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On k-Ended Spanning and Dominating Trees

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Abstract

A tree with at most k leaves is called a k-ended tree. Let t_k be the order of a largest k-ended tree in a graph. A tree T of a graph G is said to be dominating if $V(G-T)$ is an independent set of vertices. The minimum degree sum of any pair (triple) of nonadjacent vertices in G will be denoted by σ_2 (σ_3). The earliest result concerning spanning trees with few leaves (by the author, 1976) states: $(*)$ if G is a connected graph of order n with $\sigma_2 \geq n-k+1$ for some positive integer k, then G has a spanning k-ended tree. In this paper we show: (i) the connectivity condition in $(*)$ can be removed; (ii) the condition $\sigma_2 \geq n-k+1$ in $(*)$ can be relaxed by replacing n with t_{k+1}; (iii) if G is a connected graph with $\sigma_3 \geq t_{k+1}-2k+4$ for some integer $k \geq 2$, then G has a dominating k-ended tree. All results are sharp.

Keywords: Hamilton cycle, Hamilton path, Dominating cycle, Dominating path, Longest path, k-ended tree.

1. Introduction

Throughout this article we consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. A good reference for any undefined terms is in [1].

For a graph G, we use n, δ and α to denote the order (the number of vertices), the minimum degree and the independence number of G, respectively. For a subset $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S. If $\alpha \geq k$ for some integer k, let σ_k be the minimum degree sum of an independent set of k vertices; otherwise we let $\sigma_k = +\infty$.

If Q is a path or a cycle in a graph G, then the order of Q, denoted by $|Q|$, is $|V(Q)|$. Each vertex and edge in G can be interpreted as simple cycles of orders 1 and 2, respectively. The graph G is Hamiltonian if G contains a Hamilton cycle, i.e. a cycle containing every vertex of G. A cycle C of G is said to be dominating if $V(G-C)$ is an independent set of vertices.

We write a cycle Q with a given orientation by $\overrightarrow{Q}$. For $x,y \in V(Q)$, we denote by $x\overrightarrow{Q}y$ the subpath of Q in the chosen direction from x to y. For $x \in V(Q)$, we denote the successor and the predecessor of x on $\overrightarrow{Q}$ by x^+ and x^-, respectively.

A vertex of degree one is called an end-vertex, and an end-vertex of a tree is usually called a leaf. The set of end-vertices of G is denoted by $End(G)$. For a positive integer k, a

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tree T is said to be a k-ended tree if $|End(T)| \leq k$. A Hamilton path is a spanning 2-ended tree. A Hamilton cycle can be interpreted as a spanning 1-ended tree. In particular, K_2 is Hamiltonian and a 1-ended tree. We denote by t_k the order of a largest k-ended tree in G. By the definition, t_1 is the order of a longest cycle, and t_2 is the order of a longest path in G.

Our starting point is the earliest sufficient condition for a graph to be Hamiltonian due to Dirac [2].

Theorem A ([2]): *Every graph with $\delta \geq \frac{n}{2}$ is Hamiltonian.*

In 1960, Ore [3] improved Theorem A by replacing the minimum degree δ with the arithmetic mean $\frac{1}{2}\sigma_2$ of two smallest degrees among pairwise nonadjacent vertices.

Theorem B ([3]): *Every graph with $\sigma_2 \geq n$ is Hamiltonian.*

The analog of Theorem B for Hamilton paths follows easily.

Theorem C ([3]): *Every graph with $\sigma_2 \geq n-1$ has a Hamilton path.*

In 1971, Las Vergnas [4] gave a degree condition that guarantees that any forest in G of limited size and with a limited number of leaves can be extended to a spanning tree of G with a limited number of leaves in an appropriate sense. As a corollary, this result implies a degree sum condition for the existence of a tree with at most k leaves including Theorem B and Theorem C as special cases for $k = 1$ and $k = 2$, respectively.

Theorem D ([4], [5], [6]): *If G is a connected graph with $\sigma_2 \geq n-k+1$ for some positive integer k, then G has a spanning k-ended tree.*

However, Theorem D was first openly formulated and proved in 1976 by the author [6] and was reproved in 1998 by Broersma and Tuinstra [5]. Moreover, the full characterization of connected graphs without spanning k-ended trees was given in [7] when $\sigma_2 \geq n-k$ including the well-known characterization of connected graphs without Hamilton cycles when $\sigma_2 \geq n-1$. This particular result was reproved in 1980 by Nara Chie [8].

