Limits of functions – how do students handle them?

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This paper aims at formulating and analysing the development of 15 students in their creations of mental representations of limits of functions during a basic mathematics course at a Swedish university. Their concept images are sought via questionnaires and interviews. How do students respond to limits of functions? The data indicate that some students have incoherent representations, but they do not recognise it themselves.

Introduction

The notion of limits of functions is an important part of calculus. It is the foundation of many other important concepts. If students do not understand what limits are about, how can they understand concepts as, for example, derivatives and integrals?

Students can experience cognitive difficulties when they are learning about limits of functions, for example understanding the formal definition or the rules for deciding limit values (Cornu, 1991; Szydlik, 2000; Tall & Vinner, 1981; Williams, 1991). It can be hard for them to grasp the full meaning of the compact formulation with which they are faced. It is important for teachers to know the different, and perhaps unexpected, results students can get from taking a course in calculus. Students in the same class can form totally different mental representations from the same books and lectures. If we know how these different representations are formed, then we are better equipped to provide a rewarding learning environment for the students.

In this paper I will discuss the development at the first university level of 15 Swedish students' learning about limits. The results are however relevant for other countries as well since the difficulties encountered by students are similar. A browse through some South African universities' course descriptions indicates that courses at basic level comprise mainly the same topics as the Swedish courses do. The questions posed in the paper are: How do the students' mental representations of limits of functions develop during their first semester of mathematics? How do the representations change, if they change, during this time? Are the mental representations of the high achieving students different from the rest of the students? Answers to these questions provide an image of the students' development of the notion of limits of functions.

Theoretical framework

In this section I present a theoretical framework for the results of the study. It starts with a discussion about concept formation followed by a presentation of results from similar studies from other countries.

Concept image

Tall (1991) states that there are many kinds of minds. Different kinds of minds are needed in the development of mathematics and no one kind of mind is always superior to another. One way to distinguish these different minds is to look at the way the learners are thinking of a concept and its formal definition. Tall and Vinner (1981) introduce two notions called *concept image* and *concept definition*. The concept image is the total cognitive structure associated with a concept. It can be a visualisation of the concept or experiences of it or both. The concept image is individual and in different contexts the same concept name can evoke different concept images. For this the term *evoked concept image* is used (Tall & Vinner, 1981). This is not automatically all that an individual knows about a concept. The concept definition is a form of words used by the learner to define the concept. The concept definition can be learnt by heart from a book or it can be a personal re-construction of a definition to fit in with the person's mental structure. A personal concept definition can differ from the formal concept definition, which is a definition accepted by mathematicians in general.

Vinner (1991) describes a model with two "cells". One contains the definitions of a concept and the other contains the concept image. If a concept definition is memorised without meaning, the concept image cell is empty. A concept can have several concept images that are evoked at different times. After some time, two or more concept image cells can merge when the student understands the relation between the representations. Time can also have the opposite effect, the student hangs on to one concept image and forgets about the others or only gets access to some images in certain situations. When a problem is at hand and the aim is to solve it, there should be some kind of contact with the definition. It can be that the person first consults the concept image and then the definition to solve the problem, or there can be several changes between the cells. Another scenario is when only one cell is consulted. If it is the image cell that is the only one, it might be that the person's everyday life experience is misleading. In most cases the problem can be solved, but in an unfamiliar situation it might not be enough. Then the lack of the formality of the definition can be the reason for failure.

Abstraction, objects and processes

Reflective abstraction (Dubinsky, 1991) is defined as the construction of mental objects and actions upon them. These actions become processes by *interiorisation*. The objects are created by *encapsulation* of processes. A concept can be thought of both as an object and as a process depending on the current context (Dubinsky, 1991; Sfard, 1991). A *scheme* is a network of processes and objects. The schemas overlap and form a large complex web in the individual's mind. According to Dubinsky, there are five different ways of construction in reflective abstraction, two have already been mentioned:

Interiorisation: Construction of mental processes in order to understand a perceived phenomenon.

