Mathematical Problems of Computer Science 38, 77–79, 2012.

On the Algebras with Hyperidentities of the Variety of de Morgan Algebras

Yu. M. Movsisyan, V. A. Aslanyan

Yerevan State University,
Alex Manoogian 1, Yerevan 0025, Armenia
E-mail: yurimovsisyan@yahoo.com, vahagn.aslanyan@gmail.com

1 Introduction

In paper [1] the algebras with hyperidentities of the variety of Boolean algebras are characterized. In this paper the algebras with hyperidentities of the variety of De Morgan algebras are characterized. For these algebras with two binary operations we prove a structure result. As a consequence, we obtain the new finite base of the hyperidentities of the variety of De Morgan algebras, having functional and objective ranks not exceeding three.

An algebra $Q(+,\cdot,')$ with two binary and one unary operations is called a De Morgan algebra if $Q(+,\cdot)$ is a distributive lattice and $Q(+,\cdot,')$ satisfies the following identities:

$$(x+y)' = x' \cdot y',$$

$$x'' = x,$$

where $x'' = (x')'$. The standard fuzzy algebra $F = ((0,1); max(x,y), min(x,y), 1-x)$ is an example of a De Morgan algebra.

De Morgan algebras were considered by J.A.Kalman [2](as i-lattices), G.C.Moisil [3], H.Rasiowa and A.Bialynicki-Birula [4], Yu.M.Movsisyan [5], J. Berman and W. Blok [6] and others. They also related to constructive logic with strong negation (A.A.Markov [7], D.Nelson [8], N.N.Vorobev [9], I.D.Zaslavsky [10]). Except in mathematical logic and algebra, De Morgan algebras (and De Morgan bisemilattices) have applications in multi-valued simulations of digital circuits too ([11, 12]).

The hyperidentities of the variety of De Morgan algebras are characterized in [13].

Definition 1.1 *A T-algebra $A = (Q, \Sigma)$, where $T = \{1, 2\}$, is called De Morgan quasilattice if it satisfies all hyperidentities of the variety of De Morgan algebras.*

For example, the superproduct ([14, 15, 16, 17]) of the two De Morgan algebras (De Morgan quasilattices) is a De Morgan quasilattice.

2 Main result

Below we define the concept of De Morgan sum analogous to Boolean sum introduced in [1].

Definition 2.1 *Let $A = (Q, \Omega \cup \{F\})$ be an algebra with a unary operation F. Let (Q_i, Ω), $i \in I$ be subalgebras of the algebra A, and $A_i = (Q_i, \Omega \cup \{F_i\})$ be algebras with a unary operation F_i. The algebra A is called De Morgan sum of algebras A_i, if the following conditions hold true:*
1) *$Q_i \cap Q_j = \emptyset$ for all $i, j \in I, i \neq j$;*
2) *$Q = \bigcup_{i \in I} Q_i$;*
3) *Two binary operations $+, \cdot$ and a unary operation $^-$ can be defined on I such that $I(+, \cdot, ^-)$ is a De Morgan algebra;*
4) *If $i, j \in I$ and $i \leq j$ (here " $\leq$ " is the order of the lattice $I(+, \cdot)$), then there exists an isomorphism*

$$(\varphi_{i,j}, \tilde{\varepsilon}) : A_i \to A_j,$$

where $\tilde{\varepsilon}(F_i) = F_j$, $\tilde{\varepsilon}(A) = A$ for any $A \in \Omega$. Moreover, $\varphi_{i,i}$ is the identical mapping of the set Q_i, and for all $i \leq j \leq k$ we have $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$;
5) *For every $i \in I$ there exists an isomorphism*

$$(h_{i,\bar{i}}, \tilde{\varepsilon}) : A_i \to A_{\bar{i}},$$

such that $h^{-1}_{i,\bar{i}} = h_{\bar{i},i}$ and $h_{i,\bar{i}} \cdot \varphi_{\bar{i},k} = \varphi_{i,k}$ for all $k \geq i + \bar{i}$, $k \in I$;
6) *For any n-ary operation $A \in \Omega$ $(n \geq 2)$ and for any $x_1, \ldots, x_n \in Q$ we have:*

$$A(x_1, \ldots, x_n) = A(\varphi_{i_1,i_0}(x_1), \ldots, \varphi_{i_n,i_0}(x_n)),$$

where $x_j \in Q_{i_j}$, $i_j \in I$, $j = \overline{1,n}$, $i_0 = i_1 + \ldots + i_n$;
7) *For any $x \in Q$ we have:*

$$F(x) = h_{i,\bar{i}}(F_i(x)),$$

where $x \in Q_i$.

