

Preußner functions for volume estimation of *Pinus taeda* L. in Southern Brazil

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(Received for publication 30 August 2019; accepted in revised form 16 April 2021)

Abstract

Background: Taper modelling and volume estimation are key procedures in the management and planning of planted forests. The objective of this work was to evaluate the taper and volume behaviour of *Pinus taeda* L. species, along the stem in different ages, using the Preußner taper functions, compared to Schöpfer's 5th-polynomial, Kozak's and Max-Burkhardt's models. This work focused on plantations of *Pinus taeda* L., due to its wide use as a source of raw material in the forest industry of southern Brazil.

Methods: The data were collected in the last 22 years from the Midwest region of Santa Catarina, of trees ranging in age from 3.5 to 18 years. This dataset consisted of a collection of volume sections, with relative diameter measurements along the stem, used in conventional forest inventory. The total volume of the trees, obtained by integrating the Preußner taper functions, was equated by dividing the stem into four parts, in which parabolas were fitted, and compared with the total and merchantable volume estimated by Schöpfer's 5th-degree polynomial, Kozak's and Max-Burkhardt's functions.

Results: Bias, RMSE and *r* were generally better with the application of Kozak's model, and AIC and BIC for the Preußner's model.

Conclusions: Kozak functions were better to provide the merchantable volume. In terms of total volume both functions, Kozak and Preußner, provided reliable estimates. The advantage of the procedure proposed by Preußner is the flexibility of the fitted taper functions, the simplicity of volume calculations by integration, and the feasibility for interpreting their coefficients.

Keywords: modelling; taper; volume estimation; Preußner; Schöpfer's 5th-degree polynomial; Kozak's taper model; Burkhardt's taper model

Introduction

The use of wood to produce high-quality sawn timber has become increasingly prominent throughout the world, providing increased demand for forest research to calculate tree volumes in a simple and rapid manner. Wood is an essential raw material used to meet various demands in Brazil – it is important in the furniture, civil construction, agricultural, and industrial sectors and is the second leading commodity of the state of Santa Catarina (Anuário Estatístico do Estado de Santa Catarina 2016).

The rational and appropriate use of available wood volumes and their assortment is associated with the growth of a specific species, mainly through studies

on taper functions. Among these, we can highlight the dynamics of the stem, such as the evaluation of its taper, i.e. the reduction in diameter with increasing tree height. This variation can be expressed using a mathematical function (Miguel et al. 2011).

Among the variables used to evaluate wood quality for total volume and assortment purposes, the tree profile has been one of the most studied subjects in commercially valuable species. It is common to measure tree diameter at the base, i.e. at 0.3 m, and at 1.3 m aboveground and, thereafter, at fixed length intervals using Smalian's method or at relative positions using Hohenadl's method.

The use of stem equations makes it possible to describe the tree profile and, by integration, to obtain an estimation of the total and partial volumes in different sections with reasonable precision. These estimators are of paramount importance, both for quick evaluation of wood stock, in volume and biomass, and to obtain assortments of log types. The current modelling methods used to describe the profile of the tree are diverse.

Taper functions describe the shapes of trees and express diametric variation along the stem as a function of diameter at breast height (DBH) and relative heights. These variations are affected by species, at different ages, spacing and site quality.

For better performance of taper functions regarding the adjustment and the use of inflection points, Kozak (1988); Lee et al. (2003) introduced taper models in a variable way and Bi (2000) used models with trigonometric principles, represented by non-segmented regression models, which implicitly divided trees into segments, without the need for inflection points.

Taper is understood to mean the rate of decrease in diameter as the bole height increases (Newnham, 1992). The shafts of both coniferous and broad-leaved until the beginning of the canopy may take the following characteristics: (1) the base of the stem is convex to the longitudinal axis, with a variable inflection point, depending on the species, which resembles a neiloid. (2) from this inflection point until the top of the canopy, the external profile is concave to the axis. In conifers, this portion of the stem can be described by a paraboloid. (3) the terminal parts of conifers, which go from the bases of crowns to their apex, presents a profile slightly concave to the axes of trees, and can be represented by a quadratic paraboloid or cone (Assman, 1970).

Max and Burkhart (1976) proposed that the tree shapes could be represented by a segmented polynomial model, joining such segments along the stem. This idea came to be used widely by several researchers (Maguire & Batista 1996; Clark et al. 1991; Leites & Robinson, 2004; Machado et al. 2004; Jiang et al. 2007).

Taper functions have been one of the most important topics of study in forest management in the last century (Fang et al. 1999). Various profile shapes of the stem and types of models have been proposed and evaluated for their accuracy (Sterba 1980; Clutter et al. 1983; Avery & Burkhart 2002; Kozak 2004; Rojo et al. 2005).

All theoretical developments to describe the profile of the tree bole are approximate to adequately explain all variations in the shapes of trees. From an objective and practical point of view, a taper function is essential to estimate the volumes of standing trees for multiple uses, especially when establishing the production planning of the stands or detailing restrictions on the sizes of the diameters and lengths of the sections on the tree boles (Newnham 1988; Sharma & Zhang 2004; Trincado & Burkhart 2006).

According to various authors (Prodan 1965; Kozak et al. 1969; Huang et al. 2000; Kozak 2004; Adams 2005) taper models are relationships that describe the longitudinal profile of the bole of a tree with dynamic functions, because they make it possible to estimate the

total volume of the tree and logs along the bole, as well as diameters at any position of the bole, height along the trunk, volume between any two points in the trunk, in addition to following the evolution of the taper of the tree at different ages.

Although the first attempt to establish a relationship between diametric series and series of heights was presented by Höjer (1903), it was only much later that it was carried out by Altherr (1953), who built a system based on the slide rule principle, in which diameters and heights were obtained in absolute values from their relative values. Later, the possibility of relating diametric and height series by means of a polynomial function was suggested by other authors. Osumi (1959) applied this concept to the species *Criptomeria japonica*, having solved it by adjusting a polynomial of the 3rd degree. Wutt (1961), Prodan (1965), and Schöpfer (1966) proposed the adjustment of polynomials of the 4th and 5th degrees, with the argument that describing the shape of the bole with a single function, facilitating its integration and allowing to obtain reliable precision was desirable. From these results, Peters (1971) applied such modelling to *Araucaria araucana* with reliable results in Chile and Burger et al. (1979) to *Araucaria angustifolia* also with reliable results in Brazil. Researchers then tried to group the different concepts and models applied to describe the profile of tree boles and proposed a separation of them in non-segmented and segmented models as follows:

Non-segmented models

Following Demaerschalk & Kozak (1977), among the several statistical modeling techniques, we highlight the models that are non-segmented, which manage to combine efficiency with simplicity when compared to the segmented models. Of these, the first models developed and used were small order polynomials, for a relative height on the bole (Rojo et al. 2005).

In 1923, Behre used data from Ponderosa pine to determine whether or not the Höjer (1903) equation could be improved by introducing a new term, or if a different equation could be found for describing the average taper of the tree bole. Consequently, after these studies, a new equation was developed which described the bole form more consistently.

Kozak et al. (1969) developed a taper function based on the assumption that a tree bole is a quadratic paraboloid, whose equation is based on a quadratic polynomial of second order.

As some taper functions were inadequate for describing the form near the base of the bole, higher degree polynomials were used to correctly characterise the base (Rojo et al., 2005). Bruce et al. (1968) and Goulding & Murray (1976) also used high degree polynomials to characterise the bole profile.

Demaerschalk (1972; 1973) developed taper functions from integrated systems for volumetric estimates from which they were derived from volume equations based on the Schumacher and Hall (1933) model, Spurr's model (1952), Honer's model (1965) and other variations.

Many taper models were developed with the objective of having better results of accuracy for dendrometric variable estimates (Ormerod 1973; Forslund 1991; Amidon 1984; Biging 1984; Baldwin & Feduccia 1991; Sharma & Oderwald 2001). Souza et al. (2008), analyzing the performance of six non-segmented models (linear and non-linear) with data from *Eucalyptus* sp. driven to the production of sawmill wood, concluded that the Biging's model provided greater accuracy in height estimation and commercial volume, followed by the Garay model (1979). The same author compared the performance of non-segmented models with segmented ones and pointed out that, in addition to being simpler, the best results were obtained with the non-segmented models.

Segmented models

According to Demaerschalk and Kozak (1977) it is necessary to use different models for studying taper, one for the bottoms and the other for the tops of trees, in order to obtain greater precision of estimates. However, these authors stated that the number of equations to be used depended on how complex the shape of a tree profile was.

According to Bi (2000), the weaknesses shared by many taper functions are: (1) a high degree of bias in the prediction of the diameter over some portions of the bole, in particular the lower (base) and upper (apex), although the total deviation is low; and (2) an inability to take into account differences in the shapes of trunks between trees.

The study of segmented taper models is quite recent. Cao et al. (1980) and Trincado & Burkhart (2006) commented that the first segmented model developed was that of Max & Burkhart (1976) in the United States.

In Brazil, the works of Péllico Netto (1994), Figueiredo-Filho et al. (1996), Rios (1997), Figueiredo-Filho et al. (1999), de Assis et al. (2001), Fischer et al. (2001), Ferreira (2004) and Souza et al. (2008) are the most notable.

Segmented taper models use separate equations to describe various bole segments (Max & Burkhart 1976; Cao et al. 1980; Byrne & Reed 1986; Trincado & Burkhart 2006; Cao 2009; Brooks et al. 2008; Özçelik et al. 2011; Cao & Wang 2011). In most cases, the parameters of segmented models are difficult to estimate and they do not always allow predictions of height for a given diameter (Kozak 1988; Perez et al. 1990).

Iterative methods should be used to find the height for a given diameter. In addition, most taper models require more than four parameters to be estimated and involve different limitations, such as a zero diameter at the top of the trunk. Furthermore, these models have been used without testing hypotheses about how the parameters and/or the forms of equations are adapted to the bole shape (Benbrahim & Gavaland 2003).

We highlight the Kublin approach, a flexible function based on regression B-spline mixed effects, which the authors used to estimate the total volumes and assortment of the Norway spruce (*Picea abies* (L.) H. Karst.) (Kublin et al. 2013).

Scotti et al. (2014) also applied this methodology to study the maritime pine, *Pinus pinaster*, in the commune of Pattada, Sardinia region, Italy. Li et al. (2012), when comparing shape models to estimate stem taper and volume in conifer species in the Acadia region of North America, used mixed-effect nonlinear modelling, which accounts for autocorrelation between multiple observations taken on a tree stem.

An alternative approach to the application of taper curves was developed by Preußner (1974). Preußner proposed a methodology accessible to foresters, segmenting the stem into four parts and applying four parabolas to describe them. His method has advantages over others, as highlighted by Péllico Netto (1994), when he applied it to *Araucaria angustifolia* (Bertol.) Kuntze trees in southern Brazil. The subdivision of the stem, using Preußner's functions, facilitates wood volume estimation by integration at intervals that coincide with scaling points. This makes it possible to estimate the total volume, log volume, and partial assortments very easily.

To date, only two studies have used the Preußner method in Brazil, Jorge (1984) and Péllico Netto (1994). Both authors state that the methodology is appropriate for modelling tree taper. Perhaps the biggest constraint to the application of these functions is that the original study was published in German; here, we present step-by-step procedures of Preußner curve fitting to facilitate understanding of its application.

This was the first taper methodology that used form quotients related to the diameter taken at half the total tree height. This made the function smoother along the trunk, with a slight change in coefficient values, which resulted in adjusted curves for trees under different conditions. Although the proposed model is flexible, it has not been commonly used to model tree taper in Brazil.

Moreover, the results obtained from the Preußner functions were compared with those from 5th-degree polynomial functions, one of the most used methods to describe the tree taper and volume assortments in Brazil. In addition, it is of interest to compare it with other models that are extensively used, such as the models of Kozak and Max-Burkhart. Thus, we tested two hypotheses: (1) the coefficients that describe the tree taper with Preußner models, with parabolic functions, present greater stability when applied to model the taper at different ages and (2) segmented adjustments of Preußner functions are effective for modelling the stem profile, compared to the application of Schöpfer's polynomial, Kozak and Max-Burkhart's functions.