Coordination: Construction of a process by coordination of two or more other processes.

Encapsulation: Construction of an object through a process.

Generalisation: Ability to apply an existing schema on a greater range of phenomena.

Reversal: The individual is able to think of an existing internal process in reverse to construct a new process.

Generalisation is not as cognitively hard as abstraction since abstraction often means that the individual has to make a re-construction of the mental representation (Dreyfus, 1991). Crucial properties of the object at hand must be recognised and isolated from the object so that the properties are applicable in other situations. If visualisation is possible it can be a great help. The images can make the important structures and relations clear in a global manner. When an individual is generalising, he or she has a foundation of examples and experience from which to build. It is an extension of what is already there. Abstraction includes the possibility for both synthesis and generalisation. To synthesise is to create a whole from parts. When this is done, it often becomes more than its parts together. Many small disjoint parts get linked together and suddenly more things fall into place. This is a rewarding feeling and once an individual has passed this process, he or she cannot undo it. Soon all the little important pieces of the synthesising process are forgotten and the individual takes the product for granted, that is, the understanding of the notion.

There are cognitively different kinds of generalisations to do. Tall (1991) describes three of them. *Expansive generalisation* is when existing mental representations are unaltered and the new knowledge is attached as a complement. *Reconstructive generalisation* is when existing mental representations have to be changed in order to make sense. *Disjunctive generalisation* is when the individual is rote learning pieces that he or she should learn and just adds it to what is already in the mind. No integration occurs. It becomes impossible to get an overview of the notion at hand.

Learning and understanding

When a process or an object is mentioned we refer to it by a mental representation (Dreyfus, 1991). Some learners prefer visual thinking and have pictures in their mental representations while others use symbols or examples to be able to think of notions. One concept can be represented in more than one way and there can even be conflicting representations that are evoked at different times depending on the context. If the representations are not contradicting they can merge into one when the individual is able to see the connections. A coordination or a synthesis can be the result. If they are incoherent a conflict may arise. The more connections between the mental representations, the better the individual understands the concept. Then he or she can go from one to another depending on the demands of the task.

Knowledge of a concept is, according to Dubinsky (1991), the individual's tendency to bring to mind a scheme in order to be able to handle, organise or make sense of a problem situation. It can be hard to keep mathematical knowledge apart from mathematical construction. In an attempt to do so, we can observe individuals solving problems. Such observations will not explicitly reveal the objects, processes and schemas, but can indicate how knowledge is created. A part of learning is, according to Dubinsky, applying reflective abstraction to existing schemas to create new ones that provide an understanding of the concepts in question. Does this imply that learning cannot take place until at least very simplistic schemas are made? Otherwise the result can be that the students only know how to solve routine tasks and are unaware of the actual range of a concept. They can also create schemas that are incoherent and wrong. Periods of confusion are necessary when new knowledge is to be implemented with existing knowledge and this confusion is a part of the learning process (Cornu, 1991). It is the confusion that creates a need for order and the students start to adjust their schema.

James Hiebert and Thomas P. Carpenter (1992) define understanding of a mathematical concept to be something an individual has achieved when he or she can handle the concept as a part of a mental network. The degree of understanding is decided by the strength of the connections in the net and of the number of accurate connections.

Vinner (1991) states that to understand a concept is not the same as to be able to form its definition formally, but if a person has a concept image he or she can understand the concept. In this paper I will regard understanding and knowledge of a concept as Hiebert and Carpenter (1992) and Dubinsky (1991) do.

