Mathematical Problems of Computer Science 51, 21–38, 2019.

UDC 519.1

On the Manoussakis Conjecture for a Digraph to be Hamiltonian

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Abstract

Y. Manoussakis (J. Graph Theory 16, 1992, 51-59) proposed the following conjecture.

Conjecture. *Let D be a 2-strongly connected digraph of order n such that for all distinct pairs of non-adjacent vertices x, y and w, z, we have $d(x)+d(y)+d(w)+d(z) \geq 4n-3$. Then D is Hamiltonian.*

In this note, we prove that if D satisfies the conditions of this conjecture, then (i) D has a cycle factor; (ii) If $\{x, y\}$ is a pair of non-adjacent vertices of D such that $d(x) + d(y) \leq 2n - 2$, then D is Hamiltonian if and only if D contains a cycle passing through x and y; (iii) If $\{x, y\}$ a pair of non-adjacent vertices of D such that $d(x)+d(y) \leq 2n-4$, then D contains cycles of all lengths $3, 4, \ldots, n-1$; (iv) D contains a cycle of length at least $n - 1$.

Keywords: Digraph, Hamiltonian cycle, Cycle factor, Pancyclic digraph.

1. Introduction

In this paper, we consider finite digraphs (directed graphs) without loops and multiple arcs. Every cycle and path are assumed simple and directed. A digraph D is *Hamiltonian* if it contains a cycle passing through all the vertices of D. There are many conditions that guarantee that a digraph is Hamiltonian (see, e. g., [1]-[5]). In [5], the following theorem was proved.

Theorem 1.1: (Manoussakis [5]). *Let D be a strongly connected digraph of order n. Suppose that D satisfies the following condition for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geq 3n - 2$. If there is no arc from z to x, then $d(x)+d(y)+d^-(x)+d^+(z) \geq 3n-2$. Then D is Hamiltonian.*

Definition 1.2: *Let D be a digraph of order n. We say that D satisfies condition (M) when $d(x)+d(y)+d(w)+d(z) \geq 4n-3$ for all distinct pairs of non-adjacent vertices x, y and w, z.*

Manoussakis [5] proposed the following conjecture. This conjecture is an extension of Theorem 1.1

Conjecture 1.3: (Manoussakis [5]). *Let G be a 2-strongly connected digraph of order n such that for all distinct pairs of non-adjacent vertices x, y and w, z we have $d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Then D is Hamiltonian.*

This conjecture seems quite difficult to prove. Manoussakis [5] gave an example, which showed that if the conjecture is true, then the minimum degree condition is sharp. Notice that another examples can be found in [6], where for any two integers $k \geq 2$ and $m \geq 1$, the author constructed a family of k-strongly connected digraphs of order $4k + m$ with minimum degree $4k + m - 1$, which are not Hamiltonian. This result improves a conjecture of Thomassen [2] (Conjecture 1.4.1). Moreover, when $m = 1$, then from these digraphs we can obtain k-strongly connected non-Hamiltonian digraphs of order $n = 4k + 1$ with minimum degree equal to $n - 1$ and the minimal semi-degrees equal to $(n - 3)/2$. Thus, if in Conjecture 1.3 we replace $4n - 4$ instead of $4n - 3$, then for every n there are many digraphs of order n with high connection and high semi-degrees, for which Conjecture 1.3 is not true.

The author [7] proved the following theorem.

Theorem 1.4: (Darbinyan [7]). *Let D be a strongly connected digraph of order $n \geq 3$. Suppose that $d(x) + d(y) \geq 2n - 1$ for every pair of non-adjacent vertices $x, y \in V(D) \setminus \{z\}$, where z is some vertex of $V(D)$. Then either D is Hamiltonian or contains a cycle of length $n - 1$.*

It is easy to see that if a digraph D satisfies the condition (M), then it contains at most one pair of non-adjacent vertices x, y such that $d(x) + d(y) \leq 2n - 2$. From this and Theorem 1.4 immediately follows the following corollary.

Corollary 1.5: *Let G be a strongly connected digraph of order n satisfying condition (M). Then D contains a cycle of length at least $n - 1$ (in particular, D contains a Hamiltonian path).*

Corollary 1.5 was also later proved by Ning [8].

In this paper we investigate the properties those digraphs, which satisfy the condition of Conjecture 1.3. Let D be a 2-strongly connected digraph of order n satisfying the condition (M) and let $\{x, y\}$ be a pair of non-adjacent vertices of D. In Section 4 we prove:

(i) D has a cycle factor;

(ii) If $d(x) + d(y) \leq 2n - 2$, then D is Hamiltonian if and only if D contains a cycle passing through x and y;

(iii) If $d(x) + d(y) \leq 2n - 4$, then D contains cycles of all lengths $3, 4, \ldots, n - 1$;

(iv) Suppose that $x_1 x_2 \ldots x_{n-2} y x_1$ is a cycle of length $n-1$ passing through y and avoiding x. If $d(x) + d(y) \leq 2n - 2$ and $x_{n-2} \to x \to x_1$, then D is Hamiltonian.

2. Terminology and Notation

In this paper we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer to [1] for terminology and notations not discussed here. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. The *order* of D is the number of its vertices. For any $x, y \in V(D)$, we also write $x \to y$ if $xy \in A(D)$. If $xy \in A(D)$, y is an *out-neighbour* of x and x is an *in-neighbour* of y. If $x \to y$ and $y \to z$, we write $x \to y \to z$. Two distinct vertices x and y are *adjacent* if $xy \in A(D)$ or $yx \in A(D)$ (or both). If there is no arc from x to y, we shall use the notation $xy \notin A(D)$.

We let $N^+(x)$, $N^-(x)$ denote the set of *out-neighbours*, respectively the set of *in-neighbours* of a vertex x in a digraph D. If $A \subseteq V(D)$, then $N^+(x, A) = A \cap N^+(x)$ and $N^-(x, A) = A \cap N^-(x)$. The *out-degree* of x is $d^+(x) = |N^+(x)|$ and $d^-(x) = |N^-(x)|$ is the *in-degree* of x. Similarly, $d^+(x, A) = |N^+(x, A)|$ and $d^-(x, A) = |N^-(x, A)|$. The *degree* of the vertex x in D is defined as $d(x) = d^+(x) + d^-(x)$ (similarly, $d(x, A) = d^+(x, A) + d^-(x, A)$). The subdigraph of D induced by a subset A of $V(D)$ is denoted by $D\langle A\rangle$. If z is a vertex of a digraph D, then the subdigraph $D\langle V(D) \setminus \{z\}\rangle$ is denoted by $D - z$.

For integers a and b, $a \leq b$, let $[a, b]$ denote the set of all integers, which are not less than a and are not greater than b.

The path (respectively, the cycle) consisting of the distinct vertices $x_1, x_2, \ldots, x_m$ ($m \geq 2$) and the arcs $x_i x_{i+1}$, $i \in [1, m-1]$ (respectively, $x_i x_{i+1}$, $i \in [1, m-1]$, and $x_m x_1$), is denoted by $x_1 x_2 \cdots x_m$ (respectively, $x_1 x_2 \cdots x_m x_1$). The *length* of a cycle or path is the number of its arcs. We say that $x_1 x_2 \cdots x_m$ is a path from x_1 to x_m or is an (x_1, x_m)-*path*. Let x and y be two distinct vertices of a digraph D. Cycle that passing through x and y in D, we denote by $C(x, y)$.