The present work aimed to deepen our understanding of the most appropriate method to calculate the total volumes of trees and their assortments and to increase the accuracy of its application to *Pinus taeda* L., a species with large production and commercial use in southern Brazil. We used data from commercial *Pinus taeda* L. plantations, collected over the last 22 years in the Mid-West of Santa Catarina, Brazil.

Methods

Characterisation of the study area

The data used in this study was obtained from the region between the municipalities of Caçador (26°46'30", 51°00'5"), Calmon (26°35'59", 51°05'50"), Lebon Régis (26°55'44", 50°41'42"), Fraiburgo (27°01'34", 50°55'17"), and Videira (27°00'28", 51°09'07"), state of Santa Catarina, Brazil. According to the Köppen climate classification, this region, the Midwest, has a Cfb climate. In the Caçador and Videira municipalities, taken as the reference for this area, the climate is humid, with hot summers, and the three coldest months of the year have an average temperature of 15 °C. Respectively, the annual precipitation is 1,633 mm and 1,793 mm, the annual average temperature is 16 °C and 17 °C, and the altitude is 960 m and 779 m. This region is characterised by slightly undulating to undulating relief, with steep hills and basalt-derived eutrophic Cambisols, Bruna lands, which have variable natural fertility and low phosphorus content (Rocha 2016).

Data Source

We used 904 trees scaled from the continuous forest inventory of regularly spaced commercial plantations, most of them 2.5 × 2.5 m, with 3 to 18-year-old trees. Data were collected from 1998 to 2018 in *Pinus taeda* L. stands using Hohenadl's method, on different sites. The modelling was performed dividing the data in four groups ages: 5, 10, 15 and 20 years.

Preußner Taper Curve Adjustment and Application

To maintain the conditions described by Preußner (1974), the diameter measurements were restructured along the bole in relative positions, so that it was possible to obtain form quotients as a function of the diameter taken at half the bole ($d_{0.5h}$), as proposed by Hohenadl (1924).

We slightly modified the subdivision of the stem suggested by Preußner: the first segment ranged from 0 to 25% of the bole, the second segment from 25 to 50%, the third segment from 50 to 75%, and the last segment from 75 and 100%. These divisions coincide with the needs of volume estimates and other variables required for assortment studies. Thus, the following functions were adjusted to establish a continuous taper curve that joins the successive extremes of the form quotients:

$$y = a_{11} d_{0.5h} \frac{1}{\sqrt[1+b]{1+bx}} \text{ for } 0.0 < x \leq 0.25$$

$$y = a_{21} d_{0.5h} \sqrt[1-x]{1-x} \text{ for } 0.25 < x \leq 0.50$$

$$y = a_{31} d_{0.5h} \sqrt[1-x]{1-x} \text{ for } 0.50 < x \leq 0.75$$

$$y = a_{41} d_{0.5h} \sqrt[1-x]{1-x} \text{ for } 0.75 < x < 1.0$$

where, y is the diameter at position x ; $d_{0.5h}$ is the diameter at the middle of the bole; x is the relative length of the

bole (i.e. $x = h_x / h$, and "x" is a real value in the interval 0.0 and 1.0, h is the tree total height); w is the equation's exponent, equal to one for a straight line and less than one to obtain an ordinary parabola (i.e. as it approaches zero, it describes a sharp curve, and approaching one, it describes a smooth curve); and b is the accelerator of the parabolas. Thus, the previous expressions of taper curve can be envisaged by a single mathematical function or a set of successive functions taken along the bole.

Fit of the taper models proposed by Schöpfer, Kozak and Max-Burkhart

The models proposed by Schöpfer, Kozak and Max-Burkhart were fitted and compared with Preußner's model.

Adjustment and application of the Schöpfer (5th-degree polynomial)

Initially, the data were adjusted with the fractional exponent polynomial (Hradetzky 1976) and the 5th-degree polynomial (Schöpfer 1966), resulting in remarkably similar volume estimates. We applied the Hradetzky model to the data using several exponents through a stepwise regression, including four to five variables by age. However, the equations selected for volume calculation were difficult to integrate and the estimates were less accurate than those obtained by Schöpfer's 5th-degree polynomial. Consequently, we opted to compare the volumes estimates obtained with the Schöpfer's 5th-degree polynomial with those obtained with the Preußner's model, due to its wide usage in the profile description of *Pinus taeda* and *Pinus elliottii* Engelm. in southern Brazil (Figueiredo Filho 2000).

Schöpfer's 5th-degree polynomial is expressed as follows:

$$\frac{d_i}{d_{1.3}} = \beta_0 + \beta_1 \left(\frac{h_i}{h_t}\right) + \beta_2 \left(\frac{h_i}{h_t}\right)^2 + \beta_3 \left(\frac{h_i}{h_t}\right)^3 + \beta_4 \left(\frac{h_i}{h_t}\right)^4 + \beta_5 \left(\frac{h_i}{h_t}\right)^5$$

where:

β_j s are parameters to be estimated;

d_i are diameters (cm) at sequential height positions h_i ;

$d_{1.3}$ are diameters at 1.3 m aboveground (cm);

h_t is total height (m);

h_i are heights at diameters d_i .

Total tree volume was estimated by polynomial integration:

$$V = K \int_{h_1}^{h_2} d_i^2 \delta / h$$

$$V = K d_{1.3}^2 \int_{h_1}^{h_2} (c_0 + c_1 h_1^{p_1} + c_2 h_2^{p_2} + \dots + c_n h_n^{p_n})^2 \delta h$$

where:

K is $\pi/40,000$;

p_j are exponents ranging from 1 to 5

$$c_0 = \beta_0; \quad c_1 = \frac{\beta_1}{h}; \quad c_2 = \frac{\beta_2}{h^2}; \quad \dots; \quad c_{51} = \frac{\beta_5}{h^5}$$

Solving the integral we have:

$$V = Kd^2 \left[c_0^2 h_i + c_0 c_1 h_i^2 + \left(\frac{2}{3} c_0 c_2 + \frac{1}{3} c_1^2 \right) h_i^3 + \left(\frac{1}{2} c_0 c_3 + \frac{1}{2} c_1 c_2 \right) h_i^4 + \left(\frac{2}{5} c_0 c_4 + \frac{2}{5} c_1 c_3 + \frac{1}{5} c_2^2 \right) h_i^5 + \left(\frac{1}{3} c_0 c_5 + \frac{1}{3} c_1 c_4 + \frac{1}{3} c_2 c_3 \right) h_i^6 + \left(\frac{2}{7} c_1 c_5 + \frac{2}{7} c_2 c_4 + \frac{1}{7} c_3^2 \right) h_i^7 + \left(\frac{1}{4} c_2 c_5 + \frac{1}{4} c_3 c_4 \right) h_i^8 + \frac{1}{5} c_4 c_5 h_i^{10} + \frac{1}{11} c_5^2 h_i^{11} \right]_{h_1}^{h_2}$$

Adjustment and application of the Kozak taper curve
Kozak's model (2004) is defined by:

$$d_i = a_0 D^{a_1} H^{a_2} X_i^{b_1 z_i^4 + b_2 [1/e^{D/H}] + b_3 X_i^{0.1} + b_4 [1/D] + b_5 H^{Q_i} + b_6 X_i}$$

where:

$$X_i = [1.0 - (h/H)^{1/3}] / [1.0 - (p^{1/3})]$$

$$Q_i = [1.0 - (h/H)^{1/3}]$$

$$p = 1.3/H$$

To obtain the volume, a numerical integral was used, establishing the lower and upper limits for the variables.

Adjustment and application of the Max-Burkhardt taper function

$$d_i = D [b_1(X - 1) + b_2(X^2 - 1) + b_3(a_1 - X)^2 I_1 + b_4(a_2 - X)^2 I_2]^{0.5}$$

where:

$$X = h/H$$

a_1 and a_2 are the connecting points of the polynomials

$$I_1 = \begin{cases} 1 & \text{if } X \leq a_1 \\ 0 & \text{if } X > a_1 \end{cases}$$

$$I_2 = \begin{cases} 1 & \text{if } X \leq a_2 \\ 0 & \text{if } X > a_2 \end{cases}$$

$$V = Kd^2 H \left[\frac{b_2}{3} \left(\left(\frac{h_2}{H} \right)^3 - \left(\frac{h_1}{H} \right)^3 \right) + \frac{b_1}{2} \left(\left(\frac{h_2}{H} \right)^2 - \left(\frac{h_1}{H} \right)^2 \right) - (b_1 + b_2) \left(\frac{h_2}{H} - \frac{h_1}{H} \right) - \frac{b_3}{3} \left(\left(a_1 - \frac{h_2}{H} \right)^3 I_1 - \left(a_1 - \frac{h_1}{H} \right)^3 I_1 \right) - \frac{b_4}{3} \left(\left(a_2 - \frac{h_2}{H} \right)^3 I_2 - \left(a_2 - \frac{h_1}{H} \right)^3 I_2 \right) \right]$$

Assessment and Comparison of Preußner taper functions with Schöpfer's 5th-degree Polynomial, Kozak and Max- Burkhardt taper models

To assess the 5th-degree Polynomial, Kozak and Max- Burkhardt taper models an adjusted coefficient determination (R^2_{adj}), standard error of the estimate ($S_{xyz\%}$), mean square of the error (MSE), root means square of error ($RMSE$), relative root means square of error ($RMSE_{\%}$) and parameter standard error were used. To assess the Preußner's functions MSE , $RMSE$ and $RMSE_{\%}$ were used. Additionally, the following plots were inspected: absolute residuals of the diameter estimate, relative residues for the diameter estimate, observed and estimated diameters, absolute residuals of the total volume estimate, relative residuals of the total volume estimate and observed and estimated total volumes. Through this evaluation the best function between

5th-degree Polynomial, Kozak and Max-Burkhardt was selected and then compared with the Preußner functions. The comparison between the two procedures for obtaining tree taper, total volume, and merchantable volume, was evaluated through differences of observed and estimated volumes. The statistics used were:

$$RMSE = \sqrt{\frac{\sum (y_i - \hat{y}_i)^2}{n}}, RMSE_{\%} = \frac{RMSE}{\bar{y}}$$

$$Bias = \frac{\sum y_i - \hat{y}_i}{n}, Bias_{\%} = \frac{Bias}{\bar{y}}$$

$$r = \frac{COV(y_i, \hat{y}_i)}{s_{y_i} s_{\hat{y}_i}}$$

$$AIC = -2n \left[-\frac{n}{2} \ln \left(\frac{1}{n} \sum (y_i - \hat{y}_i)^2 \right) \right] + 2p$$

$$AIC_s = -2n \left[-\frac{n}{2} \ln \left(\frac{1}{n} \sum (y_i - \hat{y}_i)^2 \right) \right] + 2p \frac{n}{(n - p - 1)}$$

$$BIC = -2n \left[-\frac{n}{2} \ln \left(\frac{1}{n} \sum (y_i - \hat{y}_i)^2 \right) \right] + \ln(n) p$$

where:

y_i is estimated tree volume (m^3);

y_i is observed tree volume (m^3);

n is the number of observations.

COV is the covariance.

r is the correlation.

p is the number of parameter of the model.

AIC_s applied when $n/p < 40$.

AIC is the Akaike information criterion.

BIC is the Schwart's information criterion.

In addition, it was calculated the merchantable volume, defined by the volume of 4 logs:

- first assortment: logs size with a minimum diameter of 8 cm and height of 2.48 m.
- second assortment: logs size with a minimum diameter of 15 cm and height of 2.48 m.
- third assortment: logs size with a minimum diameter of 25 cm and height of 2.48 m.
- forth assortment: logs size with a minimum diameter of 35 cm and height of 2.48 m.

Results

Descriptive statistics

Summaries of diameter, height, and volume, for each age, are shown in Table 1. The average values were higher for older forests. The smallest trees had DBH s of 10 cm, while the largest trees were 52.5 cm in DBH . The average diameter, height and volume was 34.5 cm, 22.4 m, and

TABLE 1: Descriptive statistics for diameter at breast height (d), height (h) and total volume (v).

