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On LAO Two-stage Testing of Multiple Hypotheses Concerning Markov Chain

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Abstract

Two-stage testing of multiple hypotheses concerning Markov chain with two separate families of hypothetical transition probabilities is considered. The matrix of reliabilities of logarithmically asymptotically optimal hypotheses testing by a pair of stages is studied and compared with the case of similar one-stage testing. It is shown that two-stage testing needs less operations than one-stage testing.

Keywords: Logarithmically asymptotically optimal (LAO) test, Multiple hypotheses testing, Multistage tests, Reliabilities matrix, Error probability exponent, Markov chain.

1. Introduction

In this paper we study multiple hypotheses LAO two-stage testing by a sample of sequence of experiments concerning a Markov chain. Analogous problem was formulated and solved for the case of independent experiments in [1].

The classical problem of statistical hypotheses testing refers to two hypotheses [2]. The procedure of statistical hypotheses detection is called a test. The probability of incorrect acceptance of one hypothesis instead of the other is an error probability. We consider the case of a tests sequence, where the probabilities of error decrease exponentially as 2^{-NE}, when the number of observations N (size of sample) tends to infinity. The exponent of error probability E we call *reliability*. The test is called *logarithmically asymptotically optimal* (LAO) if for one of the given of reliabilities the constricted test provided the greatest value of the other reliability. The goal of research was to find an optimal functional relation between the error probabilities exponents of the first and the second types of error. Such optimal tests were considered first by Hoeffding [3], examined later by Csiszár and Longo [4], Tusnady [5], [6], Longo and Sgarro [7], Birgé [8] (he proposed the term LAO) and by many others. Some authors for this concept of testing [6, 9, 10] applied the terms *exponentially rate optimal* (ERO). Hoeffding's result was generalized to finite Markov chain in [11].

The need of testing of more than two hypotheses in many scientific and applied fields has increased recently. The problem of LAO testing of multiple hypotheses was investigated in [12] and was later extended for a discrete stationary Markov source and arbitrarily varying Markov source [13]–[20].

Two-stage procedures of testing are useful in applications for achieving minimal economic expenditure.

2. Definitions and Notations

This section is dedicated to necessary notations and properties. Let $\mathcal{X}$ be a finite set of states of stationary Markov chain X_0, X_1, X_2, ..., which is a stochastic process defined by matrix of transition probabilities

$$P(X_n = x|X_{n-1} = u) = P(x|u), \quad x, u \in \mathcal{X}.$$

There exists the corresponding stationary probability distribution (PD)$Q = \{Q(u)\}$, such that

$$\sum_{x\in\mathcal{X}} Q(u)P(x|u) = Q(x), \quad \sum_{u\in\mathcal{X}} Q(u) = 1.$$

Suppose L hypothetical transition probabilities $G_l = \{G_l(x|u)\}$, $x, u \in \mathcal{X}$, $l = \overline{1, L}$, with corresponding stationary distributions $Q_l = \{Q_l(u)\}$, $l = \overline{1, L}$, are given and arranged in two disjoint families. The first family $\mathcal{P}_1$ includes R hypotheses and the second family $\mathcal{P}_2$ includes $L - R$ hypotheses. It is unknown which one of those distributions is realized. On the base of the sample $\mathbf{x} = (x_0, x_1, ..., x_N) \in \mathcal{X}^{N+1}$ of the results of $N + 1$ observations the statistician is trying to make a reliable decision about real distribution.

The two-stage test on the base of the sample $\mathbf{x}$ we denote by $\Phi^N(\mathbf{x})$, it may be composed by the pair of tests $\varphi_1^N(\mathbf{x})$ and $\varphi_2^N(\mathbf{x})$ of two consecutive stages, we write $\Phi^N = (\varphi_1^N, \varphi_2^N)$. The first stage for selection of a family $\mathcal{P}_1$ or $\mathcal{P}_2$ is a non-randomized test $\varphi_1^N(\mathrm{x})$. The next stage is for making a decision in the determined family of PDs, it is made by non-randomized test $\varphi_2^N(\mathrm{x})$ based on the same sample x.

We define any necessary characteristics Shannon's entropy and Kullback-Leibler's divergences for Markov chain.