A cycle (respectively, a path) that contains all the vertices of D, is a *Hamiltonian cycle* (respectively, is a *Hamiltonian path*). A digraph is *Hamiltonian* if it contains a Hamiltonian cycle. A digraph D of order $n \geq 3$ is pancyclic if it contains cycles of all lengths m, $3 \leq m \leq n$. For a cycle $C = x_1 x_2 \cdots x_k x_1$ of length k, the subscripts considered modulo k, i.e., $x_i = x_s$ for every s and i such that $i \equiv s \pmod{k}$. If P is a path containing a subpath from x to y, we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x, y]$ denotes the subpath of C from x to y.

A digraph D is *strongly connected*, if there exists a path from x to y and a path from y to x for every pair of distinct vertices x, y. A digraph D is k-strongly ($k \geq 1$) connected if $|V(D)| \geq k + 1$ and $D\langle V(D) \setminus A\rangle$ is strongly connected for any subset $A \subset V(D)$ of at most $k - 1$ vertices.

Let H be a non-trivial proper subdigraph of a digraph D. For the subdigraph H, a H-bypass is a path of length at least two with both end-vertices in H and no other vertices in H. If C is a non-Hamiltonian cycle in D and (x, y)-path P is a C-bypass with $V(P) \cap V(C) = \{x, y\}$, then we call the length of the path $C[x, y]$ the gap of P with respect to C.

A cycle factor in D is a collection of vertex disjoint cycles $C_1, \ldots, C_l$ such that $V(C_1) \cup \ldots \cup V(C_l) = V(D)$.

For a pair of disjoint subsets A and B of $V(D)$, we define $A(A \to B) = \{xy \in A(D) | x \in A, y \in B\}$ and $A(A, B) = A(A \to B) \cup A(B \to A)$.

3. Preliminaries

Lemma 3.1: (Häggkvist, Thomassen [9]). *Let D be a digraph of ordere $n \geq 3$ containing a cycle C of length m, $m \in [2, n-1]$. Let x be a vertex not contained in this cycle. If $d(x, V(C)) \geq m+1$, then D contains a cycle of length k for all $k \in [2, m+1]$.*

The following lemma is a modification of a lemma by Bondy and Thomassen [10], its proof is almost the same.

Lemma 3.2: *Let D be a digraph of order $n \geq 3$ containing a path $P := x_1 x_2 \ldots x_m$, $m \in [2, n-1]$. Let x be a vertex not contained in this path. If one of the following statements holds:*

(i) $d(x, V(P)) \geq m+2$;

(ii) $d(x, V(P)) \geq m+1$ *and* $xx_1 \notin A(D)$ *or* $x_m x \notin A(D)$;

(iii) $d(x, V(P)) \geq m$, $xx_1 \notin A(D)$ *and* $x_m x \notin A(D)$;

then there is an $i \in [1, m-1]$ such that $x_i x, xx_{i+1} \in A(D)$, i.e., D contains a path $x_1 x_2 \ldots x_i x x_{i+1} \ldots x_m$ of length m (we say that x can be inserted into P or the path $x_1 x_2 \ldots x_i x x_{i+1} \ldots x_m$ is an extended path obtained from P with x).

It is not difficult to prove the following lemma.

Lemma 3.3: *Let D be a digraph of order n. Assume that $xy \notin A(D)$ and the vertices x, y in D satisfy the degree condition $d^+(x) + d^-(y) \geq n-2+k$, where $k \geq 1$. Then D contains at least k internally disjoint (x, y)-paths of length two.*

Lemma 3.4: (Bypass Lemma, Bondy [11]). *Let D be a strongly connected non-separable (i.e., $UG(D)$ is 2-connected) digraph and let H be a non-trivial proper subdigraph of D. Then D contains a H-bypass.*

Theorem 3.4: (Yeo [12]). *Let D be a digraph. Then D has a cycle factor if and only if $V(D)$ cannot be partitioned into subsets Y, Z, R_1, R_2 such that $A(Y \to R_1) = A(R_2 \to R_1 \cup Y) = \emptyset$, $|Y| > |Z|$ and Y is an independent set.*

Theorem 3.5: (Meyniel [4]). *Let D be a strongly connected digraph of order $n \geq 2$. If $d(x) + d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is Hamiltonian.*

Before stating the main result of [14], we need to define a family of digraphs.

Definition 3.6: *For any integers n and m, $(n+1)/2 < m \leq n-1$, let Φ_n^m denote the set of digraphs D, which satisfy the following conditions: (i) $V(D) = \{x_1, x_2, \ldots, x_n\}$; (ii) $x_n x_{n-1} \ldots x_2 x_1 x_n$ is a Hamiltonian cycle in D; (iii) for each k, $1 \leq k \leq n-m+1$, the vertices x_k and x_{k+m-1} are not adjacent; (iv) $x_j x_i \notin A(D)$ whenever $2 \leq i+1 < j \leq n$ and (v) the sum of degrees for any two distinct non-adjacent vertices at least $2n-1$.*

Theorem 3.7: (Darbinyan [13], [14]). *Let D be a strongly connected digraph of order $n \geq 3$. Suppose that $d(x) + d(y) \geq 2n-1$ for all pairs of distinct non-adjacent vertices x, y in D. Then either (a) D is pancyclic or (b) n is even and D is isomorphic to one of $K^*_{n/2,n/2}$,*

$K^*_{n/2,n/2} \setminus \{e\}$, where e is an arbitrary arc of $K^*_{n/2,n/2}$, or (c) $D \in \Phi^m_n$ (in this case D does not contain a cycle of length m).

Later on, Theorem 3.7 was also proved by Benhocine [15].

4. Proofs of the Results

From the definition of condition (M) the following lemma follows.

Lemma 4.1: *Let D be a digraph of order n satisfying condition (M). Then D contains at most one pair of non-adjacent vertices x, y such that $d(x) + d(y) \leq 2n - 2$.*

Theorem 4.2: *Let D be a 2-strongly connected digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices of D such that $d(x) + d(y) \leq 2n - 2$. Then D is Hamiltonian if and only if D contains a cycle passing through the vertices x and y.*

Proof. If D is Hamiltonian, then obviously it contain a cycle passing through x and y. Suppose that D contains a cycle passing through the vertices x and y but D is not Hamiltonian. Let C be a longest cycle, say of length m, passing through x and y. Since D is not Hamiltonian, we have that $m \leq n - 1$. From 2-connectednees of D and Bypass-Lemma it follows that there is a C-bypass, say $P = uy_1y_2 \ldots y_kv$, where $u, v \in V(C)$ and $y_1, y_2, \ldots, y_k \in V(D) \setminus V(C)$. Without loss of generality, assume that the gap $|C[u, v]| - 1$ of P is the minimum among the gaps of all C-bypasses. Then

$$A(V(C[u, v]) \setminus \{u, v\}, V(P[y_1, y_k])) = \emptyset. \tag{1}$$

Put $f := |V(C[u, v]) \setminus \{u, v\}|$. Since C is a longest cycle passing through x and y, it follows that $f \geq 1$. Now we extend the path $C[v, u]$ with the vertices of $V(C[u, v]) \setminus \{u, v\}$ as mach as possible. We obtain a (v, u)-path, say R. Then, since C is a longest cycle passing through x and y, R does not contain some vertices $u_1, u_2, \ldots, u_d$ of $V(C[u, v]) \setminus \{u, v\}$. Using (1) and Lemma 3.2(i), for all y_j and u_i we obtain

$$d(y_j, V(C)) \leq m - f + 1 \quad \text{and} \quad d(u_i, V(C)) = d(u_i, V(R)) + d(u_i, \{u_1, \ldots, u_d\}) \leq m + d - 1.$$

By the minimality of the gap $f + 1$ we also have that D contains no path of the form y_jzu_i and u_izy_j, where $z \in B := V(D) \setminus (V(C) \cup \{y_1, \ldots, y_k\})$. Therefore,

$$d(u_i, B) + d(y_j, B) \leq 2|B|.$$

Now by a simple calculation we obtain

$$d(u_i) + d(y_j) = d(u_i, V(C)) + d(y_j, V(C)) + d(u_i, B) + d(y_j, B) + d(u_i, P[y_1, y_k]) + d(y_j, P[y_1, y_k])$$

$$\leq m + d - 1 + m - f + 1 + 2|B| + 2k - 2 \leq 2n - 2,$$

a contradiction with Lemma 4.1 since u_i and y_j are not adjacent and $\{u_i, y_j\} \neq \{x, y\}$. Theorem 4.2 is proved. □

Clearly, the existence of a cycle factor is a necessary condition for a digraph to be Hamiltonian.

