Mathematical Problems of Computer Science 52, 7–13, 2019.

On Initial Segments of Turing Degrees Containing Simple T-Mitotic but not wtt-Mitotic Sets

Arsen H. Mokatsian

Institute for Informatics and Automation Problems of NAS RA
e-mail: arsenmokatsian@gmail.com

Abstract

We consider the properties of computably enumerable (c.e.) Turing degrees containing sets, which possess the property of a T-mitotic splitting but don't have a wtt-mitotic splitting.

It is proved that for any noncomputable c.e. degree $\boldsymbol{b}$ there exists a degree $\boldsymbol{a}$, such that $\boldsymbol{a} \leq \boldsymbol{b}$ and $\boldsymbol{a}$ contains a simple T-mitotic set, which is not wtt-mitotic.

Keywords: Mitotic set, T-reducibility, wtt-reducibility, Simple set, Contiguous degree.

1. Introduction

We shall use the notions and terminology introduced in Soare [1], Rogers [2].
Notations.

We deal with sets and functions over the nonnegative integers $\omega = \{0, 1, 2, \ldots\}$.

Let $\omega_{ev} = \{x : (\exists k)(x = 2k)\}$; $\omega_{od} = \{x : (\exists k)(x = 2k + 1)\}$.

Let φ_e be the e^{th} partial computable function in the standard listing (Soare [1, p. 15, p. 25]).

If $A \subseteq \omega$ and $e \in \omega$, let $\Phi_e^A(x) = \Phi_e(A : x) = \{e\}^A(x)$ (see Soare [1, pp. 48-50]).

χ_A denotes the characteristic function of A, which is often identified with A and written simply as $A(x)$.

Let $\varphi(x) \downarrow$ denotes that $\varphi(x)$ is defined, $\varphi(x) \uparrow$ denotes that $\varphi(x)$ is undefined.

$W_e = dom\, \varphi_e = \{x : \varphi_x(x) \downarrow\}$.

$\varphi_{e,\, at\, s+1}(x) \downarrow$ denotes $\varphi_{e,\, s+1}(x) \downarrow \,\&\, \varphi_{e,\, s}(x) \uparrow$.

$x \in W_{e, at\, s+1}$ denotes $x \in W_{e,s+1} - W_{e,s}$.

In the literature, Turing reducibility is usually taken as the main reducibility. If the word "reducibility" is used without a further explanation, it means, as a rule, Turing reducibility. If the term "degree of unsolvability" is used without a further explanation, the T-degree is usually meant.

Definition 1: The *use function* $u(A; e, x, s)$ is 1+ (the maximum number used in computation if $\Phi_{es}^A(x) \downarrow$), and $= 0$, otherwise. The use function $u(A; e, x)$ is $u(A; e, x, s)$ if $\Phi_{es}^A(x) \downarrow$ for some s, and is undefined if $\Phi_{es}^A(x) \uparrow$.

Definition 2: A is *computable in (Turing reducible to)* B, written $A \leq_T B$, if $A = \Phi_e^B$ for some e (Soare [1, p. 50]).

Definition 3: A is *weak truth-table reducible to* B, written $A \leq_{wtt} B$, if $(\exists e)$ $[A = \Phi_e^B$ & $(\exists$ computable $f)$ $(f(x) \geq u(B; e, x))]$ (where $u(B; e, x)$ is the use function from Definition 1) (Rogers [2, p. 158]).

Definition 4: If A is a noncomputable computably enumerable (c.e.) set , then *a splitting of* A is a pair A_1, A_2 of disjoint c.e. sets such that $A_1 \bigcup A_2 = A$ (Downey , Stob [3, p. 4]).

Definition 5: C.e. set A is *T-mitotic (wtt-mitotic)*, if there is a splitting A_1, A_2 of A such that $A_1 \equiv_T A_2 \equiv_T A$ ($A_1 \equiv_{wtt} A_2 \equiv_{wtt} A$) (Downey , Stob [3, p. 83]).

Definition 6: (i) A set is *immune*, if it is infinite but contains no infinite c.e. set.
(ii) A set is *simple*, if A is c.e. and $\bar{A}$ is immune (Soare [1, p. 78]).

Definition 7: A c.e. degree $\boldsymbol{a}$ is *contiguous* if for every pair A, B of c.e. sets in $\boldsymbol{a}$, $A \equiv_{wtt} B$ (Downey, Stob [3, p. 45]).