The Shannon's entropy for Markov chain is defined as follows:

$$H_{Q\circ P}(X) \triangleq -\sum_{x,u} Q(u)P(x|u)\log P(x|u).$$

The conditional Kullback-Leibler divergence $D(P\|W|Q)$ of PDs $Q \circ P \triangleq \{Q(u)P(x|u)),\ u \in \mathcal{X}, x \in \mathcal{X}\}$ and $Q \circ W \triangleq \{Q(u)W(x|u)),\ u \in \mathcal{X}, x \in \mathcal{X}\}$ on $\mathcal{X} \times \mathcal{X}$ is:

$$D(Q \circ P\|Q \circ W) = D(P\|W|Q) \triangleq \sum_{u\in\mathcal{X},x\in\mathcal{X}} Q(u)P(x|u)\log\frac{P(x|u)}{W(x|u)}.$$

The informational divergence of PD P and PD Q on $\mathcal{X}$ is:

$$D(Q\|Q_l) \triangleq \sum_{x\in\mathcal{X}} Q(x)\log\frac{Q(x)}{Q_l(x)}.$$

For proofs we will use the method of types, which is an important technical tool in Information Theory.

Let us name the second order type of the Markov vector $\mathbf{x}$ the square matrix of $|\mathcal{X}|^2$ relative frequencies $\{N(u, x)N^{-1}, x, u \in \mathcal{X}\}$ of the simultaneous appearance on the pairs of neighbor places of the states u and x. It is clear that $\sum_{ux} N(u, x) = N$.

Denote by $\mathcal{T}^N_{Q\circ P}$ the set of vectors from $\mathcal{X}^{N+1}$ which have such a type that for the joint PD $Q \circ P$, $N(u,x) = NQ(u)P(x|u), \quad x, u \in \mathcal{X}$.

We will use the following properties of types of second order [13]–[15]: the number $|\mathcal{T}^N_{Q\circ P}|$ of vectors in $\mathcal{T}^N_{Q\circ P}$ is the following

$$|\mathcal{T}^N_{Q\circ P}| = \exp\{NH_{Q\circ P}(X) + o(1)\}, \quad \text{with} \quad o(1) \to 0, \quad \text{when} \quad N \to \infty, \tag{1}$$

and the number of elements of sets $\mathcal{T}^N_{Q\circ P}$ for different PDs $Q \circ P$ does not exceed $(N+1)^{|\mathcal{X}|^2}$.

The probability of the vector $\mathbf{x} \in \mathcal{X}^{N+1}$ of the Markov chain with transition probabilities G_l and stationary distribution Q_l is defined as follows:

$$Q_l \circ G_l^N(\mathbf{x}) \triangleq Q_l(x_0) \prod_{n=1}^{N} G_l(x_n|x_{n-1}), \quad l = \overline{1,L},$$

$$Q_l \circ G_l^N(\mathcal{A}) \triangleq \bigcup_{\mathbf{x}\in\mathcal{A}} Q_l \circ G_l^N(\mathbf{x}), \quad \mathcal{A} \subset \mathcal{X}^{N+1}.$$

The probability of the vector $\mathbf{x}$ from $\mathcal{T}^N_{Q\circ P}$ for $l = \overline{1,L}$, can be written also in the following form:

$$\begin{aligned} Q_l \circ G_l^N(\mathbf{x}) &= Q_l(x_0) \prod_{u,x} G_l(x|u)^{NQ(u)P(x|u)} \\ &= Q_l(x_0) \exp\{N \sum_{x,u} Q(u)P(x|u) \log G_l(x|u)\} \\ &= Q_l(x_0) \exp\left\{-N \sum_{x,u} Q(u)P(x|u) \left[\log \frac{P(x|u)}{G_l(x|u)} - \log P(x|u)\right]\right\} \\ &= \exp\left\{-N\left[D(Q \circ P \| Q \circ G_l) - H_{Q\circ P}(X) - o(1)\right]\right\}, \end{aligned} \tag{2}$$

where $o(1) \to 0$, when $N \to \infty$.

