Characterization of Distributive and Standard Ideals in Semilattices

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ABSTRACT

This paper investigates the concepts of distributive ideal, dually distributive ideal and standard ideal in a join semilattice. It concerns with the property of ideals in a distributive semilattice. We obtain a characterization theorem for distributive (dually distributive) and standard ideal in a join semilattice. We establish the necessary and sufficient condition for a distributive ideal to be standard ideal. Finally, we bear out the fundamental theorem of homomorphism and Isomorphism theorem of standard ideal.

Keywords: Distributive ideal, Distributive semilattice, Dually Distributive ideal, Standard ideal, Join Semi Lattice.
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1. INTRODUCTION

The concept of distributive ideal, standard ideal and neutral ideal in a lattice L has been introduced and studied by Hashimoto (1952); and Gratzer and Schmidt (1961). Properties of distributive ideals of Birkhoff (1967) are considered in our work. In this paper we studied the notion of distributive (dually) ideal and standard ideal in a semilattice of Gratzer (1978) and produced a characterization theorem of standard ideal. The necessary and sufficient condition for a distributive ideal to be standard ideal was produced. Finally, the fundamental theorem of homomorphism and Isomorphism theorem of standard ideal were proved.

2. METHODOLOGY

Hashimoto (1952) and Gratzer and Schmidt (1961) have defined standard and distributive ideal and standard and distributive element in a lattice L and an example of standard ideal as a principal ideal. Also, they afforded a result that if "s" is a standard element and if "a" is an arbitrary element of lattice, then $a \wedge s$ is a standard element of the principal ideal (a] and this result is not valid for distributive elements. The properties of distributive ideals Birkhoff (1967) were considered for our work and we investigated the notion of distributive (dually) ideal, standard ideal in a semilattice of Gratzer (1978) and produced a characterization theorem of standard ideal. We established the necessary and sufficient condition for a distributive ideal to be standard ideal. Finally, we obtained the fundamental theorem of homomorphism and isomorphism theorem of standard ideal.

3. DISTRIBUTIVE IDEALS

3.1. Definition

A semilattice is a partially ordered set $(S, \leq)$ in which any two elements in S have the least upper bound in S.

3.2. Definition

A semilattice is a non empty set S with binary operation $\vee$ defined on it and satisfies the following:

Idempotent law	:	$a \vee a = a$ for all a in S,
Commutative law	:	$a \vee b = b \vee a$ for all a, b in S,
Associative law	:	$a \vee (b \vee c) = (a \vee b) \vee c$ for all a,b,c in S.

3.3. Theorem

In a semilattice S, define $a \leq b$ if and only if $a \vee b = b$ for all a, b in S. Then $(S, \leq)$ is an ordered set in which every elements has a least upper bound, conversely, given an ordered set P with that property, define $a \vee b = \text{l.u.b.}(a, b)$. Then $(P, \leq)$ is a semilattice.

3.4. Definition

A non empty subset D of a semilattice S is called an ideal if

(i) for x in D, y in D $\Rightarrow x \vee y$ in D , (ii) for x in D, t in S and $t \leq x \Rightarrow t$ in D.

3.5. Theorem

If I(S) denotes the set of all ideals of a semilattice S, then I(S) is a lattice with respective to the following:

(i) $D_1 \leq D_2$ if and only if $D_1 \subseteq D_2$

(ii) $D_1 \vee D_2 = \{ x \text{ in } S / x = x_1 \vee x_2$, where x_1 is in D_1, x_2 is in $D_2\}$

(iii) $D_1 \wedge D_2 = \{x \text{ in } S / x$ is in D_1 and x is in $D_2\}$; where D_1, D_2 are in I(S).

3. 6. Definition

The smallest ideal containing x in S is denoted by (x] and is given by (x] = $\{ s \text{ in } S / s \leq x\}$. Such ideal is called principal ideal generated by x.

3.7. Definition

An ideal D of a semilattice S is called distributive ideal if and only if

$D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all X, Y in I(S).

3.8. Definition

An ideal D of semilattice S is called dually distributive ideal if and only if

$D \wedge (X \vee Y) = (D \wedge X) \vee (D \wedge Y)$ for all X, Y in I(S).

3.9. Remark

The following example shows that an ideal need not be a distributive or dually distributive.

Consider the semilattice $S = \{1,a,b,c,a_n \ldots\ldots, a_1,a_0\}$ given in figure 1.