The limit concept

The limit concept causes difficulties in teaching and learning (Cornu, 1991). It is a complex concept but there is also a possible cognitive problem in the distinction between the definition and the actual concept. Students can remember the definition but it does not necessarily mean that they understand the concept. Vinner (1991) claims that mathematical definitions represent the conflict between the structure of mathematics and attainment of the concepts via cognitive processes. Mathematics is deduced from axioms and definitions. New concepts arise from previous ones in a logical manner. This is not always the way mathematics is created though. In the beginning students can meet the notion of limit in an informal intuitive way where the tasks involve situations where they can easily see the outcome. This creates a feeling of control and the students think they know what the concept is about even if they could not solve a more demanding task where they would have to master the full meaning of the definition (Cornu, 1991). One problem is the quantifiers “for every” and “there exists” in the ε – δ definition:

“For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$”

In everyday language the quantifiers can have a slightly different meaning compared to the ones used in mathematics. If the students have such conceptual obstacles they can get into trouble later on. The conceptions of an idea before any teaching occurs are called *spontaneous conceptions* by Cornu (1991) and they can be hard to get rid of even after teaching. One reason for this is the fact that when students solve problems, they tend to go to natural or spontaneous reasoning rather than scientific theories. It is as in Vinner’s (1991) model with the cells, where the individual only refers to the image cell. The obstacles start a process of re-constructing existing knowledge that is based on spontaneous conceptions and this can lead to misunderstandings.

Monaghan (1991) found that the students in his study saw the word limit as more specific than, for example, approaches and other verbs. The students had problems with the vagueness of some expressions and this led to confusion.

Cornu (1991) talks about four epistemological obstacles in the history of the limit concept. Two of them are the notions of infinitely large and infinitely small respectively and the question whether the limit is attainable or not. There are also metaphysical obstacles that students of today struggle with, such as the abstractness of the limit concept and the feeling of lack of rigor caused by the notion of infinity (Cornu, 1991; Tall, 1992).

If the obstacles are located, the teaching can be altered accordingly, not by excluding the difficult passages but by supporting and helping the students to overcome them. Tall (1991) describes the *generic extension principle*. It is when an individual is in a situation where he or she only meets examples with a specific property. If no counter-examples are present, the mind assumes that the property is valid in every context even if it is not explicitly stated. One example is if students only see examples of convergent sequences that do not reach their limits, then they might assume that convergent sequences never do.

These results indicate that there are several aspects to consider in a learning situation about limits. The study in this paper is conducted to reveal how Swedish students cope with such situations.

The study

The students and the course are described, followed by the methods and instruments used to answer the questions: How do the students' mental representations of limits of functions develop during their first semester of mathematics? How do the representations change, if they change, during this time? Are the mental representations of the high achieving students different from the rest of the students?

The sample

112 students participated in the study and 29% of them were female. They were aged 19 and up. They were enrolled in a first level university course in mathematics. The course was divided into two sub-courses. Both of them dealt with calculus and algebra. The courses were given over 20 weeks, full time (10 weeks for each course). The students had two lectures and two sessions for task solving three days per week. Each lecture and session lasted 45 minutes. Thus the total teaching time for each course was 90 hours. The notion of limits of functions was presented in the first course before derivatives. It was taught again in the second course in different settings such as integrals and series. The first course had a written examination and the second had a written examination followed by an oral one. The marks awarded were IG for not passing, G for passing and VG for passing with a good margin. I neither taught the students at any stage nor did I know any of them.

Methods

Different methods were used to collect different types of data. The students were confronted with tasks in different ways at five times during the semester, called stage A to stage E.

The students got a questionnaire at stage A at the beginning of the semester. It contained easy tasks about limits and some attitudinal queries. The scope of these and subsequent tasks is described in the instruments section. The students were also asked about the situations in which they had met the concept before they started their university studies. The attitudinal data is not presented in this paper.

After limits had been taught in the course, the students got a second questionnaire at stage B, with more limit tasks at different levels of difficulty. The aim was for the students to reveal their habits of calculating, their ability to explain what they did and their attitudes in some areas. The students were asked if they were willing to participate in two individual interviews later that semester. 38 students agreed to do so. 18 of these students were selected for two individual interviews each. The selection was done with respect to the students' responses to the questionnaires so that the sample would as much as possible resemble the whole group. The gender composition of the whole group was also considered in the choices.

