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On an Extension of the Ghouila-Houri Theorem

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Abstract

Let D be a 2-strong digraph of order $n \geq 8$ such that for every vertex $x \in \mathcal{V}(\mathcal{D})\backslash\{z\}$, $d(x) \geq n$ and $d(z) \geq n-4$, where z is a vertex in $\mathcal{V}(\mathcal{D})$. We prove that:

If D contains a cycle passing through z of length equal to $n-2$, then D is Hamiltonian.

We also give a new sufficient condition for a digraph to be Hamiltonian-connected.
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1. Introduction

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. The order of a digraph D is the number of its vertices. We shall assume that the reader is familiar with the standard terminology on digraphs. Terminology and notations not described below follow [1]. Every cycle and path is assumed simple and directed. A cycle (path) in a digraph D is called *Hamiltonian* (*Hamiltonian path*) if it includes every vertex of D. A digraph D is *Hamiltonian* if it contains a Hamiltonian cycle, and it is *Hamiltonian-connected* if for any pair of ordered vertices x and y there exists a Hamiltonian path from x to y.

There are numerous sufficient conditions for the existence of a Hamiltonian cycle in a digraph (see, [1]–[3]). Let us recall the following sufficient conditions for a digraph to be Hamiltonian.

Theorem 1: (Ghouila-Houri [4]). *Let D be a strong digraph of order $n \geq 2$. If for every vertex $x \in \mathcal{V}(\mathcal{D})$, $d(x) \geq n$, then D is Hamiltonian.*

Theorem 2: (Meyniel [5]). *Let D be a strong digraph of order $n \geq 2$. If $d(x)+d(y) \geq 2n-1$ for all pairs of non-adjacent vertices x and y in D, then D is Hamiltonian.*

Nash-Williams [6] raised the problem of describing all the extreme digraphs in Theorem 1, that is, all digraphs with minimum degree at least $|D|-1$, that do not have a Hamiltonian cycle. As a solution to this problem, Thomassen [7] proved a structural theorem on the extreme digraphs. An analogous problem for Theorem 2 was considered by the author [8]. In [8], we generalize Thomassen's structural theorem (Theorem 1, in [7]), characterizing the non-Hamiltonian strong digraphs of order n with the degree condition that $d(x) + d(y) \geq 2n - 2$ for every pair of non-adjacent distinct vertices x, y. Moreover, in [8], it was also proved that if m is the length of a longest cycle in D, then D contains cycles of all lengths $k = 2, 3, \ldots, m$. The following conjecture was suggested by Thomassen.

Conjecture 1: (Thomassen [9], see Conjecure 1.6.7 in [2]). *Every 3-strong digraph of order n and with minimum degree at least $n + 1$ is Hamiltonian-connected.*

In [10], we disprove this conjecture, by proving the following three theorems.

Theorem 3: *Every k-strong ($k \geq 1$) digraph of order n, which has $n - 1$ vertices of degrees at least n, is Hamiltonian if and only if any $(k + 1)$-strong digraph of order $n + 1$ with minimum degree at least $n + 2$ is Hamiltonian-connected.*

Theorem 4: *For every $n \geq 8$, there is a non-Hamiltonian 2-strong digraph D of order n with minimum degree equal to 4 such that D has $n - 1$ vertices of degrees at least n.*

Theorem 5: *For every $n \geq 9$, there exists a 3-strong digraph D of order n with minimum degree at least $n+1$ such that D contains two distinct vertices u, v for which $u \leftrightarrow v$, $d^+_D(u) + d^-_D(v) = 6$ and there is no (u, v)-Hamiltonian path.*

In view of Theorems 4, 5 and Conjecture 1, it is natural to pose the following problem.

Problem: *Let D be a 2-strong digraph of order $n \geq 9$. Suppose that $n - 1$ vertices of D have degrees at least n and a vertex x has degree is at least $n - m$, where $1 \leq m \leq n - 5$. Find the maximum value of m, for which D is Hamiltonian.*

Goldberg, Levitskaya and Satanovskiy [11] relaxed the conditions of the Ghouila-Houri theorem. They proved the following theorem.

Theorem 6: (Goldberg et al. [11]). *Let D be a strong digraph of order $n \geq 2$. If for every vertex $x \in \mathcal{V}(D) \setminus \{z\}$, $d(x) \geq n$ and $d(z) \geq n - 1$, then D is Hamiltonian.*

Note that Theorem 6 is an immediate consequence of Theorem 2. In [11], the authors for any $n \geq 5$ presented two examples of non-Hamiltonian strong digraphs of order n such that:

(i) In the first example, $n - 2$ vertices have degrees equal to $n + 1$ and the other two vertices have degrees equal to $n - 1$.

(ii) In the second example, $n-1$ vertices have degrees at least n and the remaining vertex has degree equal to $n - 2$.

In [12], it was reported that the following theorem holds.

Theorem 7: (Darbinyan [12]). *Let D be a 2-strong digraph of order $n \geq 9$ with minimum degree at least $n - 4$. If $n - 1$ vertices of D have degrees at least n, then D is Hamiltonian.*

In this article, we present the first part of the proof of Theorem 7, which we formulate as Theorem 9. The proof of the last theorem has never been published. It is worth mentioning that the proof presented here differs from the previous handwritten proof and is significantly shorter and more general than the previous one. The second part of the proof (i.e., the complete proof) of Theorem 7 we will present in the forthcoming paper, where we also present two examples of digraphs, which show that the bounds $n \geq 9$ and $n-4$ in Theorem 7 are sharp in a sense.