The next two results on this subject are not included in the recent survey paper [9]. We call a graph G hypo-k-ended if G has no spanning k-ended tree, but for any $v \in V(G)$, $G-v$ has a spanning k-ended tree.

Theorem E ([10]): *For each $k \geq 3$, the minimum number of vertices (edges, faces, respectively) of a simple 3-polytope without a spanning k-ended tree is $8+3k$ ($12+6k$, $6+3k$, respectively).*

Theorem F ([11]): *For each $n \geq 17k$ and $k \geq 2$, except possible for $n = 17k+1$, $17k+2$, $17k+4$ and $17k+7$, there exist hypo-k-ended graphs of order n.*

In this paper we prove that the connectivity condition in Theorem D can be removed, and the conclusion can be strengthened.

Theorem 1: *If G is a graph with $\sigma_2 \geq n-k+1$ for some positive integer k, then G has a spanning k-ended forest.*

Next, we show that Theorem D can be improved by relaxing the condition $\sigma_2 \geq n-k+1$ to $\sigma_2 \geq t_{k+1}-k+1$.

Theorem 2: *Let G be a connected graph with $\sigma_2 \geq t_{k+1} - k + 1$ for some positive integer k. Then G has a spanning k-ended tree.*

The graph $(\delta + k)K_1 + K_\delta$ shows that the bound $t_{k+1} - k + 1$ in Theorem 2 cannot be relaxed to $t_k - k + 1$. Finally, we give a dominating analog of Theorem D.

Theorem 3: *If G is a connected graph with $\sigma_3 \geq t_{k+1} - 2k + 4$ for some integer $k \geq 2$, then G has a dominating k-ended tree.*

The graph $(\delta + k - 1)K_2 + K_{\delta-1}$ shows that the bound $t_{k+1} - 2k + 4$ in Theorem 3 cannot be relaxed to $t_k - 2k + 4$.

The following corollary follows immediately.

Corollary 1: *If G is a connected graph with $\sigma_3 \geq n - 2k + 4$ for some integer $k \geq 2$, then G has a dominating k-ended tree.*

The graph $(\delta + k - 1)K_2 + K_{\delta-1}$ shows that the bound $\sigma_3 \geq t_{k+1} - 2k + 4$ in Theorem 3 cannot be relaxed to $\sigma_3 \geq t_{k+1} - 2k + 3$.

2. Proofs

Proof of Theorem 1: Let G be a graph with $\sigma_2 \geq n - k + 1$ and let $H_1, ..., H_m$ be the connected components of G. Let $\overrightarrow{P} = x\overrightarrow{P}y$ be a longest path in H_1. If $|P| \geq n - k + 2$ then $|G - P| = n - |P| \leq k - 2$, implying that G has a spanning k-ended forest. Now let $|P| \leq n - k + 1$. Since P is extreme, we have $N(x) \cup N(y) \subseteq V(P)$. Recalling also that $\sigma_2 \geq n - k + 1$, we have (by standard arguments) $N(x) \cap N^+(y) \neq \emptyset$, implying that $G[V(P)]$ is Hamiltonian. Further, if $|V(P)| < |V(H_1)|$ then we can form a path longer than P, contradicting the maximality of P. Hence, $|V(P)| = |V(H_1)|$, that is H_1 is Hamiltonian as well. By a similar argument, H_i is Hamiltonian for each $i \in \{1, ..., m\}$ and therefore, has a spanning tree with exactly one leaf. Thus, G has a spanning forest with exactly m leaves.

It remains to show that $m \leq k$. If $m = 1$ then G has a spanning 1-ended tree and therefore, has a spanning k-ended tree. Let $m \geq 2$ and let $x_i \in V(H_i)$ $(i = 1, ..., m)$. Clearly, $\{x_1, x_2, ..., x_m\}$ is an independent set of vertices. Since $d(x_i) \leq |V(H_i)| - 1$, we have

$$\sigma_2 \leq \sigma_m \leq \sum_{i=1}^{m} d(x_i) \leq \sum_{i=1}^{m} |V(H_i)| - m = n - m.$$

On the other hand, by the hypothesis, $\sigma_2 \geq n - k + 1$, implying that $m \leq k - 1$. ■

Proof of Theorem 2: Let G be a connected graph with $\sigma_2 \geq t_{k+1} - k + 1$ for some positive integer k.