Theorem 4.3: *Let D be a 2-strongly connected digraph of order n satisfying condition (M). Then D has a cycle factor.*

Proof. Suppose, on the contrary, that D has no cycle factor. By the Yeo theorem, $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $A(Y \to R_1) = A(R_2 \to R_1 \cup Y) = \emptyset$, $|Y| > |Z|$ and Y is an independent set. Using these and 2-connectedness of D, we obtain that it follows that $|Z| \geq 2$ and hence, $|Y| \geq 3$. Let x, y, z be three distinct vertices of Y. Since Y is an independent set, we have that $\{x, y\}$ and $\{x, z\}$ are two distinct pairs of non-adjacent vertices of $V(D)$. Now using condition (M), we obtain

$$4n-3 \leq 2d(x)+d(y)+d(z) \leq 8|Z|+4|R_1|+4|R_2|+4|Y|-4|Y| = 4n-4(|Y|-|Z|) \leq 4n-4,$$

a contradiction. Theorem 4.3 is proved. □

Using Yeo's theorem, it is not difficult to show that in Theorem 4.3 the minimum degree condition is sharp.

Theorem 4.4: *Let D be a 2-strongly connected digraph of order $n \geq 3$. Suppose that D contains at most one pair of non-adjacent vertices. Then D is Hamiltonian.*

Proof. Suppose, on the contrary, that D is not Hamiltonian. Therefore, D is not semicomplete and contains exactly one pair, say $\{x, y\}$, of non-adjacent vertices. Then $d(x) \geq n-2$, $d(y) \geq n-2$ and $d(z) \geq n-1$ for all $z \in V(D) - \{x, y\}$. Since D is 2-strongly connected, it follows that both subdigraphs $D - x$ and $D - y$ both are strongly connected semicomplete digraphs. Therefore, $D - x$ and $D - y$ both are Hamiltonian. Let $C_{n-1} := x_1 x_2 \dots x_{n-2} y x_1$ be a Hamiltonian cycle in $D - x$. Since D is not Hamiltonian, from $d(x) \geq n-2$ and the fact that x is adjacent to every vertex of $V(D) - \{y\}$ it follows that there exists an integer $l \in [2, n-3]$ such that

$$\{x_{l+1}, x_{l+2}, \dots, x_{n-2}\} \to x \to \{x_1, x_2, \dots, x_l\}. \tag{2}$$

Similarly, for some $k \in [2, n-3]$ we have

$$\{x_{k+1}, x_{k+2}, \dots, x_{n-2}\} \to y \to \{x_1, x_2, \dots, x_k\}. \tag{3}$$

Observe that for every pair of integers i, j, $1 \leq i < j \leq n-2$, in D there is no path of the types $x_i \to y \to x_j$ and $x_i \to z \to x_j$. By symmetry between x and y, we may assume that $k \geq l$. Then by (2), (3) and our observation we have

$$A(\{x, y\} \to \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset \quad \text{and} \quad \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\} \to \{x, y\}. \tag{4}$$

From (4) and 2-connectedness of D it follows that there exist $i \in [1, k-1]$ and $j \in [k+1, n-2]$ such that $x_i \to x_j$ (for otherwise, $A(\{x_1, x_2, \dots, x_{k-1}, x, y\} \to \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset$, which means that $D - x_k$ is not strongly connected). Note that $y \to x_{i+1}$. We choose j maximal with these properties. Using (2) and (3), it is easy to see that if $x_{j-1} \to x$, then $x_1 \dots x_i x_j \dots x_{n-2} y x_{i+1} \dots x_{j-1} x x_1$ is a Hamiltonian cycle, which contradicts our supposition. We may therefore assume that $x_{j-1} x \notin A(D)$. Then from $k \geq l$, (4) and the maximality of j it follows that $j = k+1$ and $x_kx \notin A(D)$. Hence, $l = k$ and $A(\{x_1, x_2, \ldots, x_{k-1}\} \to \{x_{k+2}, x_{k+3}, \ldots, x_{n-2}\}) = \emptyset$. This together with (4) and 2-connectedness of D implies that there is an integer $s \in [k+2, n-2]$ such that $x_k \to x_s$. Therefore, $x_1 \ldots x_i x_{k+1} \ldots x_{s-1} x x_{i+1} \ldots x_k x_s \ldots x_{n-2} y x_1$ is a Hamiltonian cycle, a contradiction. Theorem 4.4 is proved. □

Remark: *There is a strongly connected non-Hamiltonian digraph of order $n \geq 5$, which is not 2-strongly connected and has exactly one pair of non-adjacent vertices.*

To see this, consider the following digraph D defined as follows:
$V(D) = \{x_1, x_2, \ldots, x_{n-2}, y, z\}$, $x_1x_2, \ldots x_{n-2}yx_1$ is a cycle of length $n-1$ in D,

$$N^-(y) = N^-(z) = \{x_k, x_{k+1}, \ldots, x_{n-2}\} \quad \text{and} \quad N^+(y) = N^+(z) = \{x_1, x_2, \ldots, x_k\},$$

where $k \in [2, n-3]$, D also contains all the arcs x_jx_i whenever $1 \leq i < j \leq n-2$ and it contain no other arcs.

It is not difficult to check that D is neither 2-strongly connected nor Hamiltonian. □

Lemma 4.5: *Let D be a 2-strongly connected digraph of order $n \geq 3$ and let u, v be two distinct vertices in D. If D contains no cycle passing through u and v, then u, v are not adjacent and there is no path of length two between them. In particular,*

$$d^+(u) + d^-(v) \leq n-2, \quad d^-(u) + d^+(v) \leq n-2 \quad \text{and} \quad d(u) + d(v) \leq 2n-4.$$

Proof. It is obvious that u, v are not adjacent. Suppose, on the contrary, that in D there is a path of length two between the vertices u and v, say $u \to z \to v$ or $v \to z \to u$. Since D is 2-strongly connected, it follows that $D-z$ is strongly connected. Therefore, in $D-z$ there is an (u,v)- and a (v,u)-path. It is easy to see that this (u,v)-path ((v,u)-path, respectively) together with $v \to z \to u$ ($u \to z \to v$, respectively) forms a cycle passing through u and v. In both cases we have a contradiction, which proves that there is no path of length two between u and v. Therefore, by Lemma 3.3, $d^+(u)+d^-(v) \leq n-2$ and $d^-(u)+d^+(v) \leq n-2$. These imply that $d(u) + d(v) \leq 2n-4$. Lemma 4.5 is proved. □

Theorem 4.6: *Let D be a 2-strongly connected digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{u, v\}$ is a pair of non-adjacent vertices in D such that $d(u) + d(v) \leq 2n-2$. Then D is Hamiltonian or D contains a cycle of length $n-1$ passing through u and avoiding v (passing through v and avoiding u).*