Note that each contiguous degree, by definition, doesn't contain T-mitotic sets, which are not wtt-mitotic.

Lachlan proved the existence of nonmitotic c.e. set (Lachlan [4]).

Ladner proved the existence of completely mitotic c.e. degree (Ladner [5]).

Ladner and Sasso [6] proved, that for every nonzero c.e. degree $\boldsymbol{b}$ there is a nonzero c.e. degree $\boldsymbol{a} \leq \boldsymbol{b}$ such that $\boldsymbol{a}$ is contiguous (see also Downey, Stob [3]).

Thus, there is an infinite class of contiguous degrees, and these degrees, as we have mentioned, don't contain T-mitotic sets, which are not wtt-mitotic.]

Ingrassia ([7]) proved the density of nonmitotic c.e. sets (in $\boldsymbol{R}$) (see also Downey, Slaman [8]).

E. J. Griffiths ([9]) proved the following Theorem: There exists a low c.e. degree $\boldsymbol{u}$ such that if $\boldsymbol{v}$ is a c.e. degree and $\boldsymbol{u} \leq \boldsymbol{v}$, then $\boldsymbol{v}$ is not completely mitotic.

2. Preliminaries, Basic Modules

Theorem 1: *For any noncomputable c.e. degree* $\boldsymbol{b}$ *there is a degree* $\boldsymbol{a}$ *such that* $\boldsymbol{a} \leq \boldsymbol{b}$ *and* $\boldsymbol{a}$ *contains a simple T-mitotic, but not wtt-mitotic set.*

Proof. (sketch) Let h be a general computable function that maps ω to ω^2. Let (Ψ_i, ψ_i) denotes the pair $(\Phi_{i_0}, \varphi_{i_0})$ for all i, where $h(i) = (i_0, i_1)$ (note that Ψ_i is wtt-reduction with ψ_i, denoting the corresponding use function).

It is known (Ladner [10]) that the computably enumerable set A is T-mitotic, $\Leftrightarrow$ A is T-autoreducible, and similarly, the computably enumerable set A is wtt-mitotic, $\Leftrightarrow$ A is wtt-autoreducible (Downey, Stob [3], see also Trakhtenbrot [11]).

Therefore, in order to achieve non-wtt-mitoticity, it is enough for us to achieve non-wtt-autoreducibility.

Thus, to prove our theorem, it is necessary to construct such a c.e. set A, so that the following requirements are met.

R_e : $(\exists\, x)\, \neg\, (\Psi_e(A \bigcup\{x\};\, x) = A(x))$, if $(\forall z \leq y)(\psi_e(z)\ \downarrow)$.
P_e : (W_e is infinite) $\Rightarrow$ $(\exists\, x)\, (x \in W_e\ \&\ x \in A)$.
Note that satisfying the R_e requirement (for all e) provides the infinity of the set $\bar{A}$.
Order the requirements in the following priority ranking: $R_0, P_0,\ R_1, P_1, \ldots\ , R_n, P_n,\ \ldots$
Let $l(e, s) = \max\{x :\ (\forall y < x)\, (\Psi_{e,s}(A_s \bigcup\{y\}\, :\, y) = A(y)\ \&\ (\forall z \leq x)\, (\Psi_e(z)\ \downarrow))\}$.
The main strategy for satisfying R_e is the following: we select a number (the so-called follower) x (which should be accompanied by two more elements $x-2$ and $x-1$, and possibly, the third - $\hat{x}$; an exact definition of the attendant numbers of the follower x, namely $(x-2)$, $(x-1)$, $\hat{x}$, will be given hereinafter), we wait until $l(e, s) > x$ and enumerate x in A_{s+1}, if $(\forall z < x)(\psi_{e,s}(z)\ \downarrow)$, setting $r(e,\ s+1)= u(x,\ e,\ s)$, where $u(x,\ e,\ s) = u(\Psi_{e,s}(A_s \bigcup\{x\};\, x))$.
An B-permitting procedure is introduced in order to provide $A \leq_T B$ (where B is a c.e. set from degree $\boldsymbol{b}$).
To satisfy the requirement of R_e at each stage, we have a finite set of followers $x_{1,s}, < \ldots < x_{n,s}$. In this construction, a modification of the B-permitting method is used. We treat the interval $[0, \ldots,$ i] as allowing for $x_{i,s}$.
To satisfy the requirement of P_e at each stage, we have a finite set of followers $y_{1,s}, < \ldots < y_{n,s}$. For requirement P_e, the usual B-permitting method is used.
The construction will be such that if eventually we have $\Psi_e(A \bigcup \{x\};\, x) = A(x)\ \&\ \psi_e$ is a total function, then it will be possible to compute B.
The ground of satisfactions for requirements of R_e and P_e will be given below.