Age	No. trees/ha	\bar{d}	\bar{h}	\bar{v}	max v	min v	cv% v
5	78	10.0	6.4	0.0242	0.1289	0.0022	98.63
10	241	21.0	12.6	0.1648	0.9601	0.0137	82.66
15	465	28.7	18.1	0.4293	2.2631	0.0359	79.67
20	120	34.5	22.4	0.7344	2.2563	0.3322	63.97
Age	No. trees/ha	max d	min d	cv% d	max h	min h	cv% h
5	78	19.7	4.0	34.62	9.5	3.5	25.03
10	241	37.2	7.5	24.65	17.7	6.2	16.95
15	465	52.2	9.0	24.60	25.6	11.7	16.19
20	120	52.5	21.6	19.49	26.5	19.0	6.72

Max is the maximum value, min is the minimum value and cv is the variation coefficient.

0.734 m³ for the older forest, respectively. The tree with the largest measured volume was 2.263 m³. The coefficient of variation, for both variables, was smaller in the older stands, due to thinning.

Fitting of taper models proposed by Schöpfer, Kozak and Max-Burkhart

The estimated parameters of Schöpfer's 5th-degree polynomial, Kozak and Max-Burkhart, fitted for all ages, are presented in Tables 2 to 10, with the respective parameter standard error, R^2_{adj} , $S_{xy\%}$, MSE , $RMSE$ and $RMSE_{\%}$. For all models, at the evaluated ages, the values of R^2_{adj} were reliable (greater than 0.98), as well as the values of $S_{xy\%}$ were accepted as reliable (less than 10%). The values of MSE , $RMSE$ and $RMSE_{\%}$ depended on the model and age, and Kozak's model resulted in more appropriate statistics, that is, with the most reliable values when compared with the Schöpfer's and Max-Burkhart's models. For the Kozak's model, the $RMSE$ values varied between 0.0030 to 0.0656, while the value of $RMSE_{\%}$ between 12.45% to 8.93%, respectively for the ages of 5 and 20 years. Figures 1 to 6 show graphs of Kozak's model, which revealed the following:

- Absolute residuals of the diameter estimate were not biased, being the worst result obtained for the 5-year-old forest.
- The highest values of relative residuals were observed in the forests of 15 and 20 years for the diameter estimate.
- The model is flexible enough to represent the bole profile at all ages for observed and estimated diameters.
- The least reliable estimates were observed in the youngest forests (5 and 10 years) for the absolute residuals of the total volume estimate.
- The least reliable estimates were also obtained for the youngest forests (5 and 10 years) for relative residuals from the estimate of the total volume, but they were not biased.

- The model was flexible enough to detect the variability of the observed and estimated total volume at all ages.

Figures 7 to 12 show the graphics for the Schöpfer's model, which reveal the following:

- The absolute residuals for the largest diameter estimate in the 5-year-old forest were biased.
- The highest residual values for the diameter estimate were observed in the 15 and 20-year-old forests.
- The model is less flexible at representing stem profiles for observed and estimated diameters at all ages, when compared to Kozak's model.
- The absolute residuals of total volume estimate were biased.
- The least reliable estimates for relative residues of total volume estimates were obtained for the oldest forests (15 and 20 years old).
- The model is flexible enough to represent variability of observed and estimated total volumes at all ages.

Figures 13 to 18 show the graphics for the Max-Burkhart's model, which reveal the following:

- Absolute residuals of the diameter estimate: residuals of the diameter estimate were not biased, but were the worst results obtained for the 5-year-old forest.
- The highest values of relative residuals for the diameter estimate were observed in the 15 and 20-year-old forests.
- The model is less flexible at representing observed and estimated diameters on the stem profile at all ages, when compared with Kozak's model.
- Absolute residuals for total volume estimates showed a trend, especially for older forests (15 and 20 years).

TABLE 2: Fit coefficients of the models for age 5.

Model	Estimated parameters									
	\hat{a}_0	\hat{a}_1	\hat{a}_2	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$
Kozak (2004)	1.767274	0.7927	0.1120		0.2969	-0.3412	0.8002	1.2650	-0.1029	-0.1792
Schöpfer (1966)				1.3317	-2.5077	5.1290	-9.6238	8.9117	-3.2391	
Max-Burkhart (1976)		0.2469	0.5852		-2.7847	1.3517	5.8133	-0.0758		
Model	Parameter standard error									
Kozak (2004)	0.0388	0.0237	0.0331		0.0204	0.1154	0.0239	0.1816	0.0126	0.0499
Schöpfer (1966)				0.0061	0.1610	1.1206	2.9768	3.3359	1.3282	
Max-Burkhart (1976)		0.0300	0.9778		0.1315	0.0741	1.0930	0.3175		

TABLE 3: Fit statistics of the models for age 5 for total tree volume.

Model	Estimated statistics				
	R^2_{adj}	S_{yx} %	MSE	RMSE	RMSE ₁₀₀
Preußner (1974)			1.83E-05	0.0043	17.6508
Kozak (2004)	0.9898	7.4941	9.10E-06	0.0030	12.4487
Schöpfer (1966)	0.9841	9.6153	3.68E-05	0.0061	25.0410
Max-Burkhart (1976)	0.9841	9.6169	9.92E-06	0.0031	13.0004

TABLE 4: Fit coefficients of the models for age 10.

Model	Estimated parameters									
	\hat{a}_1	\hat{a}_2	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$	$\hat{\beta}_6$
Kozak (2004)	1.19797	1.0373	-0.0336		0.4102	-0.3186	0.7067	-0.1164	-0.0199	-0.3166
Schöpfer (1966)				1.2403	-2.8449	8.8225	-17.845	15.9987	-5.3722	
Max-Burkhart (1976)		0.1658	0.6348		-3.1609	1.5620	13.5017	-1.0789		
Model	Parameter standard error									
Kozak (2004)	0.0285	0.0097	0.0136		0.0123	0.0455	0.0088	0.1912	0.0056	0.0414
Schöpfer (1966)				0.0024	0.0622	0.4330	1.1503	1.2892	0.5134	
Max-Burkhart (1976)		0.0088	0.0184		0.0546	0.0306	1.2708	0.0615		

TABLE 5: Fit statistics of the models for age 10 for total tree volume.

Model	Estimated statistics				
	R^2_{adj}	S_{yx} %	MSE	RMSE	RMSE ₁₀₀
Preußner (1974)			0.000262	0.0162	9.81289
Kozak (2004)	0.9906	6.5574	0.000254	0.0159	9.67295
Schöpfer (1966)	0.9903	6.6772	0.001704	0.0413	25.0489
Max-Burkhart (1976)	0.9903	6.6741	0.000313	0.0177	10.7305

TABLE 6: Fit coefficients of the models for age 15 for total tree volume.

Estimated parameters										
Model	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$
Kozak (2004)	1.350354	1.0460	-0.0848		0.5179	-0.7605	0.6327	1.7765	-0.0355	0.0628
Schöpfer (1966)				1.2335	-3.2456	10.728	-20.150	16.467	-5.0341	
Max-Burkhart (1976)		0.1735	0.6846		-3.7667	1.8755	16.602	-1.8635		
Parameter standard error										
Kozak (2004)	0.0313	0.0075	0.0111		0.0096	0.0335	0.0068	0.1993	0.0034	0.0331
Schöpfer (1966)				0.0019	0.0510	0.3547	0.9425	1.0564	0.4206	
Max-Burkhart (1976)		0.0055	0.0091		0.0614	0.0336	0.9141	0.0439		

TABLE 7: Fit statistics of the models for age 15 for total tree volume.

Estimated statistics					
Model	R^2_{adj}	S_{yx} %	MSE	RMSE	RMSE ₁₀₀
Preußner (1974)			0.002216	0.0471	10.9655
Kozak (2004)	0.9883	7.1491	0.001916	0.0438	10.1966
Schöpfer (1966)	0.9869	7.5703	0.014876	0.1220	28.4084
Max-Burkhart (1976)	0.9869	7.5692	0.010279	0.1014	23.6147

TABLE 8: Fit coefficients of the models for age 20.

Estimated parameters										
Model	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	$\hat{\beta}_6$
Kozak (2004)	2.804767	1.0038	-0.2743		0.6255	-0.7154	0.5052	2.3690	-0.0517	0.3939
Schöpfer (1966)				1.2033	-3.0393	10.400	-19.160	14.817	-4.2206	
Max-Burkhart (1976)		0.1905	0.6781		-4.3478	2.1647	14.268	-2.7426		
Parameter standard error										
Kozak (2004)	0.3209	0.0155	0.0424		0.0203	0.1226	0.0178	1.1770	0.0065	0.0718
Schöpfer (1966)				0.0042	0.1117	0.7774	2.0657	2.3153	0.9219	
Max-Burkhart (1976)		0.0124	0.0148		0.1393	0.0765	1.4805	0.1117		

TABLE 9: Fit statistics of the models for age 20 for total tree volume.

Estimated statistics					
Model	R^2_{adj}	S_{yx} %	MSE	RMSE	RMSE ₁₀₀
Preußner (1974)			0.005903	0.0768	10.4619
Kozak (2004)	0.9855	7.3066	0.004305	0.0656	8.9339
Schöpfer (1966)	0.9824	8.1355	0.042385	0.2059	28.0330
Max-Burkhart (1976)	0.9823	8.1464	0.046220	0.2150	29.2736

TABLE 10: Observed and estimated merchantable volume (%) for four minimum diameters up the top and height of 2.48 m

Age	Source	8 cm	15 cm	25 cm	35 cm
5	Observed	90.53	9.47	0	0
10		26.12	60.47	12.73	0.68
15		9.73	38.12	40.00	12.15
20		5.08	25.15	44.07	25.70
5	Kozak	94.59	5.41	0	0
10		27.68	57.88	13.83	0.61
15		9.78	38.49	39.50	12.23
20		4.86	25.68	47.04	22.43
5	Preußner	94.31	5.69	0	0
10		23.96	54.06	19.37	2.61
15		9.69	33.31	39.46	17.55
20		4.69	18.48	35.79	41.04

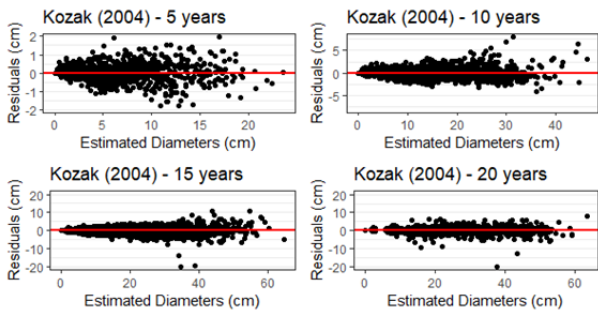


FIGURE 1: Absolute residuals of the diameter estimate for Kozak's model.

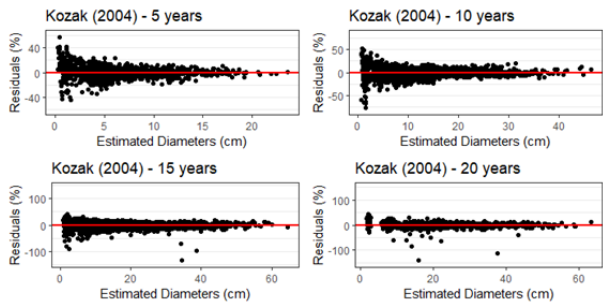


FIGURE 2: Relative residues for the diameter estimate for Kozak's model.

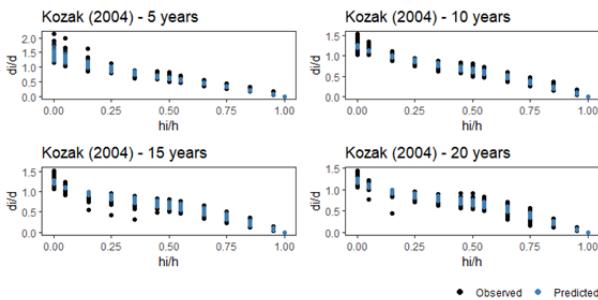


FIGURE 3: Observed and estimated diameters for Kozak's model.