The first session of interviews was held at the beginning of the second course, at stage C. Each interview lasted about 45 minutes. The students were asked about definitions of limits, both the formal one from their textbook and their individual ways of defining a limit of a function. They also solved limit tasks of various types with the purpose of revealing their perceptions of limits and they commented on their own solutions from the questionnaires to clarify their written responses where it was needed.

The students got a third questionnaire at the end of the semester, at stage D. It contained just one task. Two fictional students' discussion about a problem was described. One reasoned incorrectly and the other one objected and proposed an argument to the objection. The students in the study were asked to decide who is right and why.

A second interview was carried out at stage E after the examinations. Each interview lasted for about 20 minutes. The students commented on the last questionnaire and in connection to that, the definition was scrutinised again. The quantifiers "for every" and "there exists" in the $\varepsilon - \delta$ definition were discussed thoroughly. 15 of the 18 students were interviewed at this point.

Field notes were taken during the students' task solving sessions and at the lectures when limits were treated to give a sense of how the concept was presented to the students and how the students responded to it. Tasks and results from other parts of the study are described in more detail in other papers (Juter, 2003; Juter, 2004).

Instruments

At stage A the students solved some easy tasks about limits of functions.

Example 1: $f(x) = \dfrac{x^2}{x^2+1}$.

What happens with f(x) if x tends to infinity?

The tasks did not mention limits per se, but were designed to explore if the students could investigate functions with respect to limits.

At stage B the tasks were more demanding. Some of the tasks were influenced by Szydlik (2000) and Tall and Vinner (1981). Three tasks had the following structure:

Example 2:

a) Decide the limit: $\lim_{x\to\infty}\frac{x^3-2}{x^3+1}$.

b) Explanation.

c) Can the function f(x) = $\frac{x^3-2}{x^3+1}$ attain the limit value in 2a?

d) Why?

Example 2 is what I regard to be a routine task. There were also non-routine tasks. A solution to a task was presented to the students. It could be incomplete or wrong and the students were to make it complete and correct. There were two such tasks.

At stage C, which was the first set of interviews, the students were asked to comment on statements very similar to those used by Williams (1991) in a study about students' models of limits. The statements the students commented on are the following (translation from Swedish):

1. A limit value describes how a function moves as x tends to a certain point.
2. A limit value is a number or a point beyond which a function cannot attain values.
3. A limit value is a number which y-values of a function can get arbitrarily close through restrictions on the x-values.
4. A limit value is a number or a point which the function approaches but never reaches.
5. A limit value is an approximation which can be as accurate as desired.
6. A limit value is decided by inserting numbers closer and closer to a given number until the limit value is reached.

The reason for having these statements was to get to know the students' perceptions about the ability of functions to attain limit values and other characteristics of limits. The students were given the statements to have something to compare to their own thoughts. There were other tasks designed to make the students consider the formal definition to clarify what it really says and tasks about attainability, for example:

Example 3: Is it the same thing to say "For every $\delta > 0$ there exists an $\varepsilon > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$" as "For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$"? What is the difference if any?

As indicated before, at stage D the students got a task with a description of two students arguing over a solution to a task. The problem argued about was whether the limit value $\lim_{x\to 0}\frac{\cos(x^{-2})}{10000}$ exists or not. The task for the students in the study was to decide which argument, if any, was correct and to explain what was wrong in the erroneous argumentation.

At stage E, the second interview, the students' written responses to the task at stage D were discussed. *Example 3* was also brought up again in connection with the task at stage D.

Results

Four of the 15 students' responses are described at the five stages A, B, C, D and E above. The descriptions are summaries of fuller ones. Martin, Tommy and Frank had similar responses, Filip, Leo and Dan had similar responses, Louise, Mikael, Dennis and Oliver had similar responses and Julia and Emma had similar responses. One student's description from each group of similar responses is presented in *Table 1*. The number in brackets after each name in the table indicates the number of students with similar responses. Anna, John and David had responses which did not match any of the above. Pseudonyms have been used to retain student anonymity. The digits at stage C indicate the students' preferred choices from the six statements. The bold and larger digits are the students' choices of statements most similar to their own perceptions of limits. The first points at stage E are the students' explanations of what a limit is and the last points are about *Example 3* where the students explain the difference between the correct and incorrect definition connected to the task from stage D. *Table 2* presents the students' marks. The first letter or letters is the mark for the first course and the second is the mark for the second course. The numbers after the marks are the numbers of times the students needed to take the examination to pass.