2.1 Basic Module for R_e

The followers $x_{1,s},\ \ldots,\ x_{n,s}$ satisfy the following rules below.
Appointment. If $x_{i,s}$ is currently defined and $x_{i+1,s}$ is not, then if $l(e, s) > x_{I,s}$, declare $x_{i,s}$ as *active*, and set $x_{i+1,s+1} = \mu z(z \geq s+2\ \&\ (\exists\, k)\, (z = 2k))$. Set $\tilde{r}(e, s+1) = \max(u(x_{k,s}, e, s)\ :\ k \leq i)$. To get an idea of the restriction function $\tilde{r}(e, s)$, we give its definition, although it is not used in the basic module.
We say that $x_{i,s}$ is *superactive*, if $x_{i,s} - 2$ and $x_{i,s} - 1$ belong to A_s.
Permission. If $x_{i,s}$ is active and $i \in B_{at\, s}$, then if

(i) $(\exists\, j > i)$ $[x_{j,s}$ is superactive $\&\ x_{j,s} \notin A_s]$, let $j_0 = \mu z$ $[x_{z,s}$ is superactive $\&\ x_{j,s} \notin A_s]$. Then we enumerate the numbers $x_{j_0,\, s}, \hat{x}_{j_0}, s$ into A_{s+1} (where $\hat{x}_{j_0,s} = \psi_e(x_{j_0,s})$). Cancel $x_{k,s}$, for all $(k > j_0)$. We will do the same with the accompanying elements of the corresponding followers.

(ii) if (i) and $(\neg\exists\, j)\, [x_{j,s} \in A_s]$ does not hold, then we enumerate the numbers $x_{i,s}-2,\ x_{i,s}-1$ into A_{s+1}. Cancel $x_{k,s}$, for all $(k > i)$. We will do the same with the accompanying elements of the corresponding followers.

For any i such that the follower $x_{i,s}$ is not canceled at the end of the part "permission" of the basic module and is active, let's set $x_{i,s+1} = x_{i,s}$. We will do the same with the accompanying elements of the corresponding followers.

The "cancellation" rule, which is present in the proof of Theorem 4.8 (Downey, Slaman [8]), in this case it will be necessary to note the effect of the requirements of R_j and P_j (where $j < e$) on satisfying the requirement R_e, but not to describe the basic module itself for R_e.

2.2 Basic Module for P_e

The followers $y_{1,s}, \ldots, y_{n,s}$ satisfy the following rules below.
Appointment. If $y_{i,s}$ is currently defined and $y_{i+1,s}$ is not, then if $(\exists z)\,(z \in W_e\, z \geq y_{i,s})$, declare $y_{i,s}$ as active, and set $y_{i+1,s+1} = \mu z(z \geq s \ \&\ (\exists\, k)\,(z = 2k))$.
Permission. If $y_{i,s}$ is active and $i \in B_{at\, s}$, then enumerate the numbers $y_{i,s}$, $y_{i,s}+1$, z and $z+1$ into A_{s+1}.

The "cancellation" rule, which is present in the proof of Theorem 4.8 (Downey, Slaman [8]), in this case it will be necessary to note the effect of the requirements of R_j (where $j \leq e$) on satisfying the requirement P_e, but not to describe the basic module itself for P_e.

3. Verification of Lemmas

Lemma 1: *Suppose that ψ_e is total and $(\forall x)\,(\,\Psi_e(A \bigcup \{x\};\, x) \downarrow)$.*
Then $(\exists\, y) \neg ((\,\Psi_e(A \bigcup \{y\};\, y) = A(y))$. Thus, the requirement R_e is satisfied.

Proof. Suppose otherwise. We show that B is computable.

Note that since we only consider the satisfaction of the basis module for R_e (that is, we do not take into account the effect of the requirements R_j and P_j (where $j < e$) on the satisfaction of the requirement R_e), it is obvious that conditions (i), ..., (iv) are met.

(i) All the $x_{i,s}$ eventually become permanently defined, that is $\lim_s x_{i,s} = x_i$ exists with $x_i \notin A$.