According to (1) and (2) we obtain

$$Q_l \circ G_l^N(\mathcal{T}^N_{Q\circ P}) = \exp\left\{-N\left[D(Q \circ P \| Q \circ G_l) + o(1)\right]\right\}. \tag{3}$$

3. First Stage Test of Two Stages

The first stage of decision making consists in using the sample x for the selection of one family of two of PDs by a test $\varphi_1^N(\mathrm{x})$, which can be defined by the division of the sample space $\mathcal{X}^N$ into the pair of disjoint subsets $\mathcal{A}_i^N \triangleq \{\mathrm{x} : \varphi_1^N(\mathrm{x}) = i\}$, $i = 1, 2$. The set $\mathcal{A}_i^N$ consists of all vectors x for which i-th family $\mathcal{P}_i$ of PDs is adopted.

In fact this is the problem of two composite hypotheses testing studied in [9, 10]. At the same time the first stage test of two stages is connected with the identification procedure which was considered in [21]–[26].

The test $\varphi_1^N(\mathrm{x})$ can have two kinds of errors for the pair of hypotheses $\mathcal{P}_i, \ i = 1, 2$. Let $\alpha'_{2|1}(\varphi_1^N)$ be the probability of the erroneous acceptance of the second family $\mathcal{P}_2$ provided that the first family $\mathcal{P}_1$ is true (that is the correct PD is in the first family) and $\alpha'_{1|2}(\varphi_1^N)$ be the probability of the erroneous acceptance of $\mathcal{P}_1$ provided that the second family $\mathcal{P}_2$ is true. We define

$$\alpha'_{2|1}(\varphi_1^N) \triangleq \alpha'_{1|1}(\varphi_1^N) \triangleq \max_{r:r=\overline{1,R}} Q_r \circ G_r^N(\mathcal{A}_2^N), \tag{4}$$

$$\alpha'_{1|2}(\varphi_1^N) \triangleq \alpha'_{2|2}(\varphi_1^N) \triangleq \max_{l:l=\overline{R+1,L}} Q_l \circ G_l^N(\mathcal{A}_1^N). \tag{5}$$

The corresponding reliabilities are defined for infinite sequence φ_1 of tests:

$$E'_{i|j}(\varphi_1) \triangleq \liminf_{N\to\infty} \left\{-\frac{1}{N}\log \alpha'_{i|j}(\varphi_1^N)\right\}, \quad i,j=1,2, \quad i\neq j. \tag{6}$$

The test sequence φ_1 is considered to be LAO if for the given value of $E'^*_{2|1}$ it provides the largest value to $E'^*_{1|2}$. Our aim is to find the reliability function $E'^*_{1|2}(E'_{2|1})$.

For the given $E'^*_{2|1}$ we can define LAO test φ_1^{*N} by division of $\mathcal{X}^N$ into the following two disjoint subsets

$$\mathcal{B}_1^N = \bigcup_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^*_{2|1}} \mathcal{T}_{Q\circ P}^N, \ \text{ and } \ \mathcal{B}_2^N = \mathcal{X}^N \setminus \mathcal{B}_1^N.$$

Theorem 1: *For any $E'_{2|1} > 0$ the reliability function $E'^*_{1|2}(E'_{2|1})$ of the LAO test for testing many hypotheses G_l, $l=\overline{1,L}$, concerning Markov chains is given as follows:*

$$E'^*_{1|2}(E'^*_{2|1}) = \min_{l:\ l=\overline{R+1,L}} \ \inf_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r:\,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^*_{2|1}} D(Q\circ P\|Q\circ G_l).$$

Proof: From the estimations of the corresponding error probabilities using (3) it follows:

$$\begin{aligned}
\alpha'_{2|1}(\varphi_1^N) &= \max_{r:\ r=\overline{1,R}} Q_r\circ G_r^N(\mathcal{B}_2^N)\\
&\leq \max_{r:\ r=\overline{1,R}} (N+1)^{|\mathcal{X}|^2} \max_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)>E'^*_{2|1}} Q_r\circ G_r^N(\mathcal{T}^N_{Q\circ P})\\
&\leq \max_{r:\ r=\overline{1,R}} \max_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)>E'^*_{2|1}} \exp\left\{-N\left[D(Q\circ P\|Q\circ G_r)+o(1)\right]\right\}\\
&\leq \exp\{-N[\min_{r:\ r=\overline{1,R}} \min_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)>E'^*_{2|1}} D(Q\circ P\|Q\circ G_r)+o(1)]\}\\
&\leq \exp\left\{-N\left[E'^*_{2|1}+o(1)\right]\right\},
\end{aligned}$$