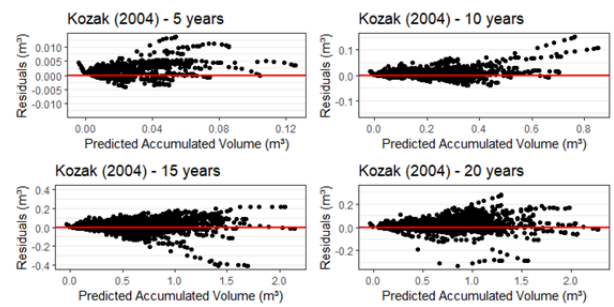


FIGURE 4: Absolute residuals of the total volume estimate for Kozak's model.

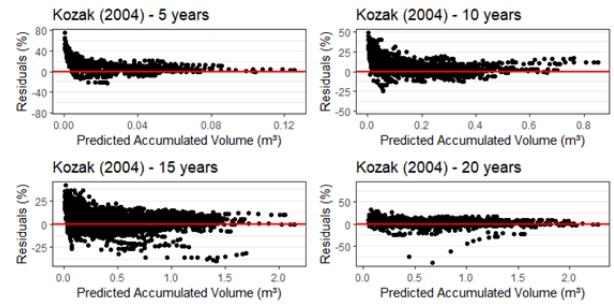


FIGURE 5: Relative residuals of the total volume estimate for Kozak's model.

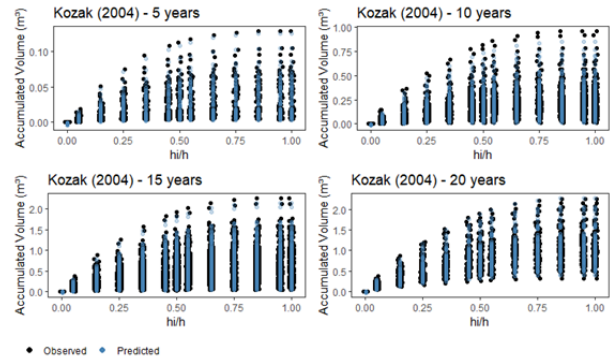


FIGURE 6: Observed and estimated total volumes for Kozak's model.

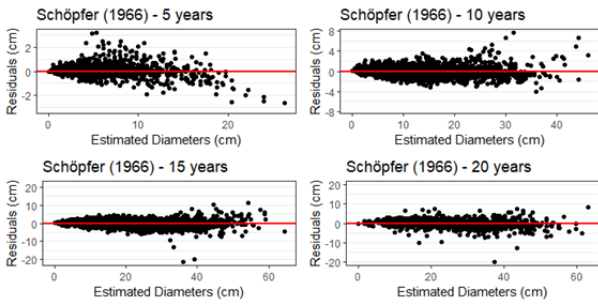


FIGURE 7: Absolute residuals of the diameter estimate for Schöpfer's model.

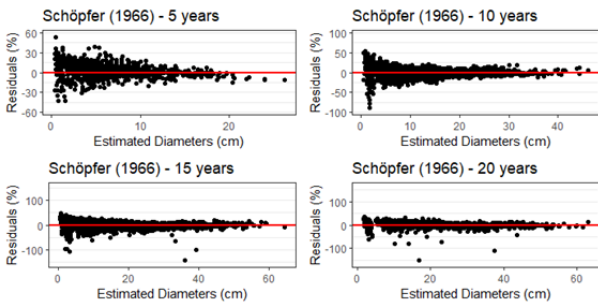


FIGURE 8: Relative residues for the diameter estimate for Schöpfer's model.

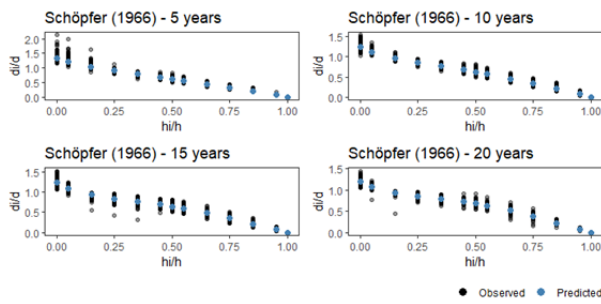


FIGURE 9: Observed and estimated diameters for Schöpfer's model.

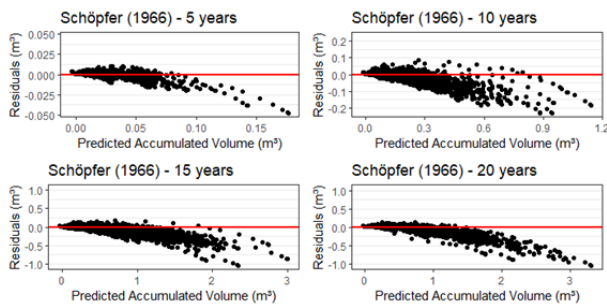


FIGURE 10: Absolute residuals of the total volume estimate for Schöpfer's model.

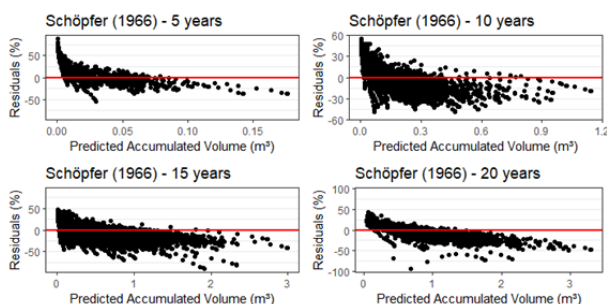


FIGURE 11: Relative residuals of the total volume estimate for Schöpfer's model.

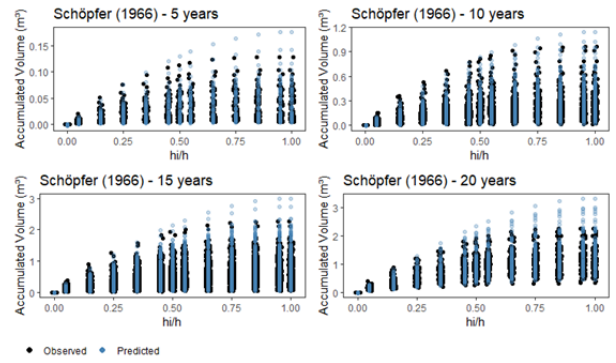


FIGURE 12: Observed and estimated total volumes for Schöpfer's model.

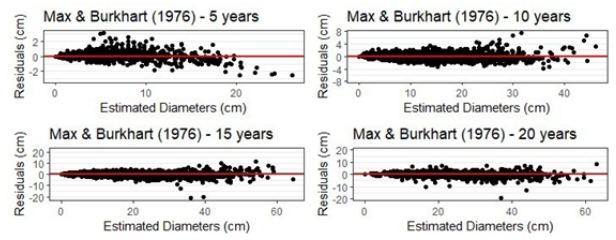


FIGURE 13: Absolute residuals of the diameter estimate for Max-Burkhardt's model.

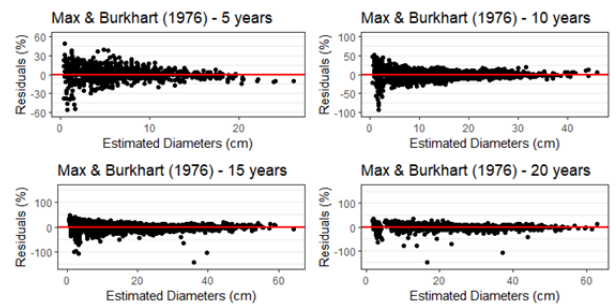


FIGURE 14: Relative residues for the diameter estimate for Max-Burkhardt's model.

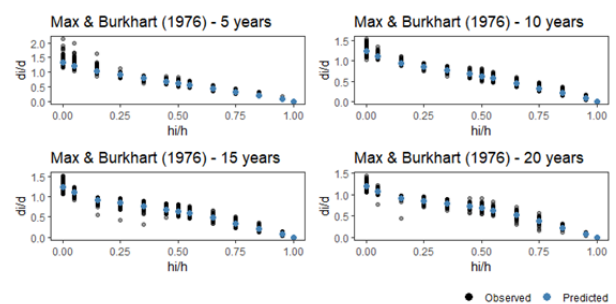


FIGURE 15: Observed and estimated diameters for Max-Burkhardt's model.

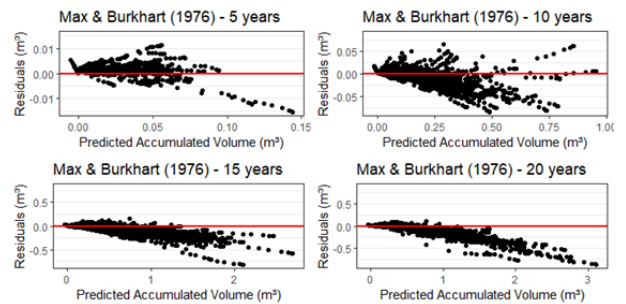


FIGURE 16: Absolute residuals of the total volume estimate for Max-Burkhardt's model.

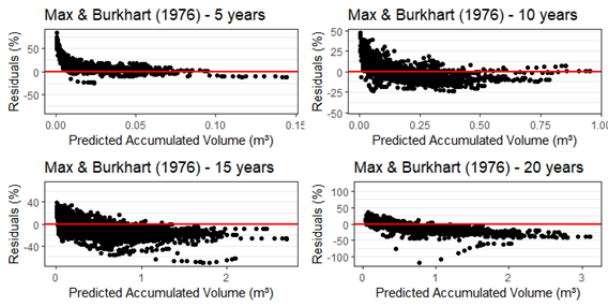


FIGURE 17: Relative residuals of the total volume estimate for Max-Burkhardt's model.

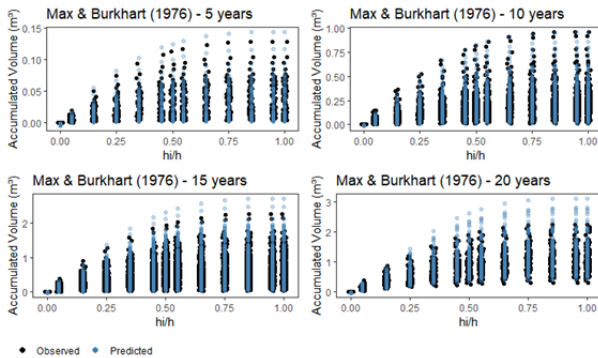


FIGURE 18: Observed and estimated total volumes for Max-Burkhardt's model.

- The least reliable estimates for residuals of the total volume estimate were observed in older forests.
- The model was flexible enough to represent variability of the total observed and estimated volumes at all ages.

Following the results of adjustment statistics and evaluations of graphical representation of residuals of observed and estimated values, the Kozak model was found to be more flexible and better explained the variability of stem profiles and total volume for forests aged between 5 and 20 years. The most reliable statistics for this model, as well as for graphical analyses, were obtained in the oldest forests (of greatest economic importance). In general, over time, trees tended to be less cylindrical, showing an increase in taper.

Preußner Functions

Steps for fitting the Preußner functions are presented in Appendix 1. The functions proposed by Preußner and fitted in the present study can be used to address various questions: what is the length of the log for a given diameter at the top? What is the diameter of the log for a given height? What is the total volume or a segmented volume within a certain range of the bole?

Although many questions can be formulated and answered, the objective of this work is only to present the calculation of the total tree volume, obtained by the sum of the integrations of the four parabolas. The steps for calculating the volumes using Preußner's functions are presented in Appendix 2.

The estimated parameters of Preußner's model, fitted for all ages, are presented in Appendix 1. BIAS approached zero, indicating that the estimates were unbiased. The RMSE values ranged from 0.0043 to 0.0768, while $RMSE_{\%}$ between 17.65% to 10.46%, respectively for the ages of 5 and 20 years.

Figures 19 to 24 show graphs of the Preußner's model and revealed the following:

- The absolute residuals of diameter estimates were mostly unbiased, with the worst performance obtained for 5-year-old forests.
- The highest values of relative residues for the diameter estimate were observed in 15 and 20-year-old forests.
- The model is slightly less flexible at representing bole profiles for observed and estimated diameters at all ages when compared to the Kozak's model.
- The absolute residuals of the total volume estimate were biased in 5-year-old forests.