(ii) Once x_k is defined at stage t, $(\forall s > t)$ $(u(x_k, e, t) = u(x_k, e, s) = u(e, x_k))$.

(iii) $(\forall i)\,(x_{i+1} > \max\{u(e, x_k) \,:\, k \leq i\,\})$.

(iv) It can be effectively recognized, when (i) occurs.

Two cases are possible:

(a) $(\exists\, m)$ $(\forall\, k > m)$ $[x_k - 2 \notin A]$;

(b) $(\forall\, m)$ $(\exists\, k > m)$ $[x_k - 2 \in A]$.

For both cases ((a) and (b)), it will be proved that B is computable (and thus, the assumption that Lemma1 is false will lead to a contradiction with the supposition of non-computability of B).

Now, if (a) holds, we prove that B is computable.

If conditions (i), ..., (iv) are satisfied, we show how to compute B (that is, the characteristic function of the set B; remind that we often identify the set B with its characteristic function).

Let $f \parallel x$ denotes the restriction of f to arguments $y < x$, and $A \parallel x$ denotes $\chi_A \parallel x$.

Let s_0 be such a stage that $B \parallel m+1 = B_{s_0} \parallel m+1$ and $A \parallel x_{m+1} = A_{s_0} \parallel x_{m+1}$.

Let $q \in \omega$ and $q > m$. Effectively compute a stage s so that x_{q+1} is defined, that is $x_{q+1} = x_{q+1,s}$ (in that case, in fact, $s > s_0$).

Then x_q is active, $x_q \in A$ and since x_{q+1} is the final value of the $q+1$-th follower, the computations of $u\,(x_j, e, s)$ are true for all $j \leq q$.

In this case $q \in B \Leftrightarrow q \in B_s$, because otherwise it would lead to the fact that $x_q - 2$

would have entered the set, contrary to our assumption that case (a) holds.

Now suppose that case (b) holds. Let us prove that in this case also B is computable.

If conditions (i), ..., (iv) are fulfilled, we show how to compute B.

Let $q \in \omega$. Effectively compute such a stage s and a number p so that $p = \mu z$ ($z \geq q$ & $x_{z-2} \in A$ & $x_{z+1} = x_{z+1,s}$).

Then x_p is active, $x_p \notin A_s$ and since x_{p+1} is the final value of the $p+1$-th follower, then $u(x_j, e, s)$ computations are true for all $j \leq p$. In this case $q \in B \Leftrightarrow q \in B_s$, since otherwise (that is, if q enters B after the stage s) this will lead to the entry p into A and satisfaction of the requirement R_e, which will contradict the initial assumption that Lemma 1 is false.

Lemma 1 is prooved.

Lemma 2: *Suppose that W_e is an infinite set. Then $(\exists\, z)$ $(z \in W_e$ & $z \in A)$. Thus, the requirement P_e is satisfied.*

Proof. Suppose otherwise.
We show that B is computable.
Let $\tilde{r}(e) = \lim_s \tilde{r}(e, s)$.
Although the use of this function in the description of the basis module for P_e is not necessary, an indication of this function clearly shows the effect of the requirements R_j (where $j \leq e$) on the satisfaction of the requirements P_e when constructing the set A.
Let t_0 be such that $(\forall s \geq t_0)\, \tilde{r}(e, s) = \tilde{r}(e)$.
Then it is obvious, that all the $y_{i,s}$ become permanently defined (i.e., $\forall i \,\exists\, (t \geq t_0)\, (\forall s)\, (y_{i,t} = y_{i,s} = y_i)$) with $y_i \notin A$.
In fact, if there existed k such that $y_k \in A$, then, by construction, there would exist z such that $z \in W_e \bigcap A$.
Assuming the opposite of the statement of the proposition, we show how B can be computed.
Let $q \in \omega$. Find $t \geq t_0$ such that y_q is permanently defined. Then $q \in B \Leftrightarrow q \in B_t$, since otherwise q's entry into B would meet P_e.

Lemma 2 is prooved.

4. Conclusion

Note that the coherence of constructions to satisfy the requirements R_e and P_e (for all e) is not difficult, since satisfying the requirements R_e and P_e (for all e) requires a finite number of steps. We also note that the indicated method of constructing the set A (based on the constructions for the basic modules) will result in the set $A \bigcap \omega_{ev}$ being T-equivalent to the set $A \bigcap \omega_{od}$.
These remarks allow us to complete the proof of the theorem. ■

Note that it follows from the above theorem that below any noncomputable c.e. degree there is an infinite number of noncomputable c.e. degrees with the abovementioned property (since the degree $\boldsymbol{a}$ (mentioned in the theorem), containing a simple set, is a noncomputable c.e. degree).