Proof. Suppose that D is not Hamiltonian. From Theorem 4.2 it follows that D contains no cycle passing through u and v. Therefore, by Lemma 4.5, $d(u) + d(v) \leq 2n-4$. Since D is 2-strongly connected, it follows that $D-u$ and $D-v$ both are strongly connected. From the last inequality and condition (M) it follows that if $\{x, y\}$ is a pair of non-adjacent vertices in $D-u$ (in $D-v$, respectively), then the following inequalities holds:

$$d(x, V(D) \setminus \{u\}) + d(y, V(D) \setminus \{u\}) \geq 2(n-1) - 1,$$

$$(d(x, V(D) \setminus \{v\}) + d(y, V(D) \setminus \{v\}) \geq 2(n-1) - 1, \quad \textit{respectively}).$$

Therefore, since $D-u$ and $D-v$ both are strongly connected, by Meyniel's theorem $D-u$ and $D-v$ both are Hamiltonian, i.e., D contains a cycle of length $n-1$ passing through u

and avoiding v (passing through v and avoiding u). Theorem 4.6 is proved. □

As an immediate corollary of Theorems 4.2 and 4.6 (respectively, Theorem 4.6 and Corollary 3.1), we obtain Corollary 4.7 (respectively, Corollary 4.8).

Corollary 4.7: *Let D be a 2-strongly connected non-Hamiltonian digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{u, v\}$ is a pair of non-adjacent vertices in D such that $d(u)+d(v) \leq 2n-2$. Then D contains at most one cycle of length two passing through u (v).*

Corollary 4.8: *Let D be a 2-strongly connected non-Hamiltonian digraph of order $n \geq 3$ satisfying condition (M). Suppose that $\{u, v\}$ is a pair of non-adjacent vertices in D such that $d(u) + d(v) \leq 2n - 2$. Then $d(u) \leq n - 1$ and $d(v) \leq n - 1$.*

Theorem 4.9: *Let D be a 2-strongly connected digraph of order $n \geq 6$ satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices in D such that $d(x) + d(y) \leq 2n - 4$. Then D contains cycles of all lengths $3, 4, \dots, n - 1$.*

Proof. Suppose first that D contains exactly one pair of non-adjacent vertices, namely $\{x, y\}$. Then $D - x$ is a strongly connected semicomplete digraph. Therefore, by the well-known theorem of Moser [16], $D - x$ contains cycles of all lengths $3, 4, \dots, n - 1$.

Suppose next that D contains at least two distinct pairs of non-adjacent vertices. Let $\{u, v\}$ be an arbitrary pair of non-adjacent vertices in $V(D) \setminus \{x\}$ (or in $V(D) \setminus \{y\}$). From condition (M) it follows that

$$d(u) + d(v) \geq 2n + 1. \tag{5}$$

Now we consider the subdigraph $H := D - x$. For the digraph H we first prove the following claim.

Claim: If $H \cong K^*_{m,m} - e$, where e is an arbitrary arc of $K^*_{m,m}$, then D contains cycles of all lengths $2, 3, \dots, n - 1$.

Proof. Let $\{u, v\}$ be an arbitrary pair of non-adjacent vertices in $K^*_{m,m} - e$. Note that $n = 2m + 1 \geq 7$. Then

$$d(u, V(D) \setminus \{x\}) + d(v, V(D) \setminus \{x\}) \leq 4m = 2n - 2.$$

Therefore, by (5), $d(x, \{u, v\}) \geq 3$. This, since $m \geq 3$, in turn, implies that every partite set of H contains at least two vertices such that each of them together with x forms a 2-cycle. Therefore, there exist two vertices $z, w \in V(H)$ such that $z \leftrightarrow w$, $z \leftrightarrow x$ and $w \leftrightarrow x$ Then, since for every k, $k \in [1, m]$ there is a cycle of length $2k$ passing through the arc $z \to w$, it follows that D contains cycles of all lengths $2, 3, \dots, n$. The claim is proved. □

We now return to the proof of Theorem 4.9. From (5) it also follows that

$$d(u, V(D) \setminus \{x\}) + d(v, V(D) \setminus \{x\}) \geq 2(n - 1) - 1.$$

Then, since H is strongly connected, from Theorem 3.7 it follows that either H contains cycles of all lengths $3, 4, \dots, n - 1$ or $H \in \{K^*_{m,m}, K^*_{m,m} - e\} \cup \Phi^k_{n-1}$, where $n/2 < k \leq n - 2$. In order to complete the proof of the theorem, by the above claim it suffices to consider only the case when $H \in \Phi^k_{n-1}$. From the definition of the set Φ^k_{n-1} it follows that H contains cycles of all lengths $2, 3, \ldots, n-1$ except the cycle of length k.

Let $x_1 x_{n-1} x_{n-2} \ldots x_2 x_1$ be a Hamiltonian cycle in H. Since $H \in \Phi^k_{n-1}$, it follows that $\{x_1, x_k\}$, $\{x_{n-k}, x_{n-1}\}$ are two distinct pairs of non-adjacent vertices other than $\{x, y\}$ and

$$d(x_1, V(H)) + d(x_k, V(H)) = d(x_{n-k}, V(H)) + d(x_{n-1}, V(H)) = 2n - 3.$$

This together with (5) implies that $d(x, \{x_1, x_{n-k}, x_k, x_{n-1}) = 8$. If $k \neq n-3$, then $x_1 \to x_{n-3}$ and $x_1 x_{n-3} x_{n-4} \ldots x_{n-k} x x_1$ is a cycle of length k. Assume that $k = n-3$. Then $\{x_1, x_{n-3}\}$, $\{x_3, x_{n-1}\}$ are two pairs of non-adjacent vertices other than $\{x, y\}$. We have that $d(x, \{x_1, x_3, x_{n-3}, x_{n-1}) = 8$, $x_1 \to x_{n-4}$ and $x_3 \to x_{n-1}$. If $x_2 \to x$, then $x_1 x_{n-4} ... x_3 x_2 x x_1$ is a cycle of length $n-3$. Assume that $x_2 x \notin A(D)$. Then, since the vertices x_2 and x_{n-2} are not adjacent and $d(x_2) + d(x_{n-2}) \geq 2n+1$, it is not difficult to see that $x_2 \to x_{n-3}$ and $x \to x_2$. Therefore, $x x_2 x_{n-3} x_{n-4} x_3 x$ is a cycle of length $n-3$. This completes the proof of the theorem. □

In view of Theorem 4.9, it is natural to set the following problem.

Problem: *Let D be a 2-strongly connected digraph of order n satisfying condition (M). Suppose that $\{x, y\}$ is a pair of non-adjacent vertices in D such that $2n-3 \leq d(x)+d(y) \leq 2n-2$. Whether D contains cycles of all lengths $3, 4, \ldots, n-1$?*

5. Remarks

In the following, we suppose, further, that D is a 2-strongly connected digraph of order n satisfying condition (M). Moreover, D contains a pair $\{y, z\}$ of non-adjacent distinct vertices y, z such that $d(y)+d(z) \leq 2n-4$. In this section, we will prove a number of properties of D.

