algebra as a method of solution to problems in geometry and use geometry to support and extend algebraic manipulations. In this way we will hold the outer threads of the abstract, internal processes in our hands.

References

APPLEBAUM, P., 1999, "The stench of perception and the cacophony of mediation", *For the Learning of Mathematics* **19**, 2, pp 11-18

ATIYAH, M., 1982, "What is geometry?" *The Mathematical Gazette* **66,** 437, pp 179-184

BERGER, M., 1998, "Graphic Calculators: an Interpretive Framework", *For the Learning of Mathematics* **18**, pp 13-20

BOYER, C. B., 1968, *A History of Mathematics,* New York: Wiley

BRUNER, J., 1986. *Actual Minds and Possible Worlds*. Cambridge: Harvard University Press.

CROWLEY, L. & TALL, D., 1999, "The Roles of Cognitive Units, Connections and Procedures in achieving Goals in College Algebra", in Zaslavsky, O., ed., *Proceedings of 23rd Conference of PME, Israel*, **2**, pp 225-232

EVES, H., 1953, *An Introduction to the History of Mathematics,* Philadelphia: Saunders College Publishing

GRABINER, J., 1995, "Descartes and problem-solving", *The Mathematical Magazine* **68**, 2, pp. 83-97

HILTON, P., 1990, "The role of geometry in the mathematics curriculum", *Pythagoras* **23**, pp 15-20

KATZ, V. J., 2004, "Stages in the history of algebra with implications for teaching". Plenary address, ICME 10, Copenhagen, Denmark

KAWSKI, M., 2002, "Interactive Visualisation in Complex analysis", in *Proceedings of the 2nd International Conference on the Teaching of Mathematics, Crete*, John Wiley, p. 286

NOSS, R. & HOYLES, C., 1996, *Windows on Mathematical Meanings*: *Learning, Cultures and Computers,* Dordrecht: Kluwer

PEA, R. D., 1987, "Cognitive technologies for mathematics education", in Schoenfeld, A. H., ed., *Cognitive Science and Mathematics Education*, pp 89-123. Hillsdale N. J.: Lawrence Erlbaum

PINTO, M. & TALL, D., 1999, "Student constructions of formal theory: giving meaning and extracting meaning", in Zaslavsky, O., ed., *Proceedings of 23rd Conference of PME*, *Israel*, **4**, pp 64-72

SIERPINSKA, A., TRGALOVÁ, J., HILLEL, J. & DREYFUS, T., 1999, " Teaching and learning linear algebra with Cabri", in Zaslavsky, O., ed., *Proceedings of 23rd Conference of PME, Israel*, **1**, pp 119-134

VYGOTSKY, L., 1978, *Mind in Society*, London: Harvard University Press

VYGOTSKY, L., 1994, "The problem of the cultural development of the child", in van der Veer, R. & Valsiner, J., eds., *The Vygotsky Reader*, pp. 57-72. Oxford: Blackwell

YERUSHALMY, M., SHTERNBERG, B. & GILEAD, S., 1999, "Visualisation as a vehicle for meaningful problem solving in algebra", in Zaslavsky, O., ed., *Proceedings of 23rd Conference of PME*, *Israel*, **1**, pp. 197 – 211

"If the human brain was so simple that we could understand it, we would be so simple that we couldn't."

Willem Hendrik Gispen