2. Further Terminology and Notation

For the sake of clarity we repeat the most impotent definition. The vertex set and the arc set of a digraph D are denoted by $\mathcal{V(D)}$ and $\mathcal{A(D)}$, respectively. The order of a digraph D is the number of its vertices. The converse digraph of D is the digraph obtained from D by reversing the direction of all arcs. The arc of a digraph D directed from x to y is denoted by xy or $x \to y$ (we also say that x *dominates* y or y is an *out-neighbour* of x and x is an *in-neighbour* of y), and $x \leftrightarrow y$ denotes that $x \to y$ and $y \to x$ ($x \leftrightarrow y$ is called *2-cycle*). If $x \to y$ and $y \to z$, we write $x \to y \to z$. If A and B are two disjoint subsets of $\mathcal{V(D)}$ such that every vertex of A dominates every vertex of B, then we say that A *dominates* B, denoted by $A \to B$. We define $\mathcal{A}(A \to B) = \{xy \in \mathcal{A}(D) \,|\, x \in A, y \in B\}$ and $\mathcal{A(A,B)} = \mathcal{A(A \to B)} \cup \mathcal{A(B \to A)}$. If $x \in \mathcal{V(D)}$ and $A = \{x\}$ we sometimes write x instead of $\{x\}$. Let $N_D^+(x)$, $N_D^-(x)$ denote the set of out-neighbors, respectively the set of in-neighbors of a vertex x in a digraph D. If $A \subseteq \mathcal{V(D)}$, then $N_D^+(x,A) = A \cap N_D^+(x)$ and $N_D^-(x,A) = A \cap N_D^-(x)$. The *out-degree* of x is $d_D^+(x) = |N_D^+(x)|$ and $d_D^-(x) = |N_D^-(x)|$ is the *in-degree* of x. Similarly, $d_D^+(x,A) = |N_D^+(x,A)|$ and $d_D^-(x,A) = |N_D^-(x,A)|$. The *degree* of the vertex x in D is defined as $d_D(x) = d_D^+(x) + d_D^-(x)$ (similarly, $d_D(x,A) = d_D^+(x,A) + d_D^-(x,A)$). We omit the subscript if the digraph is clear from the context. The subdigraph of D induced by a subset A of $\mathcal{V(D)}$ is denoted by D. In particular, $D - A = D\langle \mathcal{V(D)} \setminus \mathcal{A} \rangle$. For integers a and b, $a \leq b$, by $[a,b]$ we denote the set $\{x_a, x_{a+1}, \ldots, x_b\}$. If $j < i$, then $\{x_i, \ldots, x_j\} = \emptyset$.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). The *length* of a cycle or a path is the number of its arcs. Let D be a digraph and $z \in \mathcal{V(D)}$. By $C_m(z)$ (respectively, $C(z)$) we denote a cycle in D of length m (respectively, any cycle in D), which contains the vertex z. We say that $P = x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)*-path*. A digraph D is *strong* (*strongly connected*) if, for every pair x, y of distinct vertices in D, there exists an (x,y)-path and a (y,x)-path. A digraph D is *k-strong* (*k-strongly connected*) if, $|\mathcal{V(D)}| \geq \| + \infty$ and for any set A of at most $k-1$ vertices $D - A$ is strong. Two distinct vertices x and y are *adjacent* if $xy \in$ or $yx \in \mathcal{A(D)}$ (or both). The *converse* digraph of D is the digraph obtained from D by replacing the direction of all arcs. We will use *the principle of digraph duality*: Let D be a digraph, then D contains a subdigraps H if and only if the converse digraph of D contain the converse of subdigraph H.

3. Preliminaries

In our proofs, we will use the following well-known simple lemma.

Lemma 1: (Häggkvist and Thomassen [13]). *Let D be a digraph of order $n \geq 3$ containing a cycle C_m of length m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, \mathcal{V}(C_m)) \geq m+1$, then for every $k \in [2, m+1]$, D contains a cycle C_k including x.*

The next lemma is a slight modification of a lemma by Bondy and Thomassen [14], it is very useful and will be used extensively throughout this paper.

Lemma 2:. *Let D be a digraph of order $n \geq 3$ containing a path $P := x_1x_2\dots x_m$, $m \in [2, n-1]$. Let x be a vertex not contained in this path. If one of the following condition holds:*

(i) $d(x, \mathcal{V}(P)) \geq m+2$,

(ii) $d(x, \mathcal{V}(P)) \geq m+1$ and $xx_1 \notin \mathcal{A}(D)$ or $x_mx \notin \mathcal{A}(\mathcal{D})$,

(iii) $d(x, \mathcal{V}(P)) \geq m$ and $xx_1 \notin \mathcal{A}(\mathcal{D})$ and $x_mx \notin \mathcal{A}(\mathcal{D})$,

then there is an $i \in [1, m-1]$ such that $x_i \to x \to x_{i+1}$, i.e., D contains a path $x_1x_2\dots x_ixx_{i+1}\dots x_m$ of length m (we say that x can be inserted into P).

Using Lemma 2, we can prove the following lemma.

Lemma 3: *Let $P := x_1x_2\dots x_m$, $m \in [3, n-1]$, be a longest (x_1, x_m)-path in a digraph D of order n. Suppose that $y \in \mathcal{V}(D) \setminus \mathcal{V}(P)$ and there is no $i \in [1, m-2]$ such that $x_i \to y \to x_{i+2}$. Then the following holds:*

(i) If $yx_1 \notin \mathcal{A}(\mathcal{D})$, $x_1y \in \mathcal{A}(\mathcal{D})$ and $d(y, \mathcal{V}(P)) \geq m$, then $d(y, \mathcal{V}(P)) = m$ and $\{x_1, x_2, \dots, x_m\} \to y$;

(ii) If $x_my \notin \mathcal{A}(\mathcal{D})$, $yx_m \in \mathcal{A}(\mathcal{D})$ and $d(y, \mathcal{V}(P)) \geq m$, then $d(y, \mathcal{V}(P)) = m$ and $y \to \{x_1, x_2, \dots, x_m\}$;

(iii) If $d(y, \mathcal{V}(P)) \geq m+1$, then $d(y, \mathcal{V}(P)) = m+1$ and there exists an integer $q \in [1, m]$ such that $\{x_q, x_{q+1}, \dots, x_m\} \to y \to \{x_1, x_2, \dots, x_q\}$.