Lemma 5.1: *Let $x_1 x_2 \ldots x_{n-2} z x_1$ be a cycle of length $n-1$ in D, which does not contain y. Suppose that $x_a \to x_b$, $x_q \to y \to x_p$ and $x_t \to y \to x_s$, where $1 \leq s \leq a < p \leq q < b \leq t \leq n-2$. Then D is Hamiltonian.*

Proof. Suppose, on the contrary, that D is not Hamiltonian. By Theorem 4.2, D contains no cycle passing through y and z. Notice that there are no integers l and r, $1 \leq l < r \leq n-2$, such that $x_l \to y \to x_r$ (for otherwise, $x_1 \ldots x_l y x_r \ldots x_{n-2} z x_1$ is a cycle passing through y and z. If $z \to x_i$ with $i \in [a+1, q]$, then $C(y, z) = x_s \ldots x_a x_b \ldots x_{n-2} z x_i \ldots x_q y x_s$;

if $x_j \to z$ with $j \in [p, b-1]$, then $C(y, z) = x_1 \ldots x_a x_b \ldots x_t y x_p \ldots x_j z x_1$. Thus, in both cases we have a contradiction. Therefore,

$$d^+(z, \{x_{a+1}, \ldots, x_q\}) = d^-(z, \{x_p, \ldots, x_{b-1}\}) = 0,$$

in particular, $d(z, \{x_p, \ldots, x_q\}) = 0$ and the vertices z and x_p are not adjacent. The last equality together with the fact that D contains at most one cycle of length two passing through z (Corollary 4.7) implies that

$$d(z) = d(z, \{x_1, \ldots, x_{p-1}\}) + d(z, \{x_{q+1}, \ldots, x_{n-2}\}) \leq p-1+n-2-q+1 = n+p-q-2. \quad (6)$$

Now we consider the vertex x_p. It is easy to see that if $x_i \rightarrow x_p$ with $i \in [1, s-1]$, then $C(y,z) = x_1 \dots x_i x_p \dots x_q y x_s \dots\ x_a x_b \dots x_{n-2} z x_1$, if $x_p \rightarrow x_j$ with $j \in [t+1, n-2]$, then $C(y,z) = x_1 \dots x_a x_b \dots x_t y x_p\ x_j \dots x_{n-2} z x_1$. In both cases we have a contradiction. Therefore, we may assume that

$$d^-(x_p, \{x_1, \dots, x_{s-1}\}) = d^+(x_p, \{x_{t+1}, \dots, x_{n-2}\}) = 0.$$

This implies that

$$d(x_p) = d^+(x_p, \{x_1, \dots, x_{s-1}\}) + d^-(x_p, \{x_{t+1}, \dots, x_{n-2}\}) + d(x_p, \{x_s, \dots, x_t\}) + d(x_p, \{y\})$$
$$\leq s - 1 + n - 2 - t + 2(t - s + 1) = n + t - s - 1. \tag{7}$$

Without loss of generality, we may assume that s, q are chosen as maximal as possible and p, t are chosen as minimal as possible. Then

$$d(y, \{x_{s+1}, \dots, x_{p-1}\}) = d(y, \{x_{q+1}, \dots, x_{t-1}\}) = 0.$$

This, since D contains at most one cycle of length two passing through y, implies that

$$d(y) = d(y, \{x_1, \dots, x_s\}) + d(y, \{x_p, \dots, x_q\}) + d(y, \{x_t, \dots, x_{n-2}\})$$
$$\leq s + q - p + 1 + n - 2 - t + 1 + 1 = n + s + q - p - t + 1. \tag{8}$$

Since $\{y,z\}$ and $\{x_p, z\}$ are two distinct pairs of non-adjacent vertices, from (6), (7), (8) and condition (M) it follows that

$$4n - 3 \leq d(y) + 2d(z) + d(x_p) \leq 4n - 4 - (q - p) \leq 4n - 4,$$

which is a contradiction. Lemma 5.1 is proved. □

The following claim is an immediate consequence of Lemma 5.1.

Claim 1: Let $x_1 x_2 \dots x_{n-2} z x_1$ be a cycle of length $n-1$ in D passing through z. If $x_{n-2} \rightarrow y \rightarrow x_1$ and $x_1 \rightarrow x_{n-2}$, then D is Hamiltonian.

The following claim will be very useful in the remaining proof.

Claim 2: Let $x_1 x_2 \dots x_{n-2}$ be a Hamiltonian path in $D - \{y, z\}$. Suppose that for every pair of integers i and j, $1 \leq i < j \leq n-2$, if $x_i \rightarrow y$, then $y x_j \notin A(D)$, and if $x_i \rightarrow z$, then $z x_j \notin A(D)$. Then either D is Hamiltonian or for every $k \in [2, n-3]$, the following holds:

$$A(\{x_1, x_2, \dots, x_{k-1}\} \rightarrow \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) \neq \emptyset.$$

Proof. Suppose, on the contrary, that D is not Hamiltonian and there is an integer $k \in [2, n-3]$ such that

$$A(\{x_1, x_2, \dots, x_{k-1}\} \rightarrow \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset, \tag{9}$$

We can assume that the vertices x_m and x_l are chosen so that $y \rightarrow x_m$, $z \rightarrow x_l$ and

$$d^+(y, \{x_{m+1}, \dots, x_{n-2}\}) = d^+(z, \{x_{l+1}, \dots, x_{n-2}\}) = 0.$$

Without loss of generality, we assume that $m \leq l$. Since D is 2-strongly connected, it follows that $2 \leq m \leq l \leq n-3$. From the supposition of this claim and (9) it follows that:

(i) if $k \leq m$ or $k \geq l$, then (respectively)

$$A(\{x_1, x_2, \dots, x_{k-1}\} \to \{y, z, x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset$$

or

$$A(\{y, z, x_1, x_2, \dots, x_{k-1}\} \to \{x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset,$$

(ii) if $m+1 \leq k \leq l-1$, then

$$A(\{y, x_1, x_2, \dots, x_{k-1}\} \to \{z, x_{k+1}, x_{k+2}, \dots, x_{n-2}\}) = \emptyset.$$

Thus, in each case we have that $D - x_k$ is not strongly connected, which contradicts that D is 2-strongly connected. Claim 2 is proved. □

Lemma 5.2: *Suppose that $x_1 x_2 \dots x_{n-2} z x_1$ is a cycle in D passing through z and avoiding y. If $x_{n-2} \to y \to x_1$, then D is Hamiltonian.*

Proof. Suppose, on the contrary, that $x_{n-2} \to y \to x_1$ but D is not Hamiltonian. By Theorem 4.2, D contains no cycle passing through y and z in D. It is easy to see that the conditions of Claim 2 hold. Let $x_k \to y \to x_p$, where $2 \leq p \leq k \leq n-3$, k minimal and p maximal with this property. Since D is 2-strongly connected, from Lemma 5.1 it follows that $p \leq k-1$. This means that there is no cycle of length two passing through y. By symmetry between the vertices y and z, we may assume that also there is no cycle of length two passing through z.

Case 1. $p = k-1$. By Lemma 5.1 we have that

$$A(\{x_1, \dots, x_{p-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset.$$

Then, by Claim 2, there are some integers $i \in [1, p-1]$ and $j \in [k+1, n-2]$ such that $x_i \to x_k$ and $x_{k-1} \to x_j$. Therefore, $C(y, z) = x_1 \dots x_i x_k y x_{k-1} x_j \dots x_{n-2} z x_1$ is a cycle passing through y and z, which is a contradiction.