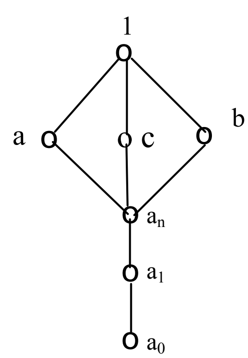

Figure 1. Semilattice ideal need not be a distributive (dually distributive).

Clearly $D=\{ a_0, a_1 \ldots.. a_n,a\}$, $X=\{ a_0, a_1 \ldots.. a_n,b\}$, and $Y=\{ a_0, a_1 \ldots.. a_n,c\}$ are ideals of S.

Now $X \wedge Y = \{a_0, a_1 \ldots\ldots.. a_n\}$, $D \vee (X \wedge Y) = \{a_0, a_1 \ldots\ldots.. a_n, a\}$,

$D \vee X = S$ and $D \vee Y = S$.

Therefore $(D \vee X) \wedge (D \vee Y) = S$ and $(D \vee X) \wedge (D \vee Y) \neq D \vee (X \wedge Y)$.

Hence D is not a distributive ideal of the semilattice S.

Also $X \vee Y = S$, $D \wedge (X \vee Y) = \{a_0, a_1 \ldots\ldots\ldots a_n, a\}$, $D \wedge X = \{a_0, a_1 \ldots\ldots\ldots a_n\}$,

$D \wedge Y = \{a_0, a_1 \ldots\ldots\ldots a_n\}$ and $(D \wedge X) \vee (D \wedge Y) = \{a_0, a_1 \ldots.\ a_n\}$.

Therefore $D \wedge (X \vee Y) \neq (D \wedge X) \vee (D \wedge Y)$

Hence D is not a dually distributive ideal of the semilattice S.

3.10. Result

If D_1 and D_2 are distributive ideals then $D_1 \vee D_2$ is also distributive.

3.10.1. Proof **:** Let D_1 and D_2 are distributive ideals of S. Then for any two ideals X and Y of S,

$(D_1 \vee D_2) \vee (X \wedge Y) = D_1 \vee (D_2 \vee (X \wedge Y))$

$= D_1 \vee [(D_2 \vee X) \wedge (D_2 \vee Y)]$ (as D_2 is distributive)

$= [(D_1 \vee (D_2 \vee X)] \wedge [D_1 \vee (D_2 \vee Y)]$ (as D_2 is distributive)

$= [(D_1 \vee D_2) \vee X] \wedge [(D_1 \vee D_2) \vee Y]$

Therefore $D_1 \vee D_2$ is a distributive ideal.

3.11. Definition

A semilattice S is said to be directed below if a, b $\in$ S, then there exists c such that $c \leq a$, $c \leq b$.

3.12. Definition

A semilattice S is called distributive if and only if $w \leq a \vee b$, where w, a, b in S $\Rightarrow$ there exists x, y in S such that $x \leq a$, $y \leq b$ and $w = x \vee y$.

3.13. Theorem

A semilattice S is distributive if and only if

(i) S is directed below.

(ii) The lattice I(S) of all ideals of S is a distributive lattice.

3.13.1. Proof **:** Suppose a semilattice S is distributive.

(i) <u>To prove that S is directed below</u>:

Let a, b are in S. Then $a \vee b \in S$. Since $a \leq a \vee b$ and S is distributive there exists x, y in S such that $x \leq a$, $y \leq b$ and $a = x \vee y$.Trivially $y \leq x \vee y = a$.

Therefore for a, b in S there exists y in S such that $y \leq a$, $y \leq b$ so that S is directed below.

(ii) <u>To prove that the lattice I(S) is distributive:</u>

Now $x \vee y \in D_1 \vee (D_2 \wedge D_3)$

$\Leftrightarrow x \in D_1, y \in (D_2 \wedge D_3) \Leftrightarrow x \in D_1, y \in D_2$ and $y \in D_3$

$\Leftrightarrow x \in D_1, y \in D_2$ and $x \in D_1, y \in D_3 \Leftrightarrow x \vee y \in D_1 \vee D_2$ and $x \vee y \in D_1 \vee D_3$

$\Leftrightarrow x \vee y \in (D_1 \vee D_2) \wedge (D_1 \vee D_3)$.

Therefore $D_1 \vee (D_2 \wedge D_3) = (D_1 \vee D_2) \wedge (D_1 \vee D_3)$.

Also $x \vee y \in (D_1 \wedge D_2) \vee (D_1 \wedge D_3)$

$\Leftrightarrow x \in D_1 \wedge D_2, y \