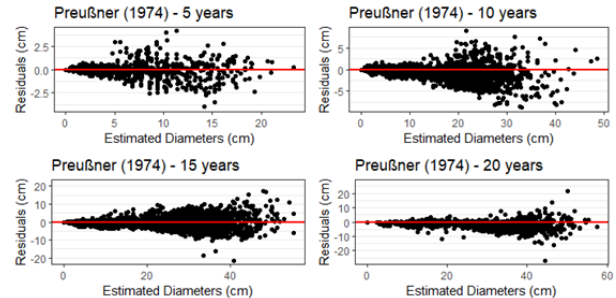


FIGURE 19: Absolute residuals of the diameter estimate for Preußner's model.

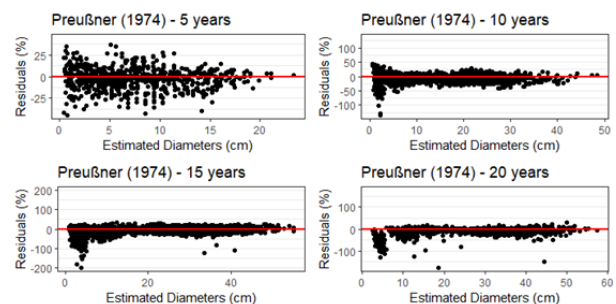


FIGURE 20: Relative residuals for the diameter estimate for Preußner's model.

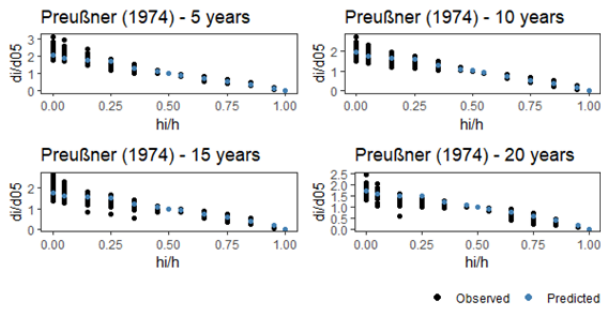


FIGURE 21: Observed and estimated diameters for Preußner's model.

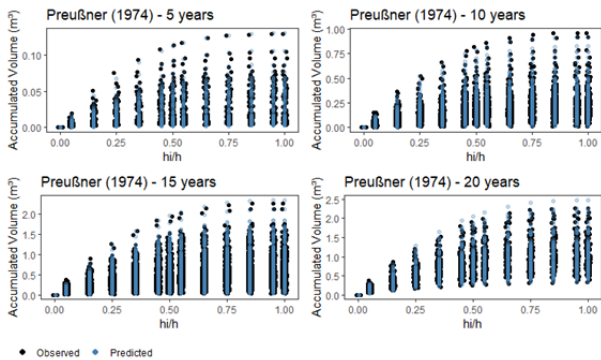


FIGURE 22: Absolute residuals of the total volume estimate for Preußner's model.

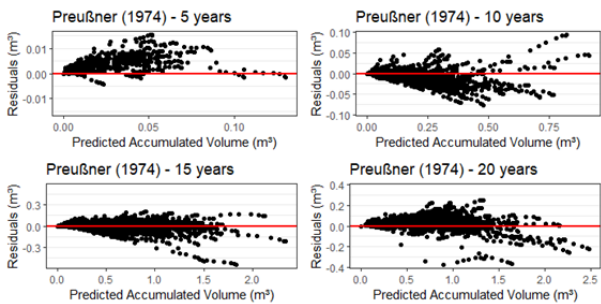


FIGURE 23: Relative residuals of the total volume estimate for Preußner's model.

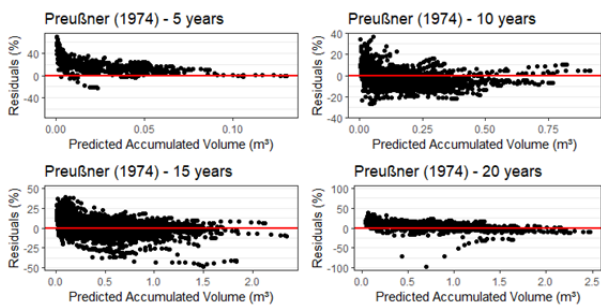


FIGURE 24: Observed and estimated total volumes for Preußner's model.

- The worst distributions of residuals for the total volume estimates were observed in the youngest forests (5 years).
- The model is flexible enough to represent variability of observed and estimated total volumes at all ages.

Discussion

Testing the performance of polynomial models, ratios of volume and cubic spline functions (Lappi 2006; Pinheiro & Bates 1995), for estimating commercial volumes, Rios (1997) found that polynomial models provided more accurate estimates for describing tree profiles, in which the fifth-degree Polynomial was better than the Polynomial of Fractional Powers than, in second place, stand volume ratios and, in third, the spline functions. The author suggested that taper equations should be adjusted by diameter classes to obtain more accurate estimates.

Lima (1986) evaluated the efficiency of the models proposed by Biging (1984), Demaerschalk (1973), Kozak et al. (1969) and Ormerod (1973) in *Pinus elliottii* trees, regarding the estimate of total and commercial volumes, and their respective diameters and heights. The author concluded that the most accurate model for estimating the commercial volume was the Kozak et al. (1969), and the least precise the Ormerod (1973). The model of Demaerschalk (1973) proved to be satisfactory for estimating all variables, except the total volume, which was most appropriate by the Biging model (1984).

Although the fifth-degree polynomial provides enough accurate results for most uses of form equations, it exhibits clear tendencies. Part of the deformations of the base are explained, but diameters are generally underestimated up to about 20% of the total height and overestimated above 80% of the total height (Gordon 1983). Hradetzky (1976) was the first to identify that a good stem representation through polynomials requires a combination of appropriate powers, being necessary, for its determination, that they are submitted to the stepwise selection process and these powers ranged from 0.005 to 25.

We assessed the quality of the adjustment of the Preußner functions (empirically fitted) and Schöpfer's 5th-degree polynomial, Kozak's and Max-Burkhardt's models (fitted by minimization of residues using the least squares method) by comparing the observed and estimated volumes. When comparing the adjustment statistics of the fitted models using regression analysis, the residuals, and the observed and estimated values, Kozak's model was selected.

Kozak's and Preußner's functions showed similar performance at all ages, with little difference in adjustment statistics and in total volume estimate. The results of Preußner's functions can be improved by minimization of residuals. In this research, we decided to find the Preußner's coefficients in an empirical way, since the coefficients varied little with age and had a logical tendency, which revealed a better understanding of the taper profile.

The value of w is the power of the function, being equal to one for obtaining a straight line and less than one for obtaining a classic parabola, i.e. if w is zero the profile (curve taper) is strong (curvature is accentuated) and when approach to one the profile (curve taper) is weak (smooth curve). The coefficient b is the accelerator of the parabola. The w values for the first segment varied between 0.06 to 0.046, respectively for forests with 5 and

20 years. Therefore, we observed that the first segment was influenced by the age of the trees, and older ones presented a more pronounced taper. The other segments were also affected by age of the trees, and the w values varied between 0.98 to 0.76, in such a way that the lowest values were observed for the oldest forest and in the last segment. The interval of the relative height $0.25 < x \leq 0.75$ resulted in the smoothest curve. The value of b was stable across the age (Figure 25).

The coefficients β_1 and β_3 revealed a trend in Kozak's model in all ages of forests. The values of the coefficient β_1 ranged from -0.7605 to 2.369. The values of the coefficient a_1 did not vary with age of the forests and was in the range of -0.2743 to 2.8047. The results revealed that the Kozak's model appropriately represented the bole profile over all ages, but it was difficult to interpret its coefficients and relate them to the bole profile.

The following characteristics for Preußner's functions can help us understand the profiles of the trees:

- The value of b is the accelerator of the parabola. Higher values of b indicate accentuated curves. If the b is equal to zero, the profile is smooth (Figure 26).
- The value of w is the form of the profile, if w is zero the profile is pronounced (curvature is accentuated) and if w is equal to one the profile is weak (smooth curve) (Figure 26).
- If all coefficients (b and w) are equal to zero, the profile of the tree approaches a cylinder (Figure 26).
- If all coefficients (b and w) are equal to one, the profile of the tree approaches a cone (Figure 26).

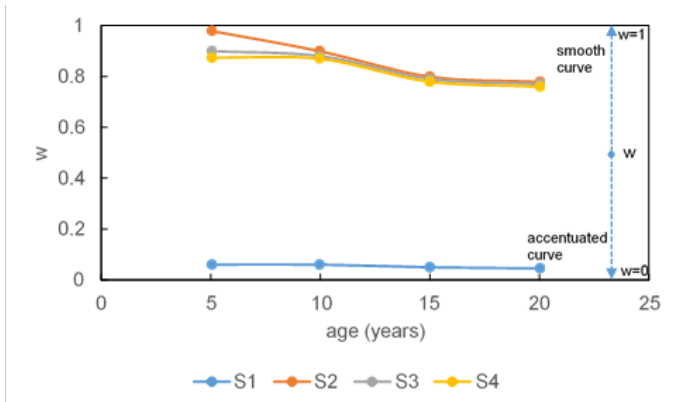


FIGURE 25: Variation of the w values across the ages; where:
 S1 is the 1st segment of the Preußner functions, i.e. $0.0 < x \leq 0.25$.
 S2 is the 2nd segment of the Preußner functions, i.e. $0.25 < x \leq 0.50$.
 S3 is the 3rd segment of the Preußner functions, i.e. $0.50 < x \leq 0.75$.
 S4 is the 4th segment of the Preußner functions, i.e. $0.75 < x \leq 1.00$.

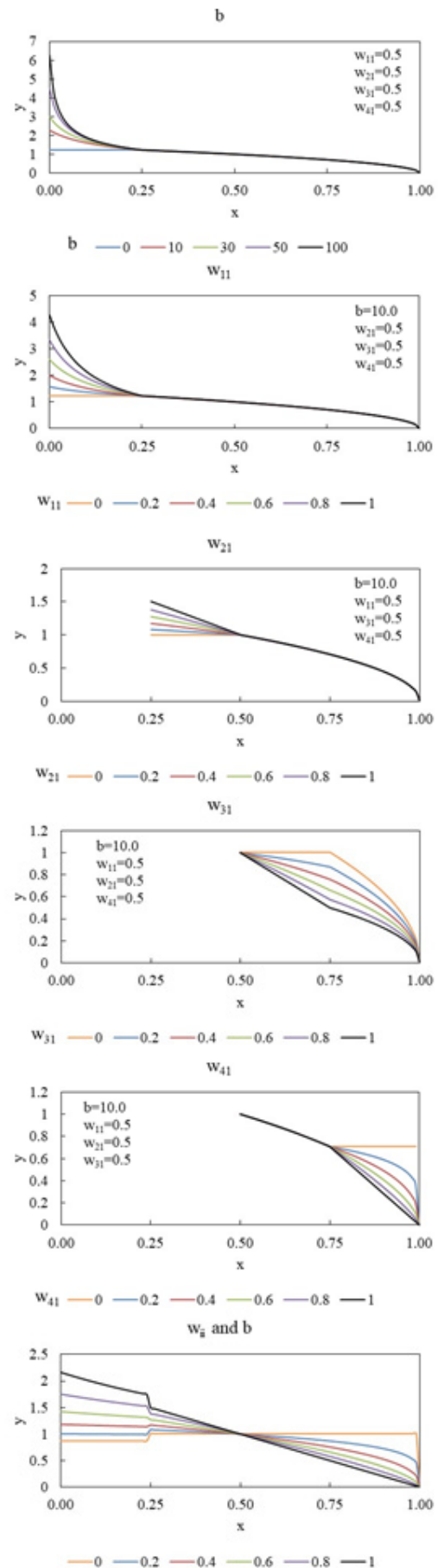


FIGURE 26: Relationship between the coefficients w and b with the taper curves.