Case 2. $p \leq k-2$. Then $d(y, \{x_{p+1}, \dots, x_{k-1}\}) = 0$. By Lemma 5.1 we have that

$$A(\{x_1, \dots, x_{p-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset. \tag{10}$$

Therefore, by Claim 2, there are $s \in [1, p-1]$, $a \in [p+1, k]$, $b \in [p, k-1]$ and $t \in [k+1, n-2]$ such that $x_s \to x_a$ and $x_b \to x_t$. If $a > b$, then $x_1 \dots x_s x_a \dots x_k y x_p \dots x_b x_t \dots x_{n-2} z x_1$ is a cycle passing through y and z, which is a contradiction. We may therefore assume that $a \leq b$. Then $p+1 \leq a \leq b \leq k-1$ and the vertices y and x_a are not adjacent. Choose (i) a and t as maximal as possible and (ii) choose b and s as minimal as possible, subject to (i). This means that

$$A(\{x_1, \dots, x_{p-1}\} \to \{x_{a+1}, \dots, x_{n-2}\}) = A(\{x_1, \dots, x_{b-1}\} \to \{x_{k+1}, \dots, x_{n-2}\}) = \emptyset. \tag{11}$$

From the minimality of s and the maximality of t we have that

$$\begin{aligned} d(x_a) &= d^+(x_a, \{x_1, \dots, x_{s-1}\}) + d^-(x_a, \{x_{t+1}, \dots, x_{n-2}\}) + d(x_a, \{x_s, \dots, x_t\}) + d(x_a, \{z\}) \\ &\leq s - 1 + n - 2 - t + 2t - 2s + 1 = n + t - s - 2. \end{aligned} \tag{12}$$

Let m be the number of vertices of the set $\{x_{s+1},\dots,x_p,x_k,\dots,x_{t-1}\}$, which are not adjacent to y. Then, since y is not on the cycle of length two and $d(y,\{x_{p+1},\dots,x_{k-1}\})=0$, it follows that

$$d(y)\le n-2-m-(k-1-p)=n+p-m-k-1. \tag{13}$$

Assume first that

$$d^+(z,\{x_{s+1},\dots,x_p\})=d^-(z,\{x_k,\dots,x_{t-1}\})=0.$$

From this and taking into account Lemma 4.5 (there is no path of length two between y and z) we obtain that

$$d(z)\le s+n-2-t+1+m+k-1-p=n+k+m+s-t-p-2. \tag{14}$$

Combining (12)-(14), $k-p\ge 2$ and $m\ge 0$, we obtain

$$2d(y)+d(x_a)+d(z)\le 4n-6-(k-p)-m\le 4n-8,$$

which is a contradiction to condition (M), since $\{y,z\}$ and $\{y,x_a\}$ are two distinct pairs of non-adjacent vertices.

Assume next that

$$d^+(z,\{x_{s+1},\dots,x_p\})\ne 0 \quad \text{or} \quad d^-(z,\{x_k,\dots,x_{t-1}\})\ne 0,$$

i.e., there is a $q\in[s+1,p]$ such that $z\to x_q$ or there is a $r\in[k,t-1]$ such that $x_r\to z$.

Using Claim 2, we obtain that $A(\{x_1,\dots,x_{a-1}\}\to\{x_{a+1},\dots,x_{n-2}\})\ne\emptyset$. Let $x_{s_1}\to x_{t_1}$, where $s_1\in[1,a-1]$ and $t_1\in[a+1,n-2]$. From (11) it follows that $s_1\in[p,a-1]$ and $t_1\in[a+1,k]$. Choose t_1 maximal with this property. Then

$$A(\{x_1,\dots,x_{a-1}\}\to\{x_{t_1+1},\dots,x_{n-2}\})=\emptyset. \tag{15}$$

Now using the facts that $z\to x_q$ or $x_r\to z$, it is not difficult to check that: if $t_1>b$, then $C(y,z)=x_1\dots x_sx_a\dots x_bx_t\dots x_{n-2}zx_q\dots x_{s_1}x_{t_1}\dots x_kyx_1$ or $C(y,z)=x_1\dots x_sx_a\dots x_bx_t\dots x_{n-2}yx_p\dots x_{s_1}x_{t_1}\dots x_rzx_1$ is a cycle passing through y and z, when $z\to x_q$ and $x_r\to z$, respectively. In both cases we have a contradiction. We may therefore assume that $t_1\le b$.

From Claim 2 we have that $A(\{x_1,\dots,x_{t_1-1}\}\to\{x_{t_1+1},\dots,x_{n-2}\})\ne\emptyset$. Let $x_{s_2}\to x_{t_2}$, where $s_2\in[1,t_1-1]$ and $t_2\in[t_1+1,n-2]$. From (11) and (15) it follows that $s_2\in[a,t_1-1]$ and $t_2\in[t_1+1,k]$. Choose t_2 maximal with this property. Then

$$A(\{x_1,\dots,x_{t_1-1}\}\to\{x_{t_2+1},\dots,x_{n-2}\})=\emptyset. \tag{16}$$

If $t_2>b$, then $C(y,z)=x_1\dots x_sx_a\dots x_{s_2}x_{t_2}\dots x_kyx_p\dots x_{s_1}x_{t_1}\dots x_bx_t\dots x_{n-2}zx_1$, a contradiction. We may therefore assume that $t_2\le b$. In particular, from $t_2\ge t_1+1$ it follows that $t_1<b$.

Using Claim 2, (11) and (16), we obtain that there are some integers $s_3\in[t_1,t_2-1]$ and $t_3\in[t_2+1,k]$ such that $x_{s_3}\to x_{t_3}$. Choose t_3 maximal with this properties. Then

$$A(\{x_1,\dots,x_{t_2-1}\}\to\{x_{t_3+1},\dots,x_{n-2}\})=\emptyset. \tag{17}$$

If $t_3 > b$, then $C(y,z) = x_1 \dots x_s x_a \dots x_{s_2} x_{t_2} \dots x_b x_t \dots x_{n-2} z x_q \dots x_{s_1} x_{t_1} \dots x_{s_3} x_{t_3} \dots x_k y x_1$ or $C(y,z) = x_1 \dots x_s x_a \dots x_{s_2} x_{t_2} \dots x_b x_t \dots x_{n-2} y x_p \dots x_{s_1} x_{t_1} \dots x_{s_3} x_{t_3} \dots x_r z x_1$, when $z \to x_q$ or $x_r \to z$, respectively. We may therefore assume that $t_3 \leq b$. Then $t_2 < b$.

Again using Claim 2, (11) and (17), we obtain that there are some integers $s_4 \in [t_2, t_3 - 1]$ and $t_4 \in [t_3 + 1, k]$ such that $x_{s_4} \to x_{t_4}$. If $t_4 > b$, then $C(x,y) = x_1 \dots x_{s_0} x_{t_0} \dots x_{s_2} x_{t_2} \dots x_{s_4} x_{t_4} \dots x_k y x_p \dots x_{s_1} x_{t_1} \dots x_{s_3} x_{t_3} \dots x_b x_t \dots x_{n-2} z x_1$, a contradiction. (Here, $x_s := x_{s_0}$ and $x_a = x_{t_0}$).

Continuing this process, we finally conclude that for some $l \geq 0$, $t_l > b$ since all the vertices $x_{t_0}, x_{t_1}, \dots, x_{t_l}$ are distinct and in $\{x_p, \dots, x_k\}$. By the above arguments we have that: if t_l is even, then $C(y,z) = x_1 \dots x_{s_0} x_{t_0} \dots x_{s_2} x_{t_2} \dots x_{s_4} x_{t_4} \dots x_{s_l} x_{t_l} \dots x_k y x_p \dots x_{s_1} x_{t_1} \dots x_{s_3}\, x_{t_3} \dots x_{s_{l-1}} x_{t_{l-1}} \dots x_b x_t \dots x_{n-2} z x_1$, if l is odd, then $C(y,z) = x_1 \dots x_{s_0} x_{t_0} \dots x_{s_2} x_{t_2} \dots x_{s_{l-1}} x_{t_{l-1}} \dots x_b x_t \dots x_{n-2} z x_q \dots x_{s_1} x_{t_1} \dots x_{s_3} x_{t_3} \dots x_{s_l} x_{t_l} \dots x_k y x_1$, or $C(y,z) = x_1 \dots x_{s_0} x_{t_0} \dots x_{s_2} x_{t_2} \dots x_{s_{l-1}} x_{t_{l-1}} \dots x_b x_t \dots x_{n-2} y x_p \dots x_{s_1} x_{t_1} \dots x_{s_3} x_{t_3} \dots x_{s_l} x_{t_l} \dots x_r z x_1$, when $z \to x_q$ or $x_r \to z$, respectively. In all cases we have a cycle passing through y and z, which contradicts our supposition. Lemma 5.2 is proved. □

From Theorem 4.4, Lemma 5.2 and Corollary 4.7 the following corollary follows.