Proof. To prove the lemma, it suffices to show that every vertex $x_i \in \mathcal{V}(\mathcal{P})$ is adjacent to y. Assume that this is not the case. (i) Let y and x_t be not adjacent. Then $t \geq 2$ since $x_1 \to y$. Since P is a longest (x_1, x_m)-path, we have that y cannot be inserted into P. Using Lemma 2(ii) and the assumption that $yx_1 \notin \mathcal{A}(\mathcal{D})$, we obtain $x_my \in \mathcal{A}(\mathcal{D})$, $2 \leq t \leq m-1$ and

$$m \leq d(y, \mathcal{V}(P)) = d(y, \{x_1, \dots, x_{t-1}\}) + d(y, \{x_{t+1}, \dots, x_m\}) \leq t-1+(m-t+1) = m.$$

This means that $d(y, \{x_1, \dots, x_{t-1}\}) = t-1$ and $d(y, \{x_{t+1}, \dots, x_m\}) = m-t+1$. Again using Lemma 2, we obtain that $x_{t-1} \to y \to x_{t+1}$, which contradicts the supposition of Lemma 3. This contradiction shows that every vertex x_i is adjacent to y.

In a similar way, one can show that if (ii) or (iii) holds, then every vertex of P also is adjacent to y. Lemma 3 is proved. □

In [10], the author proved the following theorem.

Theorem 8: (Darbinyan [12]). *Let D be a strong digraph of order $n \geq 3$. Suppouse that $d(x)+d(y) \geq 2n-1$ for all pairs of non-adjacent vertices $x, y \in \mathcal{V}(D) \setminus \{z\}$, where z is some vertex in $\mathcal{V}(\mathcal{D})$. Then D is Hamiltonian or contains a cycle of length $n-1$.*

Using Theorem 8 and Lemmas 1 and 2, it is not difficult to show that the following corollaries are true.

Corollary 1: *Let D be a strong digraph of order $n \geq 3$ satisfying the condition of Theorem 8. Then D has a cycle that contains all the vertices of D maybe except z.*

Corollary 2: *Let D be a strong digraph of order $n \geq 3$. Suppose that $n-1$ vertices of D have degrees at least n. Then D is Hamiltonian or contains a cycle of length $n-1$ (in fact, D has a cycle that contains all the vertices of degrees at least n).*

In this section, we also will prove the following lemma. We will use this lemma in the second part of the proof of Theorem 7.

Lemma 4: *Let D be a digraph of order $n \geq 4$ such that for any vertex $x \in \mathcal{V}(D)\setminus\{z\}$, $d(x) \geq n$ and $d(z) \leq n-2$, where z is some vertex in $\mathcal{V}(\mathcal{D})$. Suppose that $C_m(z) = x_1x_2\dots x_mx_1$ with $m \leq n-2$ is a longest cycle through z. If $D\langle V(D)\setminus V(C_m(z))\rangle$ is strong and D contains a $C_m(z)$-bypass $P = x_iy_1y_2\dots y_lx_j$ such that $|\mathcal{V}(C_m(z)[x_{i+1}, x_{j-1}])|$ is smallest possible over all $C_m(z)$-bypasses, then $z \in \mathcal{V}(C_m(z)[x_{i+1}, x_{j-1}])$.*

Proof. Without loss of generality, we assume that $x_j = x_1$, $x_i = x_{m-k}$, $k \geq 1$, $\mathcal{A}(\{y_1,\dots,y_l\}, \mathcal{V}(C_m(z)[x_{m-k+1}, x_m])) = \emptyset$ and k is minimum possible with this property over all $C_m(z)$-bypasses. Extending the path $C_m(z)[x_1, x_{m-k}]$ with the vertices of $\mathcal{V}(C_m(z)[x_{m-k+1}, x_m])$ as much as possible, we obtain an (x_1, x_{m-k})-path, say R. Since $C_m(z)$ is a longest cycle through z, some vertices $u_1, u_2, \dots, u_d \in \mathcal{V}(C_m(z)[x_{m-k+1}, x_m])$, $1 \leq d \leq k$, are not on the obtained extended path R. Using Lemma 2, we obtain that $d(y_i, \mathcal{V}V(C_m(z))) \leq m-k+1$ and $d(u_i, \mathcal{V}(C_m(z))) \leq m+d-1$. Put $B := \mathcal{V}(D) \setminus (\mathcal{V}(C_m(z)) \cup \mathcal{V}(\mathcal{P}))$. Note that $|B| = n-m-l$. Let v be an arbitrary vertex in B. From the minimality of k, we have that D contains no paths of the types $u_i \to v \to y_j$ and $y_j \to v \to u_i$, which in turn implies that $d^+(u_i, B) + d^-(y_j, B) \leq |B|$ and $d^-(u_i, B) + d^+(y_j, B) \leq |B|$. Therefore, $d(u_i, B) + d(y_j, B) \leq 2|B| = 2(n-m-l)$. Thus, we have

$$\begin{aligned} d(u_i) + d(y_j) &= d(u_i, \mathcal{V}(C_m(z))) + d(y_j, \mathcal{V}(C_m(z))) + d(u_i, B) + d(y_j, B) + d(y_j, \{y_1, \dots, y_l\}) \\ &\leq m+d-1+m-k+1+2n-2m-2l+2l-2 = 2n-2-(k-d) \leq 2n-2. \end{aligned}$$

This is possible if $u_i = z$. Therefore, $d = 1$ and $z \in \mathcal{V}(C_m(z)[x_{m-k+1}, x_m])$. Lemma 4 is proved. □

4. The Main Result

In this section, we prove the following theorem.

Theorem 9: *Let D be a 2-strong digraph of order $n \geq 8$. Suppose that for every $x \in \mathcal{V}(D) \setminus \{z\}$, $d(x) \geq n$ and $d(z) \geq n-4$, where z is a vertex in $\mathcal{V}(\mathcal{D})$. If D contains a cycle of length $n-2$ passing through z (i.e., a cycle $C_{n-2}(z)$), then D is Hamiltonian.*

Before we prove our main result, we will prove the following lemma.