Case 1: *G is Hamiltonian.*

By the definition, G has a spanning 1-ended tree T_1. Since $k \geq 1$, T_1 is a spanning k-ended tree.

Case 2: *G is not Hamiltonian.*

Let T_2 be a longest path in G.

Case 2.1: *$\sigma_2 \geq t_2$.*

By standard arguments, $G[V(T_2)]$ is Hamiltonian. If $t_2 < n$ then recalling that G is connected, we can form a path longer than T_2, contradicting the maximality of T_2. Otherwise G is Hamiltonian and we can argue as in Case 1.

Case 2.2: $\sigma_2 \le t_2 - 1$.

If $k = 1$ then by the hypothesis, $\sigma_2 \ge t_2$, implying that G is Hamiltonian and we can argue as in Case 1. Let $k \ge 2$. Extend T_2 to a k-ended tree T_k and assume that T_k is as large as possible. If T_k is a spanning tree then we are done. Let T_k be not spanning. Then $|End(T_k)| = k$ since otherwise we can form a new k-ended tree larger than T_k, contradicting the maximality of T_k. Now extend T_k to a largest $(k+1)$-ended tree T_{k+1}. Recalling that T_k is a largest k-ended tree, we get $|End(T_{k+1})| = k+1$ and therefore,

$$t_{k+1} \ge |T_{k+1}| = |T_2| + |T_{k+1} - T_2|.$$

Observing that $|T_2| = t_2$ and $|T_{k+1} - T_2| \ge |End(T_{k+1})| - 2 = k - 1$, we get

$$t_{k+1} \ge t_2 + k - 1 \ge \sigma_2 + k,$$

contradicting the hypothesis. ■

Proof of Theorem 3: Let G be a connected graph with $\sigma_3 \ge t_{k+1} - 2k + 4$ for some integer $k \ge 2$, and let $\overrightarrow{T_2} = x\overrightarrow{T_2}y$ be a longest path in G. If T_2 is a dominating path then we are done. Otherwise, since G is connected, we can choose a path $\overrightarrow{Q} = w\overrightarrow{Q}z$ such that $V(T_2 \cap Q) = \{w\}$ and $|Q| \ge 3$. Assume that $|Q|$ is as large as possible. Put $T_3 = T_2 \cup Q$. Since T_2 and Q are extreme, we have $N(x) \cup N(y) \subseteq V(T_2)$ and $N(z) \subseteq V(T_3)$. Let w^+ be the successor of w on T_2. If $xy \in E$ then $T_3 + xy - w^+w$ is a path longer than T_2, a contradiction. Let $xy \notin E$. By the same reason, we have $xz, yz \notin E$. Thus, $\{x, y, z\}$ is an independent set of vertices.

Claim 1: $N^-(x) \cap N^+(y) \cap N(z) = \emptyset$.
Proof: Assume the contrary.

Case 1: $v \in N^-(x) \cap N^+(y)$.

If $v = w$ then $xv^+ \in E$ and $T_3 + xv^+ - vv^+$ is a path longer than T_2, a contradiction. Suppose without loss of generality that $v \in V(w^+\overrightarrow{T_2}y)$. If $v = w^+$ then $T_3 + xv^+ - wv - vv^+$ is a path longer than T_2, a contradiction. Finally, if $v \in V(w^{+2}\overrightarrow{T_2}y)$ then

$$T_3 + xv^+ + yv^- - vv^- - vv^+ - ww^+$$

is a path longer than T_2, a contradiction.

Case 2: $v \in N^-(x) \cap N(z)$.

If $v \in V(x\overrightarrow{T_2}w^{-2})$ then

$$T_2 + xv^+ + zv - vv^+ - ww^-$$

is a path longer than T_2, a contradiction. Next, if $v = w^-$ then $T_2 + zw^- - ww^-$ is a path longer than T_2, a contradiction. Further, if $v = w$ then $T_2 + xv^+ - ww^+$ is a path longer than T_2, a contradiction. Finally, if $v \in V(w^+\overrightarrow{T_2}y)$ then

$$T_2 + xv^+ + zv - ww^+ - vv^+$$

is a path longer than T_2, a contradiction.