For $\alpha'_{1|2}(\varphi_1^N)$ we have the following estimation:

$$\begin{aligned}
\alpha'_{1|2}(\varphi_1^N) &= \max_{l:l=\overline{R+1,L}} Q_l\circ G_l^N(\mathcal{B}_2^N)\\
&\leq (N+1)^{|\mathcal{X}|^2} \max_{l:l=\overline{R+1,L}} \max_{Q\circ P:\,D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^*_{2|1}} Q_l\circ G_l^N(\mathcal{T}^N_{Q\circ P})\\
&\leq \max_{l:l=\overline{R+1,L}} \max_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^*_{2|1}} \exp\{-N[D(Q\circ P\|Q\circ G_l)+o(1)]\}\\
&= \exp\{-N[\min_{l:l=\overline{R+1,L}} \min_{Q\circ P:\,D(Q\|Q_r)<\infty,\ \min\limits_{r,r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^*_{2|1}} D(Q\circ P\|Q\circ G_l)+o(1)]\}.
\end{aligned}$$

Now let us prove the inverse inequality for $\alpha'_{1|2}(\varphi_1^N)$:

$$\begin{aligned}\alpha'_{1|2}(\varphi_1^N) &= \max_{l: l=\overline{R+1,L}} Q_l \circ G_l^N(\mathcal{B}_2^N) \\ &\geq \max_{l: l=\overline{R+1,L}} \max_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r, r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^{*}_{2|1}} Q_l \circ G_l^N(\mathcal{T}^N_{Q\circ P}) \\ &= \max_{l: l=\overline{R+1,L}} \max_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r, r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^{*}_{2|1}} \exp\{-N\left[D(Q\circ P\|Q\circ G_l)+o(1)\right]\} \\ &= \exp\{-N[\min_{l: l=\overline{R+1,L}} \min_{Q\circ P:\, D(Q\|Q_r)<\infty,\ \min\limits_{r, r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^{*}_{2|1}} D(Q\circ P\|Q\circ G_l)+o(1)]\}.\end{aligned}$$

According to the definitions of the corresponding reliabilities we obtain the proof of the theorem.

4. Second Stage Test of Two Stages

After selecting a family of PDs from the two, it is necessary to detect one PD in this family. If the first family of PDs is accepted, we consider the test $\varphi_2^N(\mathrm{x})$ which can be defined by the division of the sample space $\mathcal{B}_1$ to R disjoint subsets

$$\mathcal{C}_r^N \triangleq \{\mathrm{x} : \varphi_2^N(\mathrm{x}) = r\}, \quad r = \overline{1,R}.$$

Let $\alpha''_{l|r}\left(\varphi_2^N\right)$ be the probability of the erroneous acceptance at the second stage of the test, in which PD G_l is accepted when G_r is true:

$$\alpha''_{l|r}\left(\varphi_2^N\right) \triangleq Q_l \circ G_r^N\left(\mathcal{C}_l^N\right), \; r = \overline{1,R}, \; l = \overline{1,L}.$$

The probability to reject G_r, when it is true, is

$$\alpha''_{r|r}\left(\varphi_2^N\right) \triangleq Q_r \circ G_r^N\left(\overline{\mathcal{C}}_r^N\right) = \sum_{l=1,\, l\neq r}^{R} \alpha''_{l|r}\left(\varphi_2^N\right) + Q_r \circ G_r^N(\mathcal{B}_2^N), \quad r = \overline{1,R}. \tag{7}$$

The corresponding reliabilities for the second stage of the test are defined as:

$$E''_{l|r}(\varphi_2) \triangleq \liminf_{N\to\infty}\left\{-\frac{1}{N}\log \alpha''_{l|r}\left(\varphi_2^N\right)\right\}, \quad r = \overline{1,R}, \quad l = \overline{1,L}. \tag{8}$$