The students' connections of limits to other concepts are dominated by derivatives and various

Table 1. Students' developments through the course.

Time	**Tommy** (3)	**Leo** (3)	**Mikael** (4)	**Julia** (2)
A	-Links to derivatives -Solves easy tasks well	-Links to nutrition, physics and biology -Solves easy tasks well	-Links to prior studies, problem solving, physics -Unable to solve easy tasks	-Links to prior studies -Solves easy tasks well
B	-Limits are attainable in problem solving -Cannot state a definition -Solves routine tasks and explains	-Limits not attainable in problem solving -Cannot state a definition -Solves routine tasks and explains	-Limits are not attainable in problem solving -Cannot state a definition -Solves tasks and explains well	-Limits are attainable in problem solving -Can state a definition -Solves tasks and explains fairly well
C	-Limits are attainable in problem solving -Limits are not attainable in theory -A limit of a function is how the limit stands with respect to another function, no motion -3, **4**, 5, 6 -Cannot state or identify the definition -Not a real limit if attainable by function -Problems in solving tasks -Links to derivatives and curves -Claims to master the notion of limits	-Limits are not attainable in problem solving (hesitates) -Limits are not attainable in theory -Thinks of limits in pictures -3, **4** -Cannot state or identify the definition -Solves tasks well -Links to derivatives and number sequences -Claims to master the notion of limits fairly well	-Limits are attainable in problem solving -Limits are not attainable in theory -Thinks logically rather than explicitly defining -2, 3, **4**, 5 -Cannot state but can identify the definition after investigation -Problems in solving tasks -Links to prior studies and graphs -Claims to master the notion of limits in problem solving but not the definition	-Limits are attainable in problem solving -Limits are attainable in theory -It comes closer and closer to A as x comes closer and closer to a -1, **3**, 5 -Cannot state but can identify the definition (claims that both def in ex 3 state the same, ε and δ come in pairs) -Solves tasks well -Links to derivatives, graphs and number sequences -Claims to master the notion of limits
D	-Identifies the error	-Identifies the error, makes other error	-Identifies the error, makes other error	-Identifies the error
E	-A limit is a point to tend to but never to reach, motion -Links to integrals -Cannot identify the definition -Does not understand the quantifiers' roles	-A function tends to a value as the variable controlling the function changes, it never reaches the limit -No links -Cannot identify the definition -Does not understand the quantifiers' roles	-If the x-value tends to a value then the function tends to a value connected to the x-value -Links to Taylor expansions, series and number sequences -Can identify the definition -Can explain the difference between the definitions in ex 3	-If the function goes closer and closer to a number as x goes closer and closer to a value -Links to continuity -Can identify the definition after investigation -Can explain the difference between the definitions in ex 3 fairly well

applications in areas other than mathematics at the beginning of the course. In the middle the connections concern derivatives and functions and at the end Taylor expansions, integrals and series. The results reflect the content of the course quite well.

Table 2. Students' marks.

Name	Mark	Name	Mark
Filip	G2/IG	Dan	VG1/G1
Martin	G1/G2	Mikael	VG1/G1
Tommy	G1/G2	David	G1/VG1
Anna	G1/G2	Julia	VG1/VG1
John	G1/G1	Dennis	VG1/VG1
Frank	G1/G1	Emma	VG1/VG1
Louise	G1/G1	Oliver	VG1/VG1
Leo	VG1/G1		

Table 1 indicates that Tommy is able to solve routine tasks but not non-routine tasks at stage B. By this I mean that he solved all the routine tasks correctly but did not present any correct solutions to the non-routine tasks. Most students were able to handle the easy tasks at stage A at least partially. At stage B, where the students got the non-routine tasks, there was a clearer distinction between high achievers and the rest of the students. At stage C, there is no longer a clear distinction. The two women with highest marks managed the problems while the two men with highest marks did not. The low achievers did not show such a gender distribution and it is probably just a coincident that the high achievers did. There is overall a higher rate of correctly solved problems of all kinds among the high achievers as could be expected.