Lemma 5: *Let D be a non-Hamiltonian 2-strong digraph of order n such that for any vertex $x \in \mathcal{V}(D) \setminus \{z\}$, $d(x) \geq n$ and $d(z) \leq n-2$, where z is an arbitrary fixed vertex in $\mathcal{V}(\mathcal{D})$. Suppose that $C_{m+1}(z) = x_1x_2\dots x_mzx_1$ with $m \in [2, n-3]$ is a longest cycle in D, $d(z, Y) = 0$ and $D\langle Y\rangle$ is a strong digraph, where $Y := \mathcal{V}(D) \setminus \mathcal{V}(C_{m+1}(z))$. Let y_1, y_2 be two distinct vertices in Y. If for each $y_i \in \{y_1, y_2\}$, $d(y_i, \{x_1, x_2, \dots, x_m\}) = m+1$, then $n \geq 6$ and $d(z) \leq m-2$.*

Proof. By contradiction, suppose that $d(z) \geq m-1$. We denote by P the path $x_1x_2\dots x_m$. Note that $|Y| = n-m-1$. Since the path P cannot be extended with any vertex $y \in Y$, by Lemma 2, $d(y, \mathcal{V}(P)) \leq m+1$ and

$$n \leq d(y) = d(y, \mathcal{V}(P)) + d(y, Y) \leq m+1+d(y, Y), \ d(y, Y) \geq n-m-1 = |Y|. \qquad (1)$$

Since D is 2-strong and $C_{m+1}(z)$ is a longest cycle, using Lemma 2 and $d(y_i, \mathcal{V}(P)) = m+1$ it is not difficult to show that there is an integer $l \in [2, m-1]$ such that

$$\{x_l, x_{l+1}, \dots, x_m\} \to \{y_1, y_2\} \to \{x_1, x_2, \dots, x_l\}. \qquad (2)$$

Since $d(y,Y)\geq n-m-1=|Y|$(by (1)), and $D\langle Y\rangle$ is strong, by the Ghouila-Houri theorem, $D\langle Y\rangle$ is Hamiltonian. Put $E:=\{x_1,x_2,\ldots,x_{l-1}\}$ and $F:=\{x_{l+1},x_{l+2},\ldots,x_m\}$. Since $C_{m+1}(z)$ is a longest cycle and $D\langle Y\rangle$ is strong, from (2) it follows that

$$\mathcal{A}(\mathcal{E}\rightarrow\mathcal{Y})=\mathcal{A}(\mathcal{Y}\rightarrow\mathcal{F})=\emptyset. \tag{3}$$

Note that from $|Y|\geq 2$, $|E|\geq 1$ and $|F|\geq 1$ it follows that $n\geq 6$. We need to prove the following Claims 1-2 bellow.

Claim 1.
(i) If $d^-(z,E)\geq 1$, then $d^+(z,F)=0$.
(ii) $\mathcal{A}(\mathcal{E}\rightarrow\mathcal{F})\neq\emptyset$.

Proof. (i) By contradiction, suppose that $x_i\in E$, $x_j\in F$ and $x_i\rightarrow z\rightarrow x_j$. Then by (2), $y_1\rightarrow x_{i+1}$ and $x_{j-1}\rightarrow y_2$. Hence, $C_{m+3}(z)=x_1x_2\ldots x_izx_j\ldots x_my_1x_{i+1}\ldots x_{j-1}y_2x_1$, a contradiction.

(ii) Suppose, on the contrary, that $\mathcal{A}(\mathcal{E}\rightarrow\mathcal{F})=\emptyset$. Then using Claim 1(i) and (3), we obtain: if $d^-(z,E)\geq 1$, then $d^+(z,F)=0$ and $\mathcal{A}(E\cup Y\cup\{z\}\rightarrow F)=\emptyset$, if $d^-(z,E)=0$, then $\mathcal{A}(E\cup Y\rightarrow F\cup\{z\})=\emptyset$. Therefore, $D-x_l$ is not strong, which contradicts that D is 2-strong. □

From now on, we assume that $x_ax_b\in\mathcal{A}(\mathcal{E}\rightarrow\mathcal{F})$. Note that $1\leq a\leq l-1$ and $l+1\leq b\leq m$. We may assume that b is the maximum and a is the minimum with these properties. By (2), we have

$$x_{b-1}\rightarrow\{y_1,y_2\}\rightarrow x_{a+1}. \tag{4}$$

Since z cannot be inserted into P, using Lemma 2(ii) and Clam 1(i), we obtain

$$d(z,\{x_1,x_2,\ldots,x_a\})+d(z,\{x_b,x_{b+1},\ldots,x_m\})\leq a+m-b+2. \tag{5}$$

By $R(y_i,y_{3-i})$, where $i\in[1,2]$, we denote a longest (y_i,y_{3-i})-path in $D\langle Y\rangle$. From now on, assume that $R(y_i,y_{3-i})=R(y_1,y_2)$.

Claim 2.
(i) *If $i\in[a+1,l-1]$, then $x_iz\notin\mathcal{A}(\mathcal{D})$.*
(ii) If $j\in[l+1,b-1]$, then $zx_j\notin\mathcal{A}(\mathcal{D})$.
(iii) If $i\in[a+1,l]$ and $i-a\leq 2$, then $zx_i\notin\mathcal{A}(\mathcal{D})$.
(iv) If $j\in[l,b-1]$ and $b-j\leq 2$, then $x_jz\notin\mathcal{A}(\mathcal{D})$.

Proof. Each of claims (i)-(iv) we prove by contradiction.

(i) Assume that $i\in[a+1,l-1]$ and $x_iz\in\mathcal{A}(\mathcal{D})$. Then by (2) and (4), we have $C_{m+3}(z)=x_1x_2\ldots x_ax_b\ldots x_my_1x_{i+1}\ldots x_{b-1}y_2x_{a+1}\ldots x_izx_1$, a contradiction.