Using the properties of types the following equalities are derived for each $r = \overline{1,R}$:

$$\begin{aligned}E^I_{2|r} &\triangleq \lim_{N\to\infty}\left\{-\frac{1}{N}\log\left[Q_r \circ G_r^N(\mathcal{B}_2^N)\right]\right\} \\ &= \inf_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r, r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)>E'^{*}_{2|1}} D(Q\circ P\|Q\circ G_r).\end{aligned} \tag{9}$$

From (7)–(9) it follows that

$$E''_{r|r}(\varphi_2) = \min\left[\min_{l=\overline{1,R}} E''_{l|r}(\varphi_2), E^I_{2|r}\right], \quad r = \overline{1,R}.$$

Remark 1: *If at the first stage the first family of PDs is accepted, when our decision in the first stage is true, but we have an error in the second stage, the reliabilities of the corresponding error probabilities are $E''_{l|r}$, $r, l = \overline{1, R}$. When our wrong decision comes from the first stage, the corresponding reliabilities are $E''_{l|r}$, $l = \overline{1, R}$, $r = \overline{R+1, L}$.*

In this case the reliability matrix for the second stage of the test $\mathbf{E}''(\boldsymbol{\varphi_2})$ is the following:

$$\begin{bmatrix} E''_{1|1} & E''_{2|1} & \dots & E''_{R|1} \\ E''_{1|2} & E''_{2|2} & \dots & E''_{R|2} \\ \dots & \dots & \dots & \dots \\ E''_{1|R} & E''_{2|R} & \dots & E''_{R|R} \\ E''_{1|R+1} & E''_{2|R+1} & \dots & E''_{R|R+1} \\ E''_{1|R+2} & E''_{2|R+2} & \dots & E''_{R|R+2} \\ \dots & \dots & \dots & \dots \\ E''_{1|L} & E''_{2|L} & \dots & E''_{R|L} \end{bmatrix}.$$

If the second family of PDs is accepted, then the test $\varphi_2^N(\mathrm{x})$ is a division of the sample space $\mathcal{A}_2$ to $L-R$ disjoint subsets. The definitions of error probabilities and the corresponding reliabilities are similar to the above considered case.

Using the properties of types (like in the proof of Theorem 1) the following equalities are derived for each $l = \overline{R+1, L}$:

$$\begin{aligned} E^{II}_{2|l} &\triangleq \lim_{N\to\infty} \left\{-\frac{1}{N} \log \left[Q_l \circ G_l^N(\mathcal{B}_1^N)\right]\right\} \\ &= \inf_{Q\circ P:\ D(Q\|Q_r)<\infty,\ \min\limits_{r, r=\overline{1,R}} D(Q\circ P\|Q\circ G_r)\leq E'^{*}_{2|1}} D(Q\circ P\|Q\circ G_l). \end{aligned} \tag{10}$$

Remark 2: *If at the first stage, the second family of PDs is accepted, when our decision in the first stage is true, but we have an error in the second stage, the reliabilities of the corresponding error probabilities are $E''_{l|r}$, $r, l = \overline{R+1, l}$. When our wrong decision comes from the first stage, the corresponding reliabilities are $E''_{l|r}$, $l = \overline{R+1, l}$, $r = \overline{1, R}$.*

In this case the reliability matrix for the second stage of the test $\mathbf{E}''(\boldsymbol{\varphi_2})$ is the following:

$$\begin{bmatrix} E''_{R+1|1} & E''_{R+2|1} & \dots & E''_{L|1} \\ E''_{R+1|2} & E''_{R+2|2} & \dots & E''_{L|2} \\ \dots & \dots & \dots & \dots \\ E''_{R+1|R} & E''_{R+2|R} & \dots & E''_{L|R} \\ E''_{R+1|R+1} & E''_{R+2|R+1} & \dots & E''_{L|R+1} \\ E''_{R+1|R+2} & E''_{R+2|R+2} & \dots & E''_{L|R+2} \\ \dots & \dots & \dots & \dots \\ E''_{R+1|L} & E''_{R+2|L} & \dots & E''_{L|L} \end{bmatrix}.$$

In the following theorems we show the optimal dependence of reliabilities.