The students had a hard time formulating a definition of a limit of a function depending on x as x tends to a number at stage B. Only Julia managed to do this in a meaningful manner. Most of the other students just reformulated the words in the task to something that did not say much.

The six statements at stage C revealed that among students with marks G and IG the most frequently selected statement as most similar to their own perception was number four, which says that limits are never attained. Statement three was the most frequently selected statement among the other students. Statement three was not selected as most similar to their own perception by any of the students with marks G and IG. The students seem confused about functions' abilities to attain limit values. Most students' concept images are incoherent at this point. Theoretical situations are treated differently from problem solving situations.

Despite their lack of capability to state a definition that describes the limit of a function, many of the students still felt as if they had control over the concept. Three of the students, all without the mark VG, thought that they did not have control over the concept of limits at stage C. The other students were rather certain that they had control over the concept, despite the fact that almost none of them could formulate or in some cases even recognise the definition. Only Emma is able to formulate the formal definition. From the seven students with marks G or IG it was only Anna who recognised the correct definition in *Example 3*. Among the eight others there were five who recognised the correct definition, but among them, Emma and Julia stated that they both said the same thing. Dan, who could not recognise the definition, thought so too.

David had also indicated that the two definitions in *Example* 3 meant the same thing at stage E, but now both Julia and Emma were able to explain the difference of the definitions and thereby also the meaning of the quantifiers. Emma explained her solution to *Example 3* like this:

> "…in the real definition, δ depends on ε."
> "…it is the independent variable which so to speak forces the function value in and not the other way around."

Many students did not show such security in their explanations. The quantifiers were still a mystery to them. Dan reasoned about *Example 3* like this:

> "…in most cases I think you had been able to do it this way too, it does not seem totally unreasonable." [About the wrong definition]
> "…you can go the other way, but you have to in a way check as you go along." [Check if it is possible to make ε arbitrarily small for the δ chosen in the wrong definition]

Students with marks G or IG had trouble understanding the difference between the two definitions through the course. It was not easy for them to see connections between the formal definition and the example with the fictitious students arguing. Most students with higher marks were, after some thinking, able to tell the difference at stage E. They showed a better understanding (Dubinsky, 1991; Hiebert & Carpenter, 1992) in that they could explain the roles of the different components used in the definitions in connection to each other.

The question whether limits are attainable seems to be confusing, as other researchers have also established (Cornu, 1991; Williams, 1991). Most of the students' responses to questions and tasks about attainability presented in this paper are incoherent. This study shows that the students interpret the definition as stating that the limits can not be reached by the function. When they solved problems on the other hand, they could see that limits sometimes are attainable for functions.

Discussion

The three questions posed by this study are here answered with the results and theoretical framework presented. The students' developments are discussed, followed by the changes in their concept images. The question whether high achievers' results differ from low or average achievers' results is discussed along with the two first questions.

The students' developments

The students seem to be very pragmatic when they learn about limits. They focus on the problem solving and not so much on the theory, with the result that the theory is somewhat vaguely represented in their minds. Yet they are almost all confident about their ability to grasp the concept. This lack of awareness can perhaps be made explicit for them if the students are put in challenging and explorative situations more often.[1] It is impossible to know what the students are actually thinking (Asiala *et al*., 1996). All I can do is to see what they can accomplish and what they say and do. After reading results from other studies (Cornu, 1991; Monaghan, 1991; Tall, 1991; Tall, 1992; Vinner, 1991) I was prepared to see students having difficulties, not least with the definition, but I did not expect to see so much confusion through the course.

Looking at the students as a group there is a positive development in their achievements. They get better at problem solving and understanding the theory, but some of them do not get as far as to fully synthesise (Dreyfus, 1991) the concept during the course. Parts of the concept and its applications are known, but not enough for it to become an entity.

Many students could briefly explain what a limit is, but when it came to explaining the partial transposition of ε and δ in *Example 3* there were serious problems in most cases. Some students claimed that the transposition did not make any difference since the ε and δ come in pairs. This view is independent of the students' marks. It is hard for them to understand the role of the quantifiers as Cornu (1991) also found. Dan, for example, does not take the words "for every" into account when he says that you can check if there is an ε which can be made arbitrarily small for every choice of δ in the wrong definition.

The students do not seem to focus on theory in the first place since so few can explain what the definition says throughout the semester. Many have a split concept image which is particularly obvious in the attainability question. The incoherence in the limits' attainability shows that the representation of the definition is not evoked in the problem situation (Tall & Vinner, 1981; Vinner, 1991). A reason for this can be that since the students thought that there is no point in dealing with limits if the limits are attainable, they did not consider them to be limits if they were. It can be a case of the generic extension principle (Tall, 1991). The functions in most of the examples the students meet do not attain their limits.

Many of the students think that the definition says that functions can not attain limit values. There is a big difference in how high achievers perceive this part of the definition from how low and average achievers do. The results from the six statements in the instruments section show that the latter group of students is more likely to perceive limits as unattainable.

Cottrill, *et al*. (1996) argue that the problem of learning limits can be dependent on difficulties in creating mental schemas rather than the formal definition. The results of this study rather imply that the students have problems creating mental schemas because of the formal definition.

Changes

The representations that the students form change through the duration of the course. Some changes are rearranging entire parts of the concept image. Some changes involve just smaller modifications of, or attachments to, the concept images. Then there are parts that even though they can be wrong, are left unaltered. In some cases the students do not give up their first representations so readily. One example is the students who think that limits are unattainable for functions and stick by this apprehension through the entire course despite counter examples. Williams (1991) mentions one possible reason for this. Suppose a student is in a situation where his or her experience so far has been consistent with the theory as he or she perceives it. Then the student encounters an example or something else that does not fit in with

[1] Some students actually told me that tasks like the non-routine tasks in the study would be good to have in the course to stimulate discussion.

the theory. Then, instead of adjusting his or her interpretation of the theory, the student might just regard the example as a minor exception.

It can be hard to change a part of a mental representation. It can affect the rest of the representation in a way that requires a further modification. This can be a reason for leaving parts as they are even if the individual is aware of the error. But changes did come to pass in the study. Tommy for example stated at stage C that there is no motion in limits. At stage E he had changed his mind and altered his concept image. He had also gone from perceiving a limit as a comparison between functions to an unreachable point. John had also changed his concept image. He went from focusing on the distance between the function and the limit to seeing the limit as a border. Those changes need not make any difference in the rest of the students' mental representations since they all can be true. But they alter the way the concept is thought of. Perhaps the mental representations are not changed, but have stimuli that evoke different parts of the concept image (Tall & Vinner, 1981).

Despite the problems, most students still had a sense of control over the concept of limits. The sense of control probably comes from successful problem solving in the course (Cornu, 1991). The students can have a concept image that works in the situations they have been in so far. When they are put in a different kind of situation their concept images can be altered. As long as the students' concept images are challenged in some way, they will still have the opportunity to alter, refine and expand them. If they had a good concept definition compatible with the formal definition, the adjustments would probably be successful (Vinner, 1991). Anna is one example of this. At stage C she was solving a problem where she had a graph of a function and she had to decide a limit value at a discontinuous point. Right and left limit values were both three but the functions value at the point of discontinuity was two. Anna correctly stated the right and left limit values, but hesitated when she had to give the total limit value. She first said two but was uncomfortable with it since she felt as if she had contradicted herself. Her concept image and concept definition were in conflict with each other. Anna's concept definition, saying that functions never reach their limits, made her say that the right and left limits are three. Her concept image, saying that the functions value at the point should be the limit value if it is attainable, made her want to say it is two. After a while she got to look at the formal definition and then she was able to correctly solve all parts of the task.

The material presented here has been further analysed and more results have been published (Juter, 2003; Juter, 2004). Among the results is the impact of algebra on the students' learning of limits. The more we know about the students' abilities and perceptions, the better we can do our jobs as mathematics teachers.

References

ASIALA, M., BROWN, A., DEVRIES, D., DUBINSKY, E., MATHEWS, D. & THOMAS, K., 1996, "A framework for research and curriculum development in undergraduate mathematics education", *CBMS Issues in Mathematics Education* **6**, 1-32

CORNU, B., 1991, "Limits", in Tall, D., ed., *Advanced Mathematical Thinking*, pp 153-166, Dordrecht: Kluwer Academic Publishers

COTTRILL, J., DUBINSKY, E., NICHOLS D., SCHWINGENDORF, K. & VIDAKOVIC, D., 1996, "Understanding the limit concept: Beginning with a coordinated process scheme", *Journal of Mathematical Behaviour* **15**, pp 167-192

DREYFUS, T., 1991, "Advanced Mathematical Thinking Processes", in Tall, D., ed., *Advanced Mathematical Thinking*, pp 25-41, Dordrecht: Kluwer Academic Publishers

DUBINSKY, E., 1991, "Reflective Abstraction in Advanced Mathematical Thinking", in Tall, D., ed., *Advanced Mathematical Thinking*, pp 95-126, Dordrecht: Kluwer Academic Publishers

HIEBERT, J. & CARPENTER, T. P., 1992, "Learning and teaching with understanding", in Grouws, D. A., ed., *Handbook of Research on Mathematics Teaching and Learning*, pp 65-97, New York: National Council of Teachers of Mathematics

JUTER, K., 2003, *Learning limits of function, University students' development during a basic course in mathematics*, licentiate thesis: Luleå University of Technology, Department of Mathematics

JUTER, K., 2004, "Limits of functions – how students solve tasks", in Bergsten, C., ed., *Proceedings of MADIF 4, the 4th Swedish Mathematics Education Research Seminar*, pp 146-156, Malmö, Sweden

MONAGHAN, J., 1991, "Problems with the language of limits", *For the Learning of Mathematics* **11**, 3, pp 20-24

SFARD, A., 1991, "On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin", *Educational Studies in Mathematics* **22**, pp 1-36

SZYDLIK, J., 2000, "Mathematical beliefs and conceptual understanding of the limit of a function", *Journal for Research in Mathematics Education* **31**, 3, pp 258-276.

TALL, D., 1991, "The psychology of advanced mathematical thinking", in Tall, D., ed., *Advanced Mathematical Thinking*, pp 3-24, Dordrecht: Kluwer Academic Publishers

TALL, D., 1992, "Students' Difficulties in Calculus", in *Proceedings of Working Group 3, The 7th International Congress on Mathematical Education*, pp 13-28, Québec, Canada

TALL, D. & VINNER, S., 1981, "Concept image and concept definition in mathematics with particular reference to limits and continuity", *Educational Studies in Mathematics* **12**, pp 151-169

WILLIAMS, S., 1991, "Models of limit held by college calculus students", *Journal for Research in Mathematics Education* **22**, 3, pp 219-236

VINNER, S., 1991, "The role of definitions in the teaching and learning of mathematics", in Tall, D., ed., *Advanced Mathematical Thinking*, pp 65-81, Dordrecht: Kluwer Academic Publishers

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Parallelism

to Martin Gardner

"Lines that are parallel
meet at Infinity!"
Euclid repeatedly,
heatedly,
    urged.
Until he died,
and so reached that vicinity:
in it he
found that the damned things
    diverged.

Piet Hein