Corollary 5.3: *If D is not Hamiltonian, then $max\{d(y), d(z)\} \leq n - 2$.*

Lemma 5.4: *Let $C = x_1 x_2 \dots x_{n-3} x_1$ be a cycle of length $n-3$ in D passing through y and avoiding z. Let $V(D) \setminus V(C) = \{z, u, v\}$. If*
(i). $d(y, \{u, v\}) = 0$ and $zuvz$ is a cycle of length 3 or
(ii). $d(y) \leq n - 3$ and $z \leftrightarrow u$, then D is Hamiltonian.

Proof. Suppose, on the contrary, that D is not Hamiltonian. Then, by Theorem 4.2, D contains no cycle passing through y and z.

(i). Since $d(y, \{z, u, v\}) = 0$ and through y there is at most one cycle of length two, it follows that $d(y) \leq n - 3$. It is easy to see that for every $i \in [1, n-3]$ the following holds:

$$\overrightarrow{a}[x_i, z] + \overrightarrow{a}[v, x_{i+1}] \leq 1, \quad \overrightarrow{a}[x_i, u] + \overrightarrow{a}[z, x_{i+1}] \leq 1 \quad \text{and} \quad \overrightarrow{a}[x_i, v] + \overrightarrow{a}[u, x_{i+1}] \leq 1,$$

(for otherwise, D is Hamiltonian). Therefore,

$$d(z, V(C)) + d(u, V(C)) + d(v, V(C))$$

$$= \sum_{i=1}^{n-3} (\overrightarrow{a}[x_i, z] + \overrightarrow{a}[v, x_{i+1}] + \overrightarrow{a}[x_i, u] + \overrightarrow{a}[z, x_{i+1}] + \overrightarrow{a}[x_i, v] + \overrightarrow{a}[u, x_{i+1}]) \leq 3n - 9.$$

Then, since $d(z, \{u, v\}) \leq 3$ and $d(u, \{z, v\}) + d(v, \{u, z\}) \leq 7$ it follows that $d(z) + d(u) + d(v) \leq 3n + 1$. Therefore, since $2d(y) + d(z) + d(u) \geq 4n - 3$ and $d(v) + d(y) \geq 2n + 1$, we have

$$6n - 2 \leq 3d(y) + d(u) + d(v) + d(z) \leq 6n - 8,$$

which is a contradiction.

(ii). Then for every $i \in [1, n-3]$ we have $\overrightarrow{a}[x_i, z] + \overrightarrow{a}[u, x_{i+1}] \leq 1$ and $\overrightarrow{a}[x_i, u] + \overrightarrow{a}[z, x_{i+1}] \leq 1$. Therefore,

$$d(z, V(C)) + d(u, V(C)) = \sum_{i=1}^{n-3} (\overrightarrow{a}[x_i, z] + \overrightarrow{a}[u, x_{i+1}] + \overrightarrow{a}[x_i, u] + \overrightarrow{a}[z, x_{i+1}]) \leq 2n - 6.$$

Hence, $d(z) + d(u) \le 2n + 1$. This together with $d(y) \le n - 3$ and condition (M) gives

$$4n - 3 \le d(z) + d(u) + 2d(y) \le 2n + 1 + 2n - 6 = 2n - 5,$$

which is a contradiction. Lemma 5.4 is proved. □

Lemma 5.5: *Let $C := x_1x_2 \ldots x_{n-4}x_1$ be a cycle of length $n-4$ in D passing through y and avoiding z. Let $V(D) \setminus V(C)) = \{z, u_1, u_2, u_3\}$. If one of the following conditions holds*
(i). $d(y) \le n - 3$ and $z \leftrightarrow u_1$,
ii). zu_1u_2z is a cycle of length 3 and $d(y, \{u_1, u_2\}) = 0$,
(iii). $zu_1u_2u_3z$ is a cycle of length 4 and $d(y, \{u_1, u_2, u_3\}) = 0$, then D is Hamiltonian.

Proof. Suppose, on the contrary, that D is not Hamiltonian. By Theorem 4.2, D contains no cycle passing through y and z.

(i). Note that y and u_1 are not adjacent by Lemma 4.5. Since $z \leftrightarrow u_1$, it is easy to see that $\overrightarrow{a}[x_i, z] + \overrightarrow{a}[u_1, x_{i+1}] \le 1$ and $\overrightarrow{a}[x_i, u_1] + \overrightarrow{a}[z, x_{i+1}] \le 1$. Hence, $d(z, V(C)) + d(u_1, V(C)) \le 2n - 8$. Therefore, since, $d(u_1, \{z, u_2, u_3\}) \le 6$ and $d(z, \{u_1, u_2, u_3\}) \le 4$, we have $d(z) + d(u_1) \le 2n + 2$. This together with $d(y) \le n - 3$ and condition (M) implies that

$$4n - 3 \le d(z) + 2d(y) + d(u_1) \le 4n - 4,$$

which is a contradiction.

(ii). Then it is easy to see that

$$\overrightarrow{a}[x_i, z] + \overrightarrow{a}[u_2, x_{i+1}] \le 1, \quad \overrightarrow{a}[x_i, u_1] + \overrightarrow{a}[z, x_{i+1}] \le 1 \quad \text{and} \quad \overrightarrow{a}[x_i, u_2] + \overrightarrow{a}[u_1, x_{i+1}] \le 1.$$

Hence,

$$d(z, V(C)) + d(u_1, V(C)) + d(u_2, V(C)) \le 3n - 12.$$

Therefore, since $d(z, \{u_1, u_2, u_3\}) \le 4$ and $d(u_1, \{z, u_2, u_3\}) + d(u_2, \{z, u_1, u_3\}) \le 11$, we obtain that $d(z) + d(u_1) + d(u_2) \le 3n + 3$. This together with $d(y, \{z, u_1, u_2\}) = 0$ implies that $d(y) \le n - 3$ and $d(z) + d(u_1) + d(u_2) + 3d(y) \le 6n - 6$. On the other hand, since $d(z) + d(u_1) + 2d(y) \ge 4n - 3$ and $d(y) + d(u_2) \ge 2n + 1$, we have

$$6n - 2 \le d(z) + d(u_1) + 3d(y) + d(u_2) \le 6n - 6,$$

which is a contradiction.