Theorem 3: *If in the first stage the first family of PDs of Markov chain is accepted, then for the given positive and finite values $E''_{1|1}, E''_{2|2}, ..., E''_{R-1|R-1}$, satisfying the following compatibility conditions*

$$0 < E''_{1|1} < \min_{r=2,R} \left[D(Q_r \circ G_r\|Q_r \circ G_1)\right],$$

. . . .

$$0 < E''_{r|r} < \min[\min_{1\leq l<r} E''^{*}_{l|r},\ \min_{r<l\leq R} D(Q_l \circ G_l \| Q_l \circ G_r),\ E^{I}_{2|r}],\quad 2 \leq r \leq R-1,$$

there exists a LAO sequence of tests φ_2^*, *the elements of reliability matrix* $\mathbf{E}''(\varphi_2^*) = \{E''^{*}_{l|s}\}$ *of which are positive and given as follows:*

$$E''^{*}_{r|r} \stackrel{\triangle}{=} E''_{r|r},\ r = \overline{1, R-1}, \tag{11}$$

$$E''^{*}_{l|r} = \inf_{Q\circ P \in \mathcal{R}''_l} D(Q \circ P \| Q \circ G_r),\ l = \overline{1, R-1},\ r \neq l,\ r = \overline{1, L}, \tag{12}$$

$$E^{*}_{R|R} = \min\left[\min_{l=\overline{1,R-1}} E''_{l|R}(\varphi_2), E^{I}_{2|R}\right], \tag{13}$$

where for $l = \overline{1, R-1}$,

$$\mathcal{R}''_l \stackrel{\triangle}{=} \{Q\circ P : \min_{l, l=\overline{1,R}} D(Q\circ P \| Q\circ G_l) \leq E'_{2|1},\ D(Q\circ P \| Q\circ G_l) \leq E''_{l|l},\ D(Q_{\|} Q_l) < \infty\}, \tag{14}$$

$$\mathcal{R}''_R \stackrel{\triangle}{=} \{Q \circ P : \min_{l=\overline{1,R}} D(Q \circ P \| Q \circ G_l) \leq E'_{2|1},\ D(Q \circ P \| Q \circ G_l) \geq E''_{l|l},\ l = \overline{1, R-1}\}, \tag{15}$$

When even one of the compatibility conditions is violated, then at least one element of the matrix $\mathbf{E}''(\varphi_2^*)$ *is equal to* 0.

Theorem 4: *If in the first stage the second family of PDs of Markov chain is accepted, then for the given positive and finite values* $E''_{R+1|R+1}, E''_{R+2|R+2}, ..., E''_{L-1|L-1}$ *satisfying the following compatibility conditions*

$$0 < E''_{R+1|R+1} < \min_{r=\overline{R+2,L}} D(Q_r \circ G_r \| Q_r \circ G_{R+1}),$$

. . . .

$$0 < E''_{r|r} < \min\left[\min_{R+1\leq l<r} E''^{*}_{l|r},\ \min_{r<l\leq L} D(Q_l \circ G_l \| Q_r \circ G_r),\ E^{II}_{2|r}\right],\quad R+2 \leq r \leq L-1$$

there exists a LAO sequence of tests φ_2^*, *the elements of reliability matrix* $\mathbf{E}''(\varphi_2^*)\}$ *of which are positive and formulated as follows:*

$$E''^{*}_{r|r} \stackrel{\triangle}{=} E''_{r|r},\ r = \overline{R+1, L-1}, \tag{16}$$

$$E''^{*}_{l|r} = \inf_{Q\circ P \in \mathcal{R}''_l} D(Q \circ P \| Q \circ G_r),\ l = \overline{R+1, L-1},\ l \neq r,\ r = \overline{1, L}, \tag{17}$$

$$E^{*}_{L|L} = \min\left[\min_{l=\overline{R,L-1}} E''_{l|L}(\varphi_2), E^{II}_{2|L}\right], \tag{18}$$

where for $l = \overline{R+1, L-1}$,

$$\mathcal{R}''_l \stackrel{\triangle}{=} \{Q\circ P : \min_{r=\overline{1,R}} D(Q\circ P \| Q\circ G_r) > E'_{2|1},\ D(Q\circ P \| Q\circ G_l) \leq E''_{l|l},\ D(Q_{\|} Q_l) < \infty\}, \tag{19}$$

$$\mathcal{R}''_L \stackrel{\triangle}{=} \{Q\circ P : \min_{r=\overline{1,R}} D(Q\circ P \| Q\circ G_r) > E'_{2|1},\ D(Q\circ P \| Q\circ G_l) \geq E''_{l|l},\ l = \overline{R+1, L-R-1}\}. \tag{20}$$

When even one of the compatibility conditions is violated, then at least one element of the matrix $\mathbf{E}(\varphi_2^*)$ *is equal to* 0.