References

[1] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, 1987.

[2] H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill Book Company, 1967.

[3] R. G. Downey and M. Stob, "Splitting Theorems In Recursion Theory", *Ann. Pure Appl. Logic*, vol. 65, pp. 1-106, 1993.

[4] A. H. Lachlan, The Priority Method. I, Zeitschrift fr mathematische Logik und Grundlagen der Mathematik, vol. 13, pp. 1-10, 1967.

[5] R. Ladner, "A complete mitotic recursively enumerable degree", *Trans. Amer. Math. Soc.*, 184, pp. 497-507, 1973.

[6] R. Ladner and L. Sasso, "The weak truth-table degrees of recursively enumerable sets", *Arch. Math. Logik Grundlang Grundlagen der Mathematik*, vol. 8, pp. 429-448, 1975.

[7] M. Ingrassia, *P-generecity for recursive enumerable degrees*, PhD. Thesis, University of Illinois, 1981.

[8] R. G. Downey and T. A. Slaman, "Completely Mitotic R. E. Degrees", *Ann. Pure Appl. Logic*, vol. 41, pp. 119-152, 1989.

[9] E. J. Griffiths, *Completely Mitotic Turing Degrees, Jump Classes and Enumeration Degrees*, Ph.D. Thesis, University of Wisconsin-Madison, 1998.

[10] R. Ladner, "Mitotic Recursively Enumerable Sets", *The Journal of Symbolic Logic*, vol. 38, no. 2, pp. 199-211, 1973.

[11] B. A. Trakhtenbrot, "On autoreducibility" (in Russian), *Dokl. Akad. Nauk SSSR*, vol. 192, pp. 1224-1227, 1970.

Submitted 04.07.2019, accepted 22.11.2019.

Պարզ T-միթոտիկ, բայց ոչ wtt-միթոտիկ բազմություններ պարունակող թյուրինգյան աստիճանների որոշ հատկությունների վերաբերյալ

Արսեն Հ. Մոկացյան

ՀՀ ԳԱԱ Ինֆորմատիկայի և ավտոմատացման պրոբլեմների ինստիտուտ
e-mail: arsenmokatsian@gmail.com

Ամփոփում

Հետազոտվումեն T-միթոտիկ տրոհման հատկությամբ օժտված, բայց wtt-միթոտիկ տրոհում չունեցող բազմություններ պարունակող ռեկուրսիվորեն թվարկելի (ռ.թ.) թյուրինգյան աստիճանների հատկությունները: Ապացուցված է, որ կամայական ոչ ռեկուրսիվ (ռ.թ.) $\boldsymbol{a}$ աստիճանի համար գոյություն ունի ոչ ռեսուրսիվ (ռ.թ) այնպիսի $\boldsymbol{b}$ աստիճան, որ $\boldsymbol{b} \leq \boldsymbol{a}$ և $\boldsymbol{b}$-ն պարունակում է պարզ T- միթոտիկ, բայց ոչ wtt- միթոտիկ բազմություն:

Բանալի բառեր` միքոտիկ բազմություն, T-հանգեցում, wtt-հանգեցում, պարզ բազմություն, զուգակցված աստիճան:

О некоторых свойствах тьюринговых степеней, содержащих простые T-митотические множества, не являющиеся wtt-митотическими

Арсен А. Мокацян

Институт проблем информатики и автоматизации НАН РА
e-mail: arsenmokatsian@gmail.com

Аннотация

Исследуются свойства рекурсивно перечислимых (р.п.) тьюринговых степеней, содержащих множества, которые обладают свойством -митотического разбиения, но не имеют wtt-митотического разбиения. Доказано, что для любой нерекурсивной (р.п.) степени $\boldsymbol{a}$ существует нерекурсивная (р.п.) степень $\boldsymbol{b}$, такая что $\boldsymbol{b} \leq \boldsymbol{a}$ и $\boldsymbol{b}$ содержит простое T-митотическое множество, которое не является wtt-митотическим.

Ключевые слова: митотическое множество, T-сводимость, wtt-сводимость, простое множество, сцепленная степень.