Case 3: $v \in N^+(y) \cap N(z)$.
By a symmetric argument, we can argue as in Case 2. Claim 1 is proved. ■

By Claim 1,

$$t_3 \geq |T_3| \geq |N^-(x)| + |N^+(y)| + |N(z)| + |\{z\}|$$
$$= d(x) + d(y) + d(z) + 1 \geq \sigma_3 + 1. \quad (1)$$

If $k = 2$ then by the hypothesis, $\sigma_3 \geq t_{k+1} - 2k + 4 = t_3$, contradicting (1). Let $k \geq 3$. If T_3 is a dominating 3-ended tree then clearly we are done. Otherwise $G - T_3$ contains an edge and we can extend T_3 to a largest 4-ended tree T_4 with $|T_4| \geq |T_3| + 2$. If $k = 3$, then by the hypothesis, $\sigma_3 \geq t_{k+1} - 2k + 4 = t_4 - 2$. On the other hand, by (1), $t_4 \geq |T_4| \geq |T_3| + 2 \geq \sigma_3 + 3$, a contradiction. Hence, $k \geq 4$. If T_4 is dominating, then we are done. Otherwise we can extend T_4 to a largest 5-ended tree T_5 with $|T_5| \geq |T_4| + 2 \geq |T_3| + 4$. This procedure may be repeated until a dominating $(r + 1)$-ended tree T_{r+1} is found. If $r + 1 \leq k$ then we are done. Let $r \geq k$. Then

$$t_{k+1} \geq |T_{k+1}| \geq |T_3| + 2(k - 2)$$
$$\geq \sigma_3 + 2k - 3 \geq t_{k+1} + 1,$$

a contradiction. The proof is complete. ■

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Գրաֆում k-ավարտ կմախքային և դոմինանտ ծառերի մասին

Ժ. Նիկողոսյան

Ամփոփում

Ծառի մեկ աստիճան ունեցող գագաթը կոչվում է տերև: Գրաֆում k-ից ոչ ավել տերև ունեցող ծառը կոչվում է k-ավարտ ծառ: Գրաֆում ամենամեծ k-ավարտ ծառի գագաթների քանակը նշանակվում է t_k-ով: G գրաֆի T ծառը կոչվում է դոմինանտ, եթե V(G-T)-ն գագաթների անկախ բազմություն է: Դիցուք, σ_2-ը (σ_3 -ը) գրաֆում ոչ հարևան զույգ (եռյակ) գագաթների աստիճանների հնարավոր ամենափոքր գումարն է: Քիչ տերևներով կմախքային ծառերին առնչվող ամենավաղ արդյունքը (որը ստացվել է հեղինակի կողմից 1976-ին) պնդում է՝ (∗) եթե n գագաթանի G կապակցված գրաֆը բավարարում է $\sigma_2 \geq n-k+1$ պայմանին ինչ- որ մի k դրական ամբողջ թվի համար, ապա G-ն ունի k-ավարտ կմախքային ծառ: Ներկա աշխատանքում ապացուցվում է, որ (∗)-ում կապակցվածության պայմանը կարելի է բաց թողնել: Երկրորդ արդյունքը (∗)-ի ուժեղացումն է՝ n-ը փոխարինելով t_{k+1}-ով (ընդհանրապես $t_{k+1} \leq n$): Երրորդ արդյունքը երկրորդի տարբերակն է՝ դոմինանտ k-ավարտ ծառերի համար: Բերված բոլոր արդյունքները ենթակա չեն բարելավման:

О k-висячих остовных и доминантных деревьях

Ж. Никогосян

Аннотация

Дерево с не более чем k-висячими вершинами называется k-висячим деревом. Число вершин максимального k-висячего дерева обозначается через t_k. Через σ_2 (σ_3)обозначается минимальная сумма степеней двух (трех) попарно несмежных вершин. Дерево T в графе G называется доминантным, если V(G-T) является независимым множеством вершин. В 1976 году доказано (автором): (∗) если n вершинный связный граф G удовлетворяет условию $\sigma_2 \geq n-k+1$ для некоторого целого числа k, то G содержит k-висячое остовное дерево. В настоящей работе доказывается, что условие связности в (∗) можно опускать. Второй результат является усилением (∗) с помощью замены n через t_{k+1} (напомним, что $t_{k+1} \leq n$). Приводится также версия второго результата для доминантных k-висячих деревьев. Все результаты неулучшаемы.