The proofs of Theorems 3 and 4 are similar to the proof of Theorem 1 from [15].

5. Comparison of Reliabilities of Two-stage Test With Reliabilities of One-stage Test

In this section we will compare the reliabilities of one stage test considered in [15] and those of the two-stage test defined in this paper.

From the results of the first and the second stages we can obtain the final reliabilities $E'''_{l|r}(\Phi)$, $l = \overline{1, L}$. From Remarks 1 and 2 it follows, that when compatibility conditions in Theorems 1, 3 and 4, are satisfied then $E'''_{l|r}(\Phi) = E''_{l|r}$, $l = \overline{1, L}$.

For comparison we will take equal diagonal elements $E_{l|l} = E'''_{l|l}$, $l = \overline{1, L-1}$ of the reliability matrices of one-stage and two-stage cases.

The corresponding elements of the column $r = \overline{1, R-1} \bigcup \overline{R+1, L-1}$ of one-stage matrix and two-stage matrix are equal. The elements of those columns are functions of diagonal elements of the corresponding columns and formulas of those functions are the same for the one-stage and two stage tests. When we give the same values to diagonal elements of the reliability matrices of two-stage and one-stage cases, the values of those functions are equal.

The elements of R-th and S-th columns of two-stage reliabilities matrix can be smaller or greater than the corresponding elements of one-stage matrix.

The number of operations for realization of two-stage LAO test is less than that of one-stage LAO test.

6. Conclusion

In this paper we considered two-stage multihypotheses testing has any advantage for Markov chain. In this testing the sample set is divided into $R+2$ or $L-R+2$ subsets, so the procedure of two-stage testing is shorter, than one-stage testing, in which the sample space was divided into L disjoint subsets. We also show, that the number of preliminary given elements of the reliabilities matrix in two-stage and one-stage tests would be the same but the procedure of calculations for the first one would be shorter.

As in [1] we can also consider the two-stage LAO test by the pair of samples $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathcal{X}^N$, $\mathrm{x}_1 = (x_1, x_2, \ldots, x_{N_1})$, $\mathrm{x}_1 \in \mathcal{X}^{N_1}$, $\mathrm{x}_2 = (x_{N_1+1}, x_{N_1+2}, \ldots, x_N)$, $\mathrm{x}_2 \in \mathcal{X}^{N_2}$, $N = N_1 + N_2$, $\mathcal{X}^N = \mathcal{X}^{N_1} \times \mathcal{X}^{N_2}$. The first stage using the sample x_1 selects one of the families $\mathcal{P}_1$ or $\mathcal{P}_2$ after that using the sample x_2 the second stage selects one PD in this family.

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Մարկովյան շղթայի վերաբերյալ բազմակի վարկածների երկփուլ ԼԱՕ տեստավորում

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Ամփոփում

Դիտարկվել է անցումային բաշխումների երկու ընտանիքներով բնութագրվող Մարկովյան շղթայի վերաբերյալ բազմակի վիճակագրական վարկածների ստուգման գործընթացը: Ուսումնասիրվել է երկու փուլերի լոգարիթմորեն ասիմտոտորեն օպտիմալ տեստավորման սխալների հուսալիությունների մատրիցը և այն համեմատվել է միափուլ տեստի հուսալիության մատրիցի հետ: Ցույց է տրվել, որ երկփուլ տեստը պահանջում է ավելի քիչ գործողություն, քան միափուլ տեստը:

О двухэтапном ЛАО тестировании многих гипотез относительно Марковской цепи

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Аннотация

Рассмотрено тестирование цепи Маркова характеризующейся двумя семействами возможных переходных вероятностей. Матрица надежностей логорифмически асимптотически оптимального тестирования в два этапа изучена и сравнена со случаем аналогичного одноэтапного тестирования. Показано, что двухэтапное тестирование требует меншье операции, чем одноэтапное.