(iii). First, notice that $d(y) \le n - 4$. By an argument similar to that in the proof of (ii), we can show that

$$d(z, V(C_{n-4})) + d(u_1, V(C_{n-4})) + d(u_2, V(C_{n-4})) + d(u_3, V(C_{n-4})) \le 4n - 16.$$

Then, since

$$d(z, \{u_1, u_2, u_3\}) + d(u_1, \{z, u_2, u_3\}) + d(u_2, \{u_1, u_2, z\}) + d(u_3, \{u_1, u_2, z\}) \le 20,$$

we have $d(z) + d(u_1) + d(u_2) + d(u_3) \le 4n + 4$. Besides, from condition (M) and $d(y) \le n - 4$ it follows that

$$8n - 6 \le d(z) + 4d(y) + d(u_1) + d(u_2) + d(u_3) \le 8n - 12,$$

which is a contradiction. Lemma 5.5 is proved. □

Lemma 5.6: *Let $C := x_1 x_2 \dots x_{n-5} x_1$ be a cycle of length $n-5$ in D passing through y and avoiding z. Let $A = V(D) \setminus V(C)) = \{z, u_1, u_2, u_3, u_4\}$. If $d(y, \{u_1, u_2\}) = 0$ and zu_1u_2z is a cycle of length three, then D is Hamiltonian.*

Proof. Suppose, on the contrary, that D is not Hamiltohian. By Theorem 4.2, D contains no cycle passing through y and z. Then $d(y) \leq n-3$. It is easy to see that for all $i \in [1, n-5]$,

$$\overrightarrow{a}[x_i, z] + \overrightarrow{a}[u_2, x_{i+1}] \leq 1, \quad \overrightarrow{a}[x_i, u_1] + \overrightarrow{a}[z, x_{i+1}] \leq 1 \quad \text{and} \quad \overrightarrow{a}[x_i, u_2] + \overrightarrow{a}[u_1, x_{i+1}] \leq 1.$$

Therefore,

$$d(z, V(C)) + d(u_1, V(C)) + d(u_2, V(C)) = \sum_{i=1}^{n-5} (\overrightarrow{a}[x_i, z] + \overrightarrow{a}[u_2, x_{i+1}] + \overrightarrow{a}[x_i, u_1] + \overrightarrow{a}[z, x_{i+1}]$$
$$+ \overrightarrow{a}[x_i, u_2] + \overrightarrow{a}[u_1, x_{i+1}]) \leq 3n - 15.$$

Since $d(z, A) \leq 5$ and $d(u_1, A) + d(u_2, A) \leq 15$, it follows that $d(z) + d(u_1) + d(u_2) \leq 3n + 5$. This together with $d(y) \leq n-3$, $d(y) + d(u_2) \geq 2n+1$ and condition (M) implies that

$$6n - 2 \leq d(z) + 3d(y) + d(u_1) + d(u_2) \leq 6n - 4,$$

which is a contradiction. This proves Lemma 5.6. □

Lemma 5.7: *Suppose that $C := x_1 x_2 \dots x_{n-2} x_1$ is a cycle of length $n-2$ in D passing through y and avoiding z. Let $V(D) \setminus V(C)) = \{z, u\}$. If $u \leftrightarrow z$, then D is Hamiltonian.*

Proof. Suppose, on the contrary, that $z \leftrightarrow x$ but D is not Hamiltonian. Then, by Lemma 4.5, the vertices y and x are not adjacent. Hence, $d(y) \leq n-2$. Since D is not Hamiltonian, it follows that for every $i \in [1, n-2]$ we have $\overrightarrow{a}[x_i, z] + \overrightarrow{a}[x, x_{i+1}] \leq 1$ and $\overrightarrow{a}[x_i, x] + \overrightarrow{a}[z_0, x_{i+1}] \leq 1$. These imply that $d(z) + d(x) \leq 2n$. Therefore, by condition (M), we have

$$4n - 3 \leq d(z) + d(x) + 2d(y) \leq 4n - 4,$$

which is a contradiction. Lemma 5.7 is proved. □

6. Conclusion

In the current article, we have examined the Manoussakis conjecture for a digraph to be Hamiltonian. For a digraph with the conditions of the Manoussakis conjecture, a number of theorems and lemmas are proved. Found results may be the first step towards confirming the Manoussakis conjecture.

Added in proof. Recently, using some results of this paper, the author confirmed the Manoussakis conjecture.

Acknowledgement

The author would like to thank the referees for careful reading and many helpful remarks and suggestions.

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Submitted 14.12.2018, accepted 18.04.2019.

Կողմնորոշված գրաֆի համիլտոնյանության վերաբերյալ Մանոուսակիսի վարկածի մասին

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Ամփոփում

Մանոուսակիսը (J. of Graph Theory, vol. 16, pp. 51-59, 1992) առաջարկել է հետևյալ վարկածը:

Վարկած: Դիցուք D-ն 2-ուժեղ կապակցված n-գագաթանի կողմնորոշված գրաֆ է: Եթե D-ի ցանկացած ոչ կից գագաթների ցանկացած երկու տարբեր $\{x,y\}$ և $\{u,v\}$ զույգերի համար տեղի ունի հետևյալ $d(x)+d(y)+d(w)+d(z) \geq 4n-3$ անհավասարությունը, ապա D-ն հանդիսանում է համիլտոնյան:

Ներկա աշխատանքում ապացուցվել է, որ եթե D կողմնորոշված գրաֆը բավարարում է Մանոուսակիսի վարկածի պայմաններին, ապա

(1). D գրաֆը պարունակում է ցիկլ-ֆակտոր;

(2). Եթե D-ի ոչ կից գագաթների որևէ $\{x,y\}$ զույգի համար $d(x)+d(y) \leq 2n-2$, ապա (i) D-ն համիլտոնյան է այն և միայն այն ժամանակ, երբ D-ն պարունակում x և y գագաթներով անցնող կողմնորոշված ցիկլ; (ii) D-ն համիլտոնյան է կամ պարունակում է x (y) գագաթով անցնող $n-1$ երկարության կողմնորոշված ցիկլ, որը չի անցնում y (x) գագաթով (մասնավորապես, D-ն պարունակում է առնվազն $n-1$ երկարության ցիկլ) ;

(3). Եթե D-ի ոչ կից գագաթների որևէ $\{x,y\}$ զույգի համար $d(x)+d(y) \leq 2n-4$, ապա ցանկացած k, $2 \leq k \leq n-1$, ամբողջ թվի համար D-ն պարունակում է k երկարության կողմնորոշված ցիկլ;

(4). D գրաֆի որոշակի երկարություններ $(n-5)$-ից մինչև $(n-1)$ունեցող կողմնորոշված ցիկլերի համար ապացուցվել են մի շարք պնդումներ:

Բանալի բառեր՝ կողմնորոշված գրաֆ, համիլտոնյան ցիկլ, ֆակտոր ցիկլ, համցիկլիկ կողմնորոշված գրաֆ:

О гипотезе Маноуссакиса о гамильтоновости орграфов

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Аннотация

Маноуссакис (J. of Graph Theory, vol. 16, pp. 51-59, 1992) предложил следующую гипотезу.

Гипотеза: Пусть D является 2-сильно связным n-вершинным орграфом, в котором для любых различных пар $\{x,y\}$, $\{u,v\}$ несмежных вершин имеет место

$d(x) + d(y) + d(w) + d(z) \geq 4n - 3$. Тогда D является гамильтоновым.

В настоящей работе доказано, что если орграф D удовлетворяет условиям гипотеза Маноуссакиса, то

(1). D содержит цикл-фактор;

(2). Если для некоторой пары несмежных вершин x и y имеет место $d(x) + d(y) \leq 2n - 2$, то имеют место: (i) D является гамильтоновым тогда и только тогда, когда D содержит контур проходящий через вершин x и y, (ii) D является гамильтоновым или содержит контур длины $n - 1$, который проходит через вершину x (y) (в частности, D содержит контур длины по крайней мере $n - 1$);

(3). Если для некоторой пары несмежных вершин x и y имеет место $d(x) + d(y) \leq 2n - 4$, то D содержит контур любой длины k, $3 \leq k \leq n - 1$;

(4). Доказаны ряд свойств для контуров длины от $n - 5$ до $n - 1$.

Ключевые слова: орграф, гамильтоновый цикл, фактор цикл, панциклический орграф.