- As presented in Appendix 2, it is possible to obtain the Hohenadl form factor (HF), i.e. after integration of the function for all segments, total volume can be estimated for any tree using the resulting equation

$$v = \pi/4 \times k^2 h^2 HF$$

where HF is the average Hohenadl's natural form factor at the reference diameter $d_{0.5h}$. Note that the results of the volume from the parts of the bole (commercial volume + residual volume) will be naturally compatibility with the total volume

$$v = \pi/4 \times k^2 h^2 HF.$$

- If all coefficients (b and w) are equal to zero or one, the average Hohenadl natural form factor in the reference diameter $d_{0.5h}$ approximate one.

- The value of k is a constant for all trees on the data set, i.e., to obtain better results to estimate log volumes, the fitting should be made by stratifying the effect of HF, for example, by site.

Kozak's model resulted in the most appropriate statistics (*Bias*, *MSE*, *RMSE* and *r*) for estimating merchantable log volumes (Tables 10 and 11). The form of the variable-exponent equation improved

TABLE 11: Statistics to assessing the Kozak and Preußner's functions for merchantable volume.

Age	Kozak				Preußner			
	≥ 8 cm	≥ 15 cm	≥ 25 cm	≥ 35 cm	≥ 8 cm	≥ 15 cm	≥ 25 cm	≥ 35 cm
	RMSE							
5	0.0088	0.0077	-	-	0.0097	0.0078	-	-
10	0.0201	0.0416	0.0389	0.0030	0.0212	0.0614	0.0697	0.0358
15	0.0254	0.0603	0.0772	0.0680	0.0249	0.0808	0.1373	0.1183
20	0.0235	0.0961	0.1348	0.1359	0.0257	0.1477	0.2420	0.2233
	RMSE(%)							
5	48.20	402.94	-	-	52.96	405.49	-	-
10	34.69	31.00	137.50	196.42	36.64	45.69	246.46	2377.72
15	43.18	26.20	31.95	92.69	42.41	35.07	56.81	161.15
20	43.66	36.15	28.93	50.01	47.85	55.54	51.93	82.16
	Bias							
5	0.0008	0.0009	-	-	0.0028	0.0010	-	-
10	-0.0013	0.0102	-0.0014	0.0002	0.0039	0.0123	-0.0154	-0.0044
15	0.0012	0.0038	0.0092	0.0014	0.0000	0.0283	0.0023	-0.0331
20	0.0033	-0.0006	-0.0225	0.0390	0.0061	0.0785	0.1046	-0.1446
	Bias (%)							
5	4.48	47.78	-	-	15.35	51.17	-	-
10	-2.29	7.63	-4.79	12.65	6.79	9.19	-54.50	-291.55
15	2.07	1.66	3.79	1.94	0.03	12.28	0.93	-45.05
20	6.18	-0.24	-4.82	14.33	11.41	29.50	22.45	-53.19
	r							
5	0.93	0.76	-	-	0.68	0.43	-	-
10	0.57	0.90	0.92	1.00	0.41	0.58	0.56	0.01
15	0.42	0.77	0.94	0.95	0.16	0.02	0.54	0.79
20	0.34	0.62	0.83	0.94	0.20	0.26	0.27	0.83
	AIC							
5	-57527.30	-59152.33	-	-	-56381.61	-59076.97	-	-
10	-453732.76	-369257.15	-377184.82	-676426.73	-447380.13	-324216.00	-309408.81	-386772.83
15	-1588752.69	-1214241.75	-1107641.07	-1162346.69	-1588762.69	-1088060.57	-858765.08	-923167.36
20	-108053.42	-67428.69	-57691.04	-57457.26	-105418.52	-55064.44	-40844.97	-43156.73
	BIC							
5	-57508.74	-59133.76	-	-	-56383.42	-59078.77	-	-
10	-453702.18	-369226.56	-377154.23	-676396.14	-447366.19	-324202.06	-309394.87	-386758.89
15	-1588715.41	-1214204.48	-1107603.80	-1162309.41	-1596534.21	-1088044.01	-858748.51	-923150.79
20	-108029.97	-67405.24	-57667.59	-57433.81	-105418.16	-55064.07	-40844.60	-43156.36

estimates of diameter up to tree tops, assimilating different site conditions at each age. The profiles were constant for trees described by Preußner's model, which were represented by HF. In this way, the application for Preußner's model can be improved; the fitting can be performed considering site or even h/d ratios, for example. In addition, to improve the results of Preußner's model, MSE should be minimised using least squares.

Preußner's model is not derived by the same least squares procedure as Kozak's model, even though in both cases the average bias approximated zero. On the other hand, when assessing the AIC and BIC statistics, the best results were obtained with the Preußner's model. This resulted from the complexity of the Kozak's model when compared to Preußner's model.

Additionally, the coefficients of Preußner's functions are obtained analytically, whereas in the other functions they are obtained by regression adjustment. The coefficients of the parabolas are also interpretative, and cannot be similarly analysed in other functions. In Preußner's model, this interpretation can be done by section. Although the analysis of the relation between coefficients of the functions and the profile of the tree may be interesting, for Kozak's model these analyses will be appropriate only if the fundamental assumption of independence of errors within each tree, normality and homogeneity of the residuals are considered.

The operational ease of applying mathematical models is an issue for debate, as they must meet two basic conditions: efficiency and practicality of use. Preußner's functions are easily integrated by sections, which is not the case for other tested models, once the integrals are complex (they are adjusted for the entire bole). Preußner's model presupposes steps for adjustment of the coefficients:

- (1) the solution of "x" (relative position) as a function of "y" (diameter), adjustment of the coefficients as a function of $d_{1,3}$, solution of "x" as a function of "y" for $d_{1,3}$, and the integration of the four sections to estimate the total and assortment volumes;
- (2) fractional coefficients, close to 1 and with two decimal places for the first section of the bole, decrease in the other sections and (1) through (17) constitute the first step (Appendix 1);
- (3) as a result, decreasing coefficients are also observed in equations (17) to (32) (Appendix 1), but the magnitude of change increases with age. This behaviour is also observed in the constants;
- (4) the exponents used to obtain the constant "k" show little variation and decrease slightly with age, as shown in equations (44) to (47) (Appendix 1).

These behaviours observed in the constants and exponents in the calculation of $d_{0,5h}$ are repeated in the adjustment solution of $d_{1,3}$, as a function of $d_{0,5h}$.

Conclusions

- Kozak's model is the most accurate function to estimate merchantable volume of 3 to 18-year-old *Pinus taeda* in commercial plantations in the middle-west region of Santa Catarina, Brazil.
- Preußner's functions are adequate and accurate to estimate the total and assortment volumes.
- The advantages of the procedure proposed by Preußner include: flexible functions, simple volume calculations, and the flexibility to interpret their coefficients.
- *Bias*, *RMSE* and *r* were generally better with the application of Kozak's model, and *AIC* and *BIC* for the Preußner's model.
- Preußner's functions fitted by age and site-stratified subpopulations can significantly increase the accuracy and efficiency of tree volume and merchantable volume estimates.

Competing interests

The authors have no competing interests.

Authors' contributions

All the authors contributed to all aspects of the study and manuscript preparation.

Acknowledgements

Acknowledgment to CAPES for the support by granting a study scholarship for first author.

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Appendix 1: Steps for fitting the Preußner functions at different ages.

The Preußner taper curve as a function of the average diameter $d_{0.5h}$

Fitting the functions on the four segments of the stem at different ages resulted in:

$$\begin{array}{l}
 \text{5 years} \left\{ \begin{array}{l}
 1^\circ \frac{y}{d_{0.5h}} = 2.0770 \frac{1}{(1+100x)^{0.06}} \\
 2^\circ \frac{y}{d_{0.5h}} = 1.9725 (1-x)^{0.98} \\
 3^\circ \frac{y}{d_{0.5h}} = 1.8660 (1-x)^{0.9} \\
 4^\circ \frac{y}{d_{0.5h}} = 1.8000 (1-x)^{0.874}
 \end{array} \right. \text{for} \left\{ \begin{array}{l}
 0.0 < x \leq 0.25 \\
 0.25 < x \leq 0.5 \\
 0.5 < x \leq 0.75 \\
 0.75 < x \leq 1.0
 \end{array} \right. \begin{array}{l}
 (1) \\
 (2) \\
 (3) \\
 (4)
 \end{array} \\
 \\
 \text{10 years} \left\{ \begin{array}{l}
 1^\circ \frac{y}{d_{0.5h}} = 1.9650 \frac{1}{(1+100x)^{0.06}} \\
 2^\circ \frac{y}{d_{0.5h}} = 1.8661 (1-x)^{0.9} \\
 3^\circ \frac{y}{d_{0.5h}} = 1.8404 (1-x)^{0.88} \\
 4^\circ \frac{y}{d_{0.5h}} = 1.8150 (1-x)^{0.87}
 \end{array} \right. \text{for} \left\{ \begin{array}{l}
 0.0 < x \leq 0.25 \\
 0.25 < x \leq 0.5 \\
 0.5 < x \leq 0.75 \\
 0.75 < x \leq 1.0
 \end{array} \right. \begin{array}{l}
 (5) \\
 (6) \\
 (7) \\
 (8)
 \end{array} \\
 \\
 \text{15 years} \left\{ \begin{array}{l}
 1^\circ \frac{y}{d_{0.5h}} = 1.7746 \frac{1}{(1+100x)^{0.05}} \\
 2^\circ \frac{y}{d_{0.5h}} = 1.7411 (1-x)^{0.8} \\
 3^\circ \frac{y}{d_{0.5h}} = 1.7291 (1-x)^{0.79} \\
 4^\circ \frac{y}{d_{0.5h}} = 1.7053 (1-x)^{0.78}
 \end{array} \right. \text{for} \left\{ \begin{array}{l}
 0.0 < x \leq 0.25 \\
 0.25 < x \leq 0.5 \\
 0.5 < x \leq 0.75 \\
 0.75 < x \leq 1.0
 \end{array} \right. \begin{array}{l}
 (9) \\
 (10) \\
 (11) \\
 (12)
 \end{array} \\
 \\
 \text{20 years} \left\{ \begin{array}{l}
 1^\circ \frac{y}{d_{0.5h}} = 1.7275 \frac{1}{(1+100x)^{0.046}} \\
 2^\circ \frac{y}{d_{0.5h}} = 1.7171 (1-x)^{0.78} \\
 3^\circ \frac{y}{d_{0.5h}} = 1.7053 (1-x)^{0.77} \\
 4^\circ \frac{y}{d_{0.5h}} = 1.6818 (1-x)^{0.76}
 \end{array} \right. \text{for} \left\{ \begin{array}{l}
 0.0 < x \leq 0.25 \\
 0.25 < x \leq 0.5 \\
 0.5 < x \leq 0.75 \\
 0.75 < x \leq 1.0
 \end{array} \right. \begin{array}{l}
 (13) \\
 (14) \\
 (15) \\
 (16)
 \end{array}
 \end{array}$$

The results obtained by the functions (1 to 16) were compared with the original data. We calculated the percentage error at each reference point, where the form quotients were taken, and found that it was less than 3% (although direction, positive or negative, varied).