(ii) Assume that $j\in[l+1,b-1]$ and $zx_j\in\mathcal{A}(\mathcal{D})$. Then by (2) and (4), we have $C_{m+3}(z)=x_1x_2\ldots x_ax_b\ldots x_mzx_j\ldots x_{b-1}y_1x_{a+1}\ldots x_{j-1}y_2x_1$, a contradiction.

(iii) Assume that $i\in[a+1,l]$, $i-a\leq 2$ and $zx_i\in\mathcal{A}(\mathcal{D})$. Then $C(z)=x_1x_2\ldots x_ax_b\ldots x_mzx_i\ldots x_{b-1}R(y_1,y_2)x_1$ is a cycle of length at least $m+2$, a contradiction.

(iv) Assume that $j\in[l,b-1]$, $b-j\leq 2$ and $x_jz\in\mathcal{A}(\mathcal{D})$. Then $C(z)=x_1x_2\ldots x_ax_b\ldots x_mR(y_1,y_2)x_{a+1}\ldots x_jzx_1$ is a cycle of length at least $m+2$, a contradiction. Claim 2 is proved. □

Now we will consider the following cases depending on the values of a and b with respect to l.

Case 1. $a \leq l-3$ and $b \geq l+3$.

Then by Claim 2, $d(z, \{x_{a+1}, x_{a+2}, x_{b-2}, x_{b-1}\}) = 0$. Therefore, since z cannot be inserted into P, using (5) and Lemma 2, we obtain

$$m-1 \leq d(z, \{x_1, x_2, \ldots, x_a, x_b, x_{b+1}, \ldots, x_m\}) + d(z, \{x_{a+3}, \ldots, x_{b-3}\})$$

$$\leq a+m-b+2+b-3-a-2+1 = m-2,$$

which is a contradiction.

Case 2. $a \leq l-3$ and $b = l+2$.

Then by Claim 2, $d(z, \{x_{a+1}, x_{a+2}, x_{l+1}\}) = 0$ and $x_l z \notin \mathcal{A}(\mathcal{D})$. Therefore, since z cannot be inserted into P, using (5) and Lemma 2, we obtain

$$m-1 \leq d(z, \{x_1, x_2, \ldots, x_a, x_b, x_{b+1}, \ldots, x_m\}) + d(z, \{x_{a+3}, \ldots, x_l\})$$

$$\leq a+m-b+2+l-a-2 = m-(l+2)+l = m-2,$$

which is a contradiction.

Case 3. $a \leq l-3$ and $b = l+1$.

Then by Claim 2, $d(z, \{x_{a+1}, x_{a+2}\}) = 0$ and $x_l z \notin \mathcal{A}(\mathcal{D})$. Similar to Case 2, we obtain

$$m-1 \leq d(z, \{x_1, x_2, \ldots, x_a, x_b, x_{b+1}, \ldots, x_m\}) + d(z, \{x_{a+3}, \ldots, x_l\})$$

$$\leq a+m-b+2+l-a-2 = m-b+l = m-(l+1) = m-1.$$

This implies that $d(z, \{x_{a+3}, \ldots, x_l\}) = l-a-2$. Hence, by Claim 2(i) and $x_l z \notin \mathcal{A}(\mathcal{D})$, $z \to \{x_{a+3}, \ldots, x_l\}$. From this and (4), we see that the cycle $Q(z) = x_1 x_2 \ldots x_a x_b \ldots x_m z$ $x_{a+3} \ldots x_l R(y_1, y_2) x_1$ has length equal to $m-1+|\mathcal{V}(R(y_1, y_2))|$. Since $C_{m+1}(z)$ is a longest cycle and $D\langle Y \rangle$ is Hamiltonian, it follows that $|\mathcal{V}(R(y_1, y_2))| = |Y| = 2$. Then $m = n-3$, $y_1 \leftrightarrow y_2$, $x_{a+1} \leftrightarrow x_{a+2}$ and x_{a+1} (x_{a+2}) is adjacent to each vertex $x_i \in \{x_1, x_2, \ldots x_m\}$, as $d(x_{a+1}) \geq n$ $(d(x_{a+2}) \geq n)$ and x_{a+1} (x_{a+2}) cannot be inserted into $Q(z)$.

We will distinguish two subcases.

Subcase 3.1. $m \geq l+2$. From the minimality of a and the maximality of b, it follows that

$$\mathcal{A}(\{x_1, x_2, \ldots, x_a\} \to \{x_{b+1}, x_{b+2}, \ldots, x_m\}) = \emptyset. \tag{6}$$

Assume that $x_i \to x_j$ with $i \in [a+1, l]$ and $j \in [l+2, m]$. Using (4) and the fact that $z x_{a+3} \in \mathcal{A}(\mathcal{D})$, it is not difficult to see that if $i \in [a+1, a+2]$, then $C(z) = x_1 x_2 \ldots x_{a+1}(x_{a+2}) x_j \ldots x_m z x_{a+3} \ldots x_{j-1} y_1 y_2 x_1$ is a cycle of length at least $m+2$, if $i \in [a+3, l-1]$, then $C_{m+3}(z) = x_1 x_2 \ldots x_i x_j \ldots x_m z x_{i+1} \ldots x_{j-1} y_1 y_2 x_1$, if $i = l$, then $C_{m+3}(z) = x_1 x_2 \ldots x_a x_{l+1} \ldots x_{j-1} y_1 y_2 x_{a+1} \ldots x_l x_j \ldots x_m z x_1$. Thus, in all cases, we have a contradiction. We may, therefore, assume that (recall that $b = l+1$)

$$\mathcal{A}(\{x_{a+1}, x_{a+2}, \ldots, x_l\} \to \{x_{b+1}, x_{b+2}, \ldots, x_m\}) = \emptyset.$$

Combining this with (6), we obtain

$$\mathcal{A}(\{x_1, x_2, \ldots, x_l\} \to \{x_{b+1}, x_{b+2}, \ldots, x_m\}) = \emptyset. \tag{7}$$

Assume first that $d^-(z, E) \geq 1$. Then by Claim 1(i), $d^+(z, F) = 0$. This together with (3) and (7) implies that $\mathcal{A}(\{z, x_1, x_2, \ldots, x_l\} \cup Y \to \{x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$. Assume second that $d^-(z, E) = 0$. Since $x_l z \notin \mathcal{A}(D)$, we obtain $\mathcal{A}(\{x_1, x_2, \ldots, x_l\} \cup Y \to \{z, x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$. So, in both cases we have that the subdigraph $D - x_{l+1}$ is not strong, which contradicts that D is 2-strong.

Subcase 3.2. $b = l + 1 = m$.

Assume that $a \geq 2$. As mentioned above, either $x_1 \to x_{a+1}$ or $x_{a+1} \to x_1$. Therefore, $C_{m+3}(z) = x_1 x_{a+1} \ldots x_{m-1} y_1 y_2 x_2 \ldots x_a x_m z x_1$ or $C_{m+2}(z) = x_1 \ldots x_a x_m z x_{a+3} \ldots x_{m-1} y_1 y_2 x_{a+1} x_1$. So, in both cases, we have a contradiction.

Assume next that $a = 1$. Then from $d^-(z, \{x_2, x_3, \ldots, x_{m-1}\}) = 0$ (by Claims 2(i) and 2(iv)) and $d^-(z) \geq 2$ it follows that $x_1 \to z$. We know that $z \to \{x_{a+3}, \ldots, x_l\}$. Using this, it is not difficult to see that if $x_i \to x_m$ with $i \in [2, m-2]$, then for $i = 2$, $C_{m+2}(z) = x_1 x_2 x_m z x_4 \ldots x_{m-1} y_1 y_2 x_1$, and for $i \in [3, m-2]$, $C_{m+3}(z) = x_1 x_2 \ldots x_i x_m z x_{i+1} \ldots x_{m-1} y_1 y_2 x_1$, a contradiction. We may, therefore, assume that

$$d^-(x_m, \{x_2, x_3, \ldots, x_{m-2}\}) = 0. \tag{8}$$

Now we consider the vertex x_1. If $x_j \to x_1$ with $j \in [2, m-2]$, then for $j = 2$, $C_{m+2}(z) = x_1 x_m z x_4 \ldots x_{m-1} y_1 y_2 x_2 x_1$, and for $j \in [3, m-2]$, $C_{m+3}(z) = x_1 x_m z x_{j+1} \ldots x_{m-1} y_1 y_2 x_2 \ldots x_j x_1$. Thus, in both cases, we have a contradiction. We may, therefore, assume that $d^-(x_1, \{x_2, x_3, \ldots, x_{m-2}\}) = 0$. This together with (3), (8) and $d^-(z, \{x_2, x_3, \ldots, x_{m-1}\}) = 0$ implies that

$$\mathcal{A}(\{x_2, x_3, \ldots, x_{m-2}\} \to Y \cup \{z, x_1, x_m\}) = \emptyset.$$

This means that $D - x_{m-1}$ is not strong, which contradicts that D is 2-strong.

Case 4. $a = l - 2$. Taking into account Case 2 and the digraph duality, we may assume that $b \leq l + 2$.

Subcase 4.1. $a = l - 2$ and $b = l + 2$. Then by Claim 2, $d(z, \{x_{l-1}, x_l, x_{l+1}\}) = 0$. This together with (5) implies that

$$m - 1 \leq d(z, \{x_1, x_2, \ldots, x_a, x_b, x_{b+1}, \ldots, x_m\}) \leq a + m - b + 2$$

$$= m + l - 2 - l - 2 + 2 = m - 2,$$

a contradiction.

Subcase 4.2. $a = l - 2$ and $b = l + 1$. Then by Claim 2, $d(z, \{x_{l-1}, x_l\}) = 0$.

Assume first that $m \geq l + 2$. If there exist $i \in [l-1, l]$ and $j \in [l+2, m]$ such that $x_i \to x_j$, then $C(z) = x_1 x_2 \ldots x_{l-2} x_{l+1} \ldots x_{j-1} R(y_1, y_2) x_i x_j \ldots x_m z x_1$ is a cycle of length at least $m + 2$, a contradiction. We may, therefore, assume that $\mathcal{A}(\{x_{l-1}, x_l\} \to \{x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$. This together with (3), the minimality of a and the maximality of b implies that $\mathcal{A}(\{x_1, x_2, \ldots, x_l\} \to \{x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$. Therefore, if $d^-(z, E) = 0$, then $\mathcal{A}(\{x_1, x_2, \ldots, x_l\} \cup Y \to \{z, x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$, and if $d^-(z, E) \geq 1$, then $d^+(z, F) = 0$ (Claim 1(i)) and $\mathcal{A}(\{z, x_1, x_2, \ldots, x_l\} \cup Y \to \{x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$. Thus, in both cases, we have that $D - x_{l+1}$ is not strong, a contradiction.

Assume next that $m = l + 1$. Then $a = l - 2 = m - 3$. Let $a \geq 2$. From the minimality of a it follows that $d^-(x_m, \{x_1, x_2, \ldots, x_{a-1}\}) = 0$. If there exist $i \in [1, a-1]$ and $j \in [a+1, a+2]$ such that $x_i \to x_j$, then it is easy to see that $C(z) = x_1 x_2 \ldots x_i x_j \ldots x_{m-1} R(y_1, y_2) x_{i+1} \ldots x_a x_m z x_1$ is a cycle of length at least $m+2$, a contradiction. We may, therefore, assume that $\mathcal{A}(\{x_1, x_2, \ldots, x_{a-1}\} \to \{x_{a+1}, x_{a+2}, x_{a+3} = x_m\}) = \emptyset$.

From this we have: if $d^-(z, \{x_1, x_2, \ldots, x_{a-1}) = 0$, then

$$\mathcal{A}(\{x_1, x_2, \ldots, x_{a-1}\} \to Y \cup \{z, x_{a+1}, x_{a+2}, x_{a+3}\}) = \emptyset,$$

if $d^-(z, \{x_1, x_2, \ldots, x_{a-1}) \geq 1$, then by Claim 1(i), $zx_m \notin \mathcal{A}(D)$ and

$$\mathcal{A}(\{x_1, x_2, \ldots, x_{a-1}\} \cup \{z\} \to Y \cup \{x_{a+1}, x_{a+2}, x_{a+3}\}) = \emptyset.$$

So, in both cases, we have that $D - x_a$ is not strong, which contradicts that D is 2-strong. Let now $a = 1$. Then $m = 4 = b = l + 1$ and $d(z, \{x_2, x_3\}) = 0$. This together with $d(z, Y) = 0$, $d^+(z) \geq 2$ and $d^-(z) \geq 2$ implies that $x_1 \to z \to x_4$, which contradicts Claim 1(i).

Case 5. $a = l - 1$. Taking into account Cases 3 and 4, we may assume that $b = l + 1$. Then $d(z, \{x_l\}) = 0$, and from (3), the minimality of a and the maximality of b it follows that

$$\begin{aligned} &\mathcal{A}(\{x_1, x_2, \ldots, x_{l-1}\} \to Y \cup \{x_{l+2}, x_{l+3}, \ldots, x_m\}) \\ &= \mathcal{A}(\{x_1, x_2, \ldots, x_{l-2}\} \to Y \cup \{x_{l+1}, x_{l+2}, \ldots, x_m\}) = \emptyset. \end{aligned} \tag{9}$$

It is not difficult see that: if $x_l \to x_j$ with $j \in [l + 2, m]$, then $C(z) = x_1 x_2 \ldots x_{l-1} x_{l+1} \ldots x_{j-1} R(y_1, y_2) x_l x_j \ldots x_m z x_1$ is a cycle of length at least $m + 3$, if $x_i \to x_l$ with $i \in [1, l - 2]$, then $C(z) = x_1 x_2 \ldots x_i x_l R(y_1, y_2) x_{i+1} \ldots x_{l-1} x_{l+1} \ldots x_m z x_1$ is a cycle of length at least $m + 3$. So, in both cases we have a contradiction. We may, therefore, assume that $d^+(x_l, \{x_{l+2} x_{l+3}, \ldots, x_m\}) = d^-(x_l, \{x_1, \ldots, x_{l-2}\}) = 0$. Then by (9),

$$\begin{aligned} &\mathcal{A}(\{x_1, x_2, \ldots, x_{l-2}\} \to \{x_l, x_{l+1}, \ldots, x_m\}) \\ &= \mathcal{A}(\{x_1, x_2, \ldots, x_l\} \to \{x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset. \end{aligned} \tag{10}$$

Assume that $m \geq l + 2$. If $d^-(z, E) \geq 1$, then $d^+(z, F) = 0$ (Claim 1(i)). This together with (3), (10), $d(z, \{x_l\}) = 0$ and $d(z, Y) = 0$ implies that $\mathcal{A}(\{z, x_1, x_2, \ldots, x_l\} \cup Y \to \{x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset$, which in turn implies that $D - x_{l+1}$ is not strong, a contradiction. We may, therefore, assume that $d^-(z, E) = 0$. Now it is not difficult to see that

$$\mathcal{A}(\{x_1, x_2, \ldots, x_l\} \cup Y \to \{z, x_{l+2}, x_{l+3}, \ldots, x_m\}) = \emptyset.$$

This means that $D - x_{l+1}$ is not strong, a contradiction.

Assume now that $m = l + 1$. By the digraph duality, we may assume that $a = l - 1 = 1$. Hence, $b = l + 1 = m = 3$. Then, since $d^+(z) \geq 2$ and $d^-(z) \geq 2$, $x_1 \to z \to x_m$, which contradicts Claim 1(i). The discussion of Case 5 is completed. Lemma 5 is proved. □

Now we are ready to prove the main result. For the convenience of the reader, we restate it here.

Theorem 9: *Let D be a 2-strong digraph of order $n \geq 8$ and z be a fixed vertex in $\mathcal{V}(D)$. Suppose that for any vertex $x \in \mathcal{V}(D) \setminus \{z\}$, $d(x) \geq n$, $d(z) \geq n - 4$, and D contains a cycle of length $n - 2$ passing through z. Then D is Hamiltonian.*

Proof. Suppose, on the contrary, that D contains a cycle $C_{n-2}(z) := x_1 x_2 \ldots x_{n-2} x_1$ but it is not Hamiltonian. By Theorem 3 (or by Theorem 2), $d(z) \leq n - 2$. Let $\{y_1, y_2\} = \mathcal{V}(D) \setminus \mathcal{V}(C_{n-2}(z))$. Since $z \in \mathcal{V}(C_{n-2}(z))$, we have that $d(y_i) \geq n$. Using Lemma 1, it is easy to show that D contains no $C_{n-1}(z)$, $d(y_1) = d(y_2) = n$, $d(y_1, \mathcal{V}(C_{n-2}(z))) =$

$d(y_2, \mathcal{V}(C_{n-2}(z))) = n-2$ and $y_1 \leftrightarrow y_2$. If y_1 or y_2 is adjacent to every vertex x_i, $i \in [1, n-2]$, then D contains a cycle $C(z)$ of length at least $n-1$, a contradiction. We may, therefore, assume that y_1 and some vertex of $C_{n-2}(z)$ are not adjacent, say x_{n-2}. Then $d(y_1, \{x_1, x_2, \ldots, x_{n-3}\}) = n-2$. Since y_1 cannot be inserted into $x_1x_2\ldots x_{n-3}$, using Lemma 2, we obtain that $x_{n-3} \to y_1 \to x_1$. This together with $y_1 \leftrightarrow y_2$ implies that $d(x_{n-2}, \{y_1, y_2\}) = 0$ (for otherwise, D contains a cycle of length at least $n-1$ through z, which is a contradiction). Therefore, $d(y_2, \{x_1, x_2, \ldots, x_{n-3}\}) = n-2$, and by Lemma 2, $x_{n-3} \to y_2 \to x_1$. Then $C_{n-1} = x_1x_2\ldots x_{n-3}y_1y_2x_1$ is a cycle of length $n-1$. We know that C_{n-1} does not contain the vertex z. Therefore, $z = x_{n-2}$. Thus, we have that the conditions of Lemma 5 hold. Therefore, $d(z) \leq n-5$, which contradicts that $d(z) \geq n-4$. The theorem is proved. □

In [15], Overbeck-Larisch proved the following sufficient condition for a digraph to be Hamiltonian-connected.

Theorem 10: (Overbeck-Larisch [15]). *Let D be a 2-strong digraph of order $n \geq 3$ such that, for each two non-adjacent distinct vertices x, y we have $d(x) + d(y) \geq 2n + 1$. Then for each two distinct vertices u, v with $d^+(u) + d^-(v) \geq n + 1$ there is a Hamiltonian (u, v)-path.*

Let D be a digraph of order $n \geq 3$ and let u and v be two distinct vertices in $\mathcal{V}(D)$. Follows Overbeck-Larisch [15], we define a new digraph $H_D(u, v)$ as follows: $\mathcal{V}(H_D(u, v)) = \mathcal{V}(D - \{u, v\}) \cup \{z\}$ (z a new vertex) and $\mathcal{A}(H_D(u, v)) = \mathcal{A}(D - \{u, v\}) \cup \{zy \,|\, y \in N^+_{D-v}(u)\} \cup \{yz \,|\, y \in N^-_{D-u}(v)\}$.

Now, using Theorem 7, we will prove the following theorem, which is an analogue of the Overbeck-Larisch theorem.

Theorem 11: *Let D be a 3-strong digraph of order $n + 1 \geq 10$ with minimum degree at least $n + 2$. If for two distinct vertices u, v, $d^+_D(u) + d^-_D(v) \geq n - 2$ or $d^+_D(u) + d^-_D(v) \geq n - 4$ with $uv \notin \mathcal{A}(\mathcal{D})$, then there is a Hamiltonian (u, v)-path in D.*

Proof. Let D be a 3-strong digraph of order $n + 1 \geq 10$ and let u, v be two distinct vertices in $\mathcal{V}(D)$. Suppose that D and u, v satisfy the degree conditions of the theorem. Now we consider the digraph $H := H_D(u, v)$ of order $n \geq 9$. By an easy computation, we obtain that the minimum degree of H is at least $n - 4$, and H has $n - 1$ vertices of degrees at least n. Moreover, we know that H is 2-strong (see [10]). Thus, the digraph H satisfies the conditions of Theorem 7. Therefore, H is Hamiltonian, which in turn implies that in D there is a Hamiltonian (u, v)-path. □

5. Conclusion

For Hamiltonicity of a graph G (undirected graph), there are numerous sufficient conditions in terms of the number $k(G)$ of connectivity, where $k(G) \geq 3$ (recall that for a graph G to be Hamiltonian, $k(G) \geq 2$ is a necessary condition) and the minimum degree $\delta(G)$ (or the sum of degrees of some vertices with certain properties), see the survey papers by Gould, e.g. [16]. This is not the case for the general digraphs. In [17], the author proved that: For every pair of integers $k \geq 2$ and $n \geq 4k + 1$ (respectively, $n = 4k + 1$), there exists a k-strong $(n-1)$-regular (respectively, with minimum degree at least $n-1$ and with minimum semi-degrees at least $2k - 1 = (n-3)/2$) a non-Hamiltonian digraph of order n. In [1] (Page

253), it was showed that there is no k such that every k-strong multipartite tournament with a cycle factor has Hamiltonian cycle.

Based on the evidence from Theorem 9, we raise the following conjecture, the truth of which in the case $k = 0$ follows from Theorem 9.

Conjecture 2: *Let D be a 2-strong digraph of order n and z be a fixed vertex in $\mathcal{V}(\mathcal{D})$. Suppose that for any vertex $x \in \mathcal{V}(D) \setminus \{z\}$, $d(x) \geq n+k$ and $d(z) \geq n-k-4$, where $k \geq 0$ is an integer. Then D is Hamiltonian.*

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Գուիիլա-Հուրիի թեորեմի մի ընդլայնման մասին

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Ամփոփում

Ներկա աշխատանքում ապացուցվել է հետևյալ թեորեմը:
Թեորեմ: Դիցուք D-ն 2-ուժեղ կապակցված n-գագաթանի ($n \geq 8$) կողմնորոշված գրաֆ է, որի $n-1$ գագաթների աստիճանները փոքր չեն n թվից, իսկ z գագաթի աստիճանը փոքր չէ $n-4$ թվից: Եթե D-ն ն պարունակում է $n-2$ երկարությամբ ցիկլ, որը անցնում է z գագաթով, ապա D պարունակում է համիլտոնյան ցիկլ:

Բանալի բառեր՝ հակադարձ մատրից, երեքանկյունագծային մատրից, հերմիտյան մատրից, տյոպլիցյան մատրից:

Об одном расширении теоремы Гуйя-Ури

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Аннотация

В настоящей работе доказана следующая теорема. Теорема. Пусть D есть 2-сильно связный $n \geq 8$ вершинный орграф, в котором $n-1$ вершин имеют степень не меньше чем n , а вершина z имеет степень не меньше чем $n-4$. Если D содержит контур длины $n-2$, которий содержит вершину z, то D содержит гамильтонов контур.

Ключевые слова: орграф, гамильтонов контур, 2-сильно, гамильтоново-связныий.