Theorem 2.1 *An algebra $A = (Q, \{+, \cdot, ^-\})$ with two binary operations $+, \cdot$ and one unary operation $^-$ is a De Morgan quasilattice iff it is a De Morgan algebra or De Morgan sum of De Morgan algebras.* ∎

Corollary 2.1 *The variety of De Morgan algebras has a finite base of hyperidentities having functional and objective ranks not exceeding three.* ∎

References

[1] Yu.M. Movsisyan, *Algebras with hyperidentities of the variety of Boolean algebras.* Izvestiya Rossiyskoy Akademii Nauk: Seriya Matematicheskaya 60, 127-168, 1996. English translation in Russian Academy of Science Izvestiya Matematischeskaya, 60, 1219-1260, 1996.

[2] J. A. Kalman, Lattices with involution, *Trans. Amer. Math. Soc.* 87, (1958), 485-491.

[3] G.C.Moisil, Recherches sur l'algebre de la logique, *Annales scientifiques de l'universite de Jassy*, 22, (1935), 1-117.

[4] A.Bialynicki-Birula, H.Rasiowa, On the representation of quasi-Boolean algebras, *Bull. Acad. Polon. Sci., Ser. Math. Astronom. Phys.*, 5, (1957), 259-261.

[5] Yu.M.Movsisyan, Binary representations of algebras with at most two binary operations. A Cayley theorem for distributive lattices, *International Journal of Algebra and Computation*, Vol.19, 1(2009), 97-106.

[6] J. Berman, W. Blok, Stipulations, multi-valued logic and De Morgan algebras, *Multi-valued Logic* 7 (5-6) (2001), 391-416.

[7] A.A.Markov, Constructive Logic (in Russian),*Uspekhi Mat. Nauk,* 5(1950),187-188.

[8] D.Nelson, Constructible falsity, *J. Symbolic Logic*, 14(1959), 16-26.

[9] N.N.Vorobev, A constructive propositional calculus with strong negation (in Russian), *Dokl. Akad. Nauk SSR,* 85(1952), 465-468.

[10] I.D.Zaslavsky, *Symmetric Constructive Logic* (in Russian), Publishing House of Academy of Sciences of Armenian SSR (1978).

[11] J.A. Brzozowski, De Morgan bisemilattices, *Proceedings of the 30th IEEE International Symposium on Multiple-Valued Logic* (ISMVL 2000), May 23-25, (2000), p.173.

[12] J.A. Brzozowski, Partially ordered structures for hazard detection, *Special Session: The Many Lives of Lattice Theory, Joint Mathematics Meetings*, San Diego, CA, January 6-9, (2002).

[13] Yu.M.Movsisyan, V.A.Aslanyan, *Hyperidentities of De Morgan algebras*, Logic Journal of the IGPL.(doi:10.1093/jigpal/jzr053)

[14] Yu.M. Movsisyan, *Introduction to the theory of algebras with hyperidentities* (in Russian), Yerevan State University Press, Yerevan, 1986.

[15] Yu.M. Movsisyan, Hyperidentities in algebras and varieties, *Uspekhi Matematicheskikh Nauk*, Vol.53, No. 1(319), (1998), 61–114. English translation in Russian Mathematical Surveys 53, 1, (1998), 57–108.

[16] Yu.M.Movsisyan, Bilattices and hyperidentities , *Proceedings of the Steklov Institute of Mathematics*, Vol. 274, (2011), pp. 174-192.

[17] Yu.M.Movsisyan, A.B.Romanowska, J.D.H Smith, Superproducts, hyperidentities, and algebraic structures of logic programming, *Comb. Math. And Comb. Comp.*, 2006, v.58, p.101-111.