One advantage of this approach over other taper functions is that the solution of variable “x” as a function of “y” can be easily obtained. The operative transformations for obtaining the height in each diameter and the transformations as a function of d are summarised in (17) to (32).

$$\begin{array}{l}
 \text{5 years} \left\{ \begin{array}{l}
 x = a_{12} \frac{1}{\left(\frac{y}{d_{0.5h}}\right)^{16.6667}} - \frac{1}{(b)} \\
 x = 1 - a_{22} \left(\frac{y}{d_{0.5h}}\right)^{1.0204} \\
 x = 1 - a_{32} \left(\frac{y}{d_{0.5h}}\right)^{1.1111} \\
 x = 1 - a_{42} \left(\frac{y}{d_{0.5h}}\right)^{1.1442}
 \end{array} \right. \text{for} \left\{ \begin{array}{l}
 1.4879 \leq \frac{y}{d_{0.5h}} \leq 2.0770 \\
 1.0000 \leq \frac{y}{d_{0.5h}} \leq 1.4879 \\
 0.5359 \leq \frac{y}{d_{0.5h}} \leq 1.0000 \\
 0.0000 \leq \frac{y}{d_{0.5h}} \leq 0.5359
 \end{array} \right. \begin{array}{l}
 (17) \\
 (18) \\
 (19) \\
 (20)
 \end{array} \\
 \\
 \text{10 years} \left\{ \begin{array}{l}
 x = a_{12} \frac{1}{\left(\frac{y}{d_{0.5h}}\right)^{16.6667}} - \frac{1}{(b)} \\
 x = 1 - a_{22} \left(\frac{y}{d_{0.5h}}\right)^{1.1111} \\
 x = 1 - a_{32} \left(\frac{y}{d_{0.5h}}\right)^{1.1364} \\
 x = 1 - a_{42} \left(\frac{y}{d_{0.5h}}\right)^{1.1494} \\
 x = a_{12} \frac{1}{\left(\frac{y}{d_{0.5h}}\right)^{20}} - \frac{1}{(b)}
 \end{array} \right. \text{for} \left\{ \begin{array}{l}
 1.4404 \leq \frac{y}{d_{0.5h}} \leq 1.9650 \\
 1.0000 \leq \frac{y}{d_{0.5h}} \leq 1.4404 \\
 0.5434 \leq \frac{y}{d_{0.5h}} \leq 1.0000 \\
 0.0000 \leq \frac{y}{d_{0.5h}} \leq 0.5434 \\
 1.3832 \leq \frac{y}{d_{0.5h}} \leq 1.7746
 \end{array} \right. \begin{array}{l}
 (21) \\
 (22) \\
 (23) \\
 (24) \\
 (25)
 \end{array}
 \end{array}$$

$$\begin{cases}
 \text{15 years} \left\{ \begin{array}{l} x = 1 - a_{22} \left(\frac{y}{d_{0.5h}}\right)^{1.25} \\ x = 1 - a_{32} \left(\frac{y}{d_{0.5h}}\right)^{1.2658} \\ x = 1 - a_{42} \left(\frac{y}{d_{0.5h}}\right)^{1.2821} \end{array} \right. & \text{for} & \begin{cases} 1.0000 \leq \frac{y}{d_{0.5h}} \leq 1.3832 & (26) \\ 0.5783 \leq \frac{y}{d_{0.5h}} \leq 1.0000 & (27) \\ 0.0000 \leq \frac{y}{d_{0.5h}} \leq 0.5783 & (28) \end{cases} \\
 \text{20 years} \left\{ \begin{array}{l} x = a_{12} \frac{1}{\left(\frac{y}{d_{0.5h}}\right)^{21.7391}} - \frac{1}{(b)} \\ x = 1 - a_{22} \left(\frac{y}{d_{0.5h}}\right)^{1.2821} \\ x = 1 - a_{32} \left(\frac{y}{d_{0.5h}}\right)^{1.2987} \\ x = 1 - a_{42} \left(\frac{y}{d_{0.5h}}\right)^{1.3158} \end{array} \right. & \text{for} & \begin{cases} 1.3720 \leq \frac{y}{d_{0.5h}} \leq 1.7275 & (29) \\ 1.0000 \leq \frac{y}{d_{0.5h}} \leq 1.3720 & (30) \\ 0.5864 \leq \frac{y}{d_{0.5h}} \leq 1.0000 & (31) \\ 0.0000 \leq \frac{y}{d_{0.5h}} \leq 0.5864 & (32) \end{cases}
 \end{cases}$$

The transformed constants resulted in:

For age up to 5 years:

$$a_{12} = \frac{a_{11}^{16.6667}}{b} = 1,952.7222; a_{22} = \frac{1}{a_{21}^{1.0204}} = 0.5; a_{32} = \frac{1}{a_{31}^{1.1111}} = 0.5; a_{42} = \frac{1}{a_{41}^{1.1442}} = 0.5104$$

For ages from 5 to 10 years:

$$a_{12} = \frac{a_{11}^{16.6667}}{b} = 774.9383; a_{22} = \frac{1}{a_{21}^{1.1111}} = 0.5; a_{32} = \frac{1}{a_{31}^{1.1364}} = 0.5; a_{42} = \frac{1}{a_{41}^{1.1494}} = 0.504$$

For ages from 10 to 15 years:

$$a_{12} = \frac{a_{11}^{20}}{b} = 959.5463; a_{22} = \frac{1}{a_{21}^{1.25}} = 0.5; a_{32} = \frac{1}{a_{31}^{1.2658}} = 0.5; a_{42} = \frac{1}{a_{41}^{1.2821}} = 0.5045$$

For ages from 15 to 20 years:

$$a_{12} = \frac{a_{11}^{21.7391}}{b} = 1,449.9909; a_{22} = \frac{1}{a_{21}^{1.2821}} = 0.5; a_{32} = \frac{1}{a_{31}^{1.2987}} = 0.5; a_{42} = \frac{1}{a_{41}^{1.3158}} = 0.5046$$

Using equations (17) to (32), it was possible to obtain the relative position in the shaft “x”, for a given diameter “y”. Note: to choose the proper equation, one must first calculate the value of $y/d_{0.5h}$ for the chosen diameter “y”.

The taper curve as a function of diameter at breast height

From a practical point of view, taper curves should be presented as a function of DBH. Importantly, the conversion of the functions to DBH will depend on the height of the tree. Thus, in the present case, as most trees have heights greater than 5.2 m, the conversion will take place in the first segment, using the equations pertinent to this section. In trees with heights less than 5.2 m, the conversion will occur in the second segment.

The relative diameter at breast height is given by $x = 1.3/h$, and, with conversion in the second segment, we have:

$$5 \text{ years old: } d_{1.3} = a_{11}d_{0.5h}(1 - x)^{0.98} \tag{33}$$

For the first segment:

$$5 \text{ years old: } d_{1.3} = a_{11}d_{0.5h} \left(\frac{1}{(1+bx)^{0.06}}\right) \tag{34}$$

$$10 \text{ years old: } d_{1.3} = a_{11}d_{0.5h} \left(\frac{1}{(1+bx)^{0.06}}\right) \tag{35}$$

$$15 \text{ years old: } d_{1.3} = a_{11}d_{0.5h} \left(\frac{1}{(1+bx)^{0.05}}\right) \tag{36}$$

$$20 \text{ years old: } d_{1.3} = a_{11} d_{0.5h} \left(\frac{1}{(1+bx)^{0.046}} \right) \tag{37}$$

Consequently, for the second segment:

$$5 \text{ years old: } d_{0.5h} = \frac{d_{1.3} \frac{1}{\left(1-\frac{1.3}{h}\right)^{0.98}}}{a_{21}} \tag{38}$$

For the first segment:

$$5 \text{ years old: } d_{0.5h} = \frac{d_{1.3}(1+bx)^{0.06}}{a_{11}} \tag{39}$$

$$10 \text{ years old: } d_{0.5h} = \frac{d_{1.3}(1+bx)^{0.06}}{a_{11}} \tag{40}$$

$$15 \text{ years old: } d_{0.5h} = \frac{d_{1.3}(1+bx)^{0.05}}{a_{11}} \tag{41}$$

$$20 \text{ years old: } d_{0.5h} = \frac{d_{1.3}(1+bx)^{0.046}}{a_{11}} \tag{42}$$

Once the diameters and heights along the stem of a given tree, as well as $d_{1.3}$ and the total height (h), are known, then equations (38) and (39) can be considered as constant k , i.e.:

$$\text{For the age of 5 years and 2nd segment: } k = d_{1.3} \frac{1}{\left(1-\frac{1.3}{h}\right)^{0.98}} \tag{43}$$

$$\text{For the age of 5 years and 1st segment: } k = d_{1.3}(1+bx)^{0.060} \tag{44}$$

$$\text{For the age of 10 years and 1st segment: } k = d_{1.3}(1+bx)^{0.060} \tag{45}$$

$$\text{For the age of 15 years and 1st segment: } k = d_{1.3}(1+bx)^{0.050} \tag{46}$$

$$\text{For the age of 20 years and 1st segment: } k = d_{1.3}(1+bx)^{0.046} \tag{47}$$

Tree height must be considered in metres, which ensures that the diameter $d_{0.5h}$ is obtained as a function of $d_{1.3}$. Therefore:

$$\text{For 2nd segment: } d_{0.5h} = \frac{k}{a_{21}} \tag{48}$$

$$\text{For 1st segment: } d_{0.5h} = \frac{k}{a_{11}} \tag{49}$$

By substitution of (49) in (1) to (16) is obtained:

5 years	$\left\{ \begin{array}{l} 1^\circ y = a_{13} k \frac{1}{(1+100x)^{0.06}} \\ 2^\circ y = k(1-x)^{0.98} \\ 3^\circ y = a_{33} k(1-x)^{0.9} \\ 4^\circ y = a_{43} k(1-x)^{0.874} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.0 < x \leq 0.25 \\ 0.25 < x \leq 0.5 \\ 0.5 < x \leq 0.75 \\ 0.75 < x \leq 1.0 \end{array} \right.$	$\tag{32}$ $\tag{33}$ $\tag{34}$ $\tag{35}$
10 years	$\left\{ \begin{array}{l} 1^\circ y = a_{13} k \frac{1}{(1+100x)^{0.06}} \\ 2^\circ y = k(1-x)^{0.9} \\ 3^\circ y = a_{33} k(1-x)^{0.88} \\ 4^\circ y = a_{43} k(1-x)^{0.87} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.0 < x \leq 0.25 \\ 0.25 < x \leq 0.5 \\ 0.5 < x \leq 0.75 \\ 0.75 < x \leq 1.0 \end{array} \right.$	$\tag{36}$ $\tag{37}$ $\tag{38}$ $\tag{39}$
15 years	$\left\{ \begin{array}{l} 1^\circ y = a_{13} k \frac{1}{(1+100x)^{0.05}} \\ 2^\circ y = k(1-x)^{0.8} \\ 3^\circ y = a_{33} k(1-x)^{0.79} \\ 4^\circ y = a_{43} k(1-x)^{0.78} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.0 < x \leq 0.25 \\ 0.25 < x \leq 0.5 \\ 0.5 < x \leq 0.75 \\ 0.75 < x \leq 1.0 \end{array} \right.$	$\tag{40}$ $\tag{41}$ $\tag{42}$ $\tag{43}$
20 years	$\left\{ \begin{array}{l} 1^\circ y = a_{13} k \frac{1}{(1+100x)^{0.046}} \\ 2^\circ y = k(1-x)^{0.78} \\ 3^\circ y = a_{33} k(1-x)^{0.77} \\ 4^\circ y = a_{43} k(1-x)^{0.76} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.0 < x \leq 0.25 \\ 0.25 < x \leq 0.5 \\ 0.5 < x \leq 0.75 \\ 0.75 < x \leq 1.0 \end{array} \right.$	$\tag{40}$ $\tag{41}$ $\tag{42}$ $\tag{43}$

The constants resulted from:

For 5 years:

$$a_{13} = \frac{a_{11}}{a_{21}} = 1.0530; a_{33} = \frac{a_{31}}{a_{21}} = 0.9461; a_{43} = \frac{a_{41}}{a_{21}} = 0.9126; k = d_{1.3}(1+bx)^{0.06}$$

For 10 years:

$$a_{13} = \frac{a_{11}}{a_{21}} = 1.0530; a_{33} = \frac{a_{31}}{a_{21}} = 0.9862; a_{43} = \frac{a_{41}}{a_{21}} = 0.9727; \quad k = d_{1.3}(1 + bx)^{0.06}$$

For 15 years:

$$a_{13} = \frac{a_{11}}{a_{21}} = 1.0192; a_{33} = \frac{a_{31}}{a_{21}} = 0.9931; a_{43} = \frac{a_{41}}{a_{21}} = 0.9794; \quad k = d_{1.3}(1 + bx)^{0.05}$$

For 20 years:

$$a_{13} = \frac{a_{11}}{a_{21}} = 1.0061; a_{33} = \frac{a_{31}}{a_{21}} = 0.9931; a_{43} = \frac{a_{41}}{a_{21}} = 0.9794; \quad k = d_{1.3}(1 + bx)^{0.046}$$

To obtain heights at a given diameter, equations (32) to (43) can also be solved. Therefore, the constants resulted from:

5 years	$\left\{ \begin{array}{l} x = a_{14} \frac{1}{\left(\frac{y}{k}\right)^{16.6667}} - \frac{1}{(b)} \\ x = 1 - \left(\frac{y}{k}\right)^{1.0000} \\ x = 1 - a_{34} \left(\frac{y}{k}\right)^{1.1111} \\ x = 1 - a_{44} \left(\frac{y}{k}\right)^{1.1442} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.7543 \leq \frac{y}{k} \leq 1.0530 \\ 0.5070 \leq \frac{y}{k} \leq 0.7543 \\ 0.2717 \leq \frac{y}{k} \leq 0.5070 \\ 0.0000 \leq \frac{y}{k} \leq 0.2717 \end{array} \right.$	(44) (45) (46) (47)
10 years	$\left\{ \begin{array}{l} x = a_{14} \frac{1}{\left(\frac{y}{k}\right)^{16.6667}} - \frac{1}{(b)} \\ x = 1 - \left(\frac{y}{k}\right)^{1.0000} \\ x = 1 - a_{34} \left(\frac{y}{k}\right)^{1.1364} \\ x = 1 - a_{44} \left(\frac{y}{k}\right)^{1.1494} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.7719 \leq \frac{y}{k} \leq 1.0530 \\ 0.5359 \leq \frac{y}{k} \leq 0.7719 \\ 0.2912 \leq \frac{y}{k} \leq 0.5359 \\ 0.0000 \leq \frac{y}{k} \leq 0.2912 \end{array} \right.$	(48) (49) (50) (51)
15 years	$\left\{ \begin{array}{l} x = a_{14} \frac{1}{\left(\frac{y}{k}\right)^{20}} - \frac{1}{(b)} \\ x = 1 - \left(\frac{y}{k}\right)^{1.0000} \\ x = 1 - a_{34} \left(\frac{y}{k}\right)^{1.2658} \\ x = 1 - a_{44} \left(\frac{y}{k}\right)^{1.2821} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.7944 \leq \frac{y}{k} \leq 1.0192 \\ 0.5743 \leq \frac{y}{k} \leq 0.7944 \\ 0.3322 \leq \frac{y}{k} \leq 0.5743 \\ 0.0000 \leq \frac{y}{k} \leq 0.3322 \end{array} \right.$	(52) (53) (54) (55)
20 years	$\left\{ \begin{array}{l} x = a_{14} \frac{1}{\left(\frac{y}{k}\right)^{21.7391}} - \frac{1}{(b)} \\ x = 1 - \left(\frac{y}{k}\right)^{1.0000} \\ x = 1 - a_{34} \left(\frac{y}{k}\right)^{1.2987} \\ x = 1 - a_{44} \left(\frac{y}{k}\right)^{1.3158} \end{array} \right.$	for	$\left\{ \begin{array}{l} 0.7990 \leq \frac{y}{k} \leq 1.0060 \\ 0.5824 \leq \frac{y}{k} \leq 0.7990 \\ 0.3415 \leq \frac{y}{k} \leq 0.5824 \\ 0.0000 \leq \frac{y}{k} \leq 0.3415 \end{array} \right.$	(52) (53) (54) (55)

The constants resulted from:

For 5 years:

$$a_{14} = \frac{a_{13}^{16.6667}}{b} = 0.0236; a_{34} = \frac{1}{a_{33}^{1.1111}} = 1.0636; a_{44} = \frac{1}{a_{43}^{1.1442}} = 1.1104$$

For 10 years:

$$a_{14} = \frac{a_{13}^{16.6667}}{b} = 0.0236; a_{34} = \frac{1}{a_{33}^{1.1364}} = 1.0159; a_{44} = \frac{1}{a_{43}^{1.1494}} = 1.0324$$

For 15 years:

$$a_{14} = \frac{a_{13}^{20}}{b} = 0.0146; a_{34} = \frac{1}{a_{33}^{1.2658}} = 1.0088; a_{44} = \frac{1}{a_{43}^{1.2821}} = 1.0270$$

For 20 years:

$$a_{14} = \frac{a_{13}^{21.7391}}{b} = 0.0114; a_{34} = \frac{1}{a_{33}^{1.2987}} = 1.0090; a_{44} = \frac{1}{a_{43}^{1.3158}} = 1.0277$$

Functions (44) to (55) are applicable when the value of “k” is known, which depends on the value of the DBH and the height of each tree. Thus, the variable y/k can be entered in the equations.

Appendix 2: Example for obtaining tree volume using the Preußner function.

For a relative function, the integration can be performed for the relative variable “x” in the range $0 \leq x \leq 1$. The variable “y” provides *d*; therefore, for the integration to result in the volume, it is necessary to multiply *y* by $\pi/4$ to obtain the area of the circle and by the height (*h*) to obtain the volume of the solid (*v*). Finally, “y” is squared (y^2) to get the cross-sectional area for a given diameter. Thus, we have:

$$v = \frac{\pi}{4} h \int_0^1 y^2 dx$$

Therefore, taking the functions for “y” (which describe the range from zero to one), for the age of 5 years, we have:

$$v = \frac{\pi}{4} h \left[\int_0^{0.25} a_{13} k^2 \frac{1}{(1 + 100x)^{0.12}} dx + \int_{0.25}^{0.50} k^2 (1 - x)^{1.96} dx + \int_{0.50}^{0.75} a_{33} k^2 (1 - x)^{1.8} dx + \int_{0.75}^{1.0} a_{43} k^2 (1 - x)^{1.748} dx \right]$$

For age of 10 years:

$$v = \frac{\pi}{4} h \left[\int_0^{0.25} a_{13} k^2 \frac{1}{(1 + 100x)^{0.12}} dx + \int_{0.25}^{0.50} k^2 (1 - x)^{1.8} dx + \int_{0.50}^{0.75} a_{33} k^2 (1 - x)^{1.76} dx + \int_{0.75}^{1.0} a_{43} k^2 (1 - x)^{1.74} dx \right]$$

For age of 15 years:

$$v = \frac{\pi}{4} h \left[\int_0^{0.25} a_{13} k^2 \frac{1}{(1 + 100x)^{0.1}} dx + \int_{0.25}^{0.50} k^2 (1 - x)^{1.6} dx + \int_{0.50}^{0.75} a_{33} k^2 (1 - x)^{1.58} dx + \int_{0.75}^{1.0} a_{43} k^2 (1 - x)^{1.56} dx \right]$$

For age of 20 years:

$$v = \frac{\pi}{4} h \left[\int_0^{0.25} a_{13} k^2 \frac{1}{(1 + 100x)^{0.092}} dx + \int_{0.25}^{0.50} k^2 (1 - x)^{1.56} dx + \int_{0.50}^{0.75} a_{33} k^2 (1 - x)^{1.54} dx + \int_{0.75}^{1.0} a_{43} k^2 (1 - x)^{1.52} dx \right]$$

Integration for age of 5 years results:

$$v = \frac{\pi}{4} h k^2 \left[a_{13}^2 \frac{1}{b} 1.2664 (1+bx)^{0.7897} \Big|_0^{0.25} 0.3657 (1-x)^{2.7347} \Big|_{0.25}^{0.50} a_{33}^2 0.3449 (1-x)^{2.8992} \Big|_{0.50}^{0.75} a_{43}^2 0.3854 (1-x)^{2.5946} \Big|_{0.75}^{1.0} \right]$$

For the age of 10 years:

$$v = \frac{\pi}{4} h k^2 \left[a_{13}^2 \frac{1}{b} 1.1364 (1+bx)^{0.88} \Big|_0^{0.25} 0.3571 (1-x)^{2.8} \Big|_{0.25}^{0.50} a_{33}^2 0.3623 (1-x)^{2.76} \Big|_{0.50}^{0.75} a_{43}^2 0.3650 (1-x)^{2.74} \Big|_{0.75}^{1.0} \right]$$

For the age of 15 years:

$$v = \frac{\pi}{4} h k^2 \left[a_{13}^2 \frac{1}{b} 1.1111 (1+bx)^{0.9} \Big|_0^{0.25} 0.3846 (1-x)^{2.6} \Big|_{0.25}^{0.50} a_{33}^2 0.3876 (1-x)^{2.58} \Big|_{0.50}^{0.75} a_{43}^2 0.3906 (1-x)^{2.56} \Big|_{0.75}^{1.0} \right]$$

For the age of 20 years:

$$v = \frac{\pi}{4} h k^2 \left[a_{13}^2 \frac{1}{b} 1.1013 (1+bx)^{0.908} \Big|_0^{0.25} 0.3906 (1-x)^{2.56} \Big|_{0.25}^{0.50} a_{33}^2 0.3937 (1-x)^{2.54} \Big|_{0.50}^{0.75} a_{43}^2 0.3968 (1-x)^{2.52} \Big|_{0.75}^{1.0} \right]$$

From the previously obtained integrals, it is possible to estimate the total or partial volume for a desired range of a segment of the stem. To illustrate the estimation of volume, we use a tree with a diameter equal to 38 cm at breast height and a total height of 22.8 m.

First, the value of *k* is calculated as: $1.3/22.8 = 0.0573$. This value is in the first segment, and applying it in the equation we have:

$$k = 38 \left(1 + 100 \left(\frac{1.3}{22.9} \right) \right)^{0.046} = 41.475 \text{ cm}$$

Using k in meters, the tree volume can be easily obtained:

$$v = \frac{\pi}{4} (22.9) (0.41)^2 \left[1.01^2 \frac{1}{100} 1.1 (1+100x)^{0.06} \Big|_0^{0.25} 0.39 (1-x)^{2.56} \Big|_{0.25}^{0.50} 0.99^2 0.39 (1-x)^{2.54} \Big|_{0.50}^{0.75} 0.98^2 0.39 (1-x)^{2.52} \Big|_{0.75}^{1.0} \right]$$

$$v = 3.080359 (0.203611 + 0.120792 + 0.055283 + 0.011570)$$

$$v = 3.080359(0.391256)$$

$$v = 1.205209 \text{ m}^3$$

The observed volume of the selected *Pinus taeda* tree was 1.189460 m³ and we estimated the volume to be 1.205209 m³ (with bark).

Note that after integration, we can calculate the total volume of any tree using the resulting equation $v = \pi/4 k^2 h^2 (0.391256)$; where, 0.391256 is the average Hohenadl natural form factor in the reference diameter $d_{0.5h}$.

For volume estimation at a specified interval, the integral calculation procedure will be the same as that previously presented, except that the integration will be performed on the interval defined for the variable "x" or $x_1 \leq x \leq x_2$. Therefore, $v_{x_1x_2} = \pi/4 h \int_{x_1}^{x_2} y^2 dx$. Using the same tree from the example above, the volume of wood between sections 0-25%, 25-50%, 50-75%, and 75-100% of the total height can be calculated as follows:

$$v_{x_0x_{0.25}} = \frac{\pi}{4} (22.9)(0.4147518)^2 \left[1.0061^2 \frac{1}{100} 1.1013 (1 + 100x)^{0.06} \Big|_0^{0.25} \right] = 0.627194 \text{ m}^3$$

$$v_{x_{0.25}x_{0.50}} = \frac{\pi}{4} (22.9)(0.4147518)^2 [0.3906 (1 - x)^{2.56} \Big|_{0.25}^{0.50}] = 0.372083 \text{ m}^3$$

$$v_{x_{0.50}x_{0.75}} = \frac{\pi}{4} (22.9)(0.4147518)^2 [0.9931^2 0.3937 (1 - x)^{2.54} \Big|_{0.50}^{0.75}] = 0.170291 \text{ m}^3$$

$$v_{x_{0.75}x_{1.0}} = \frac{\pi}{4} (22.9)(0.4147518)^2 [0.9794^2 0.3968 (1 - x)^{2.52} \Big|_{0.75}^{1.0}] = 0.035641 \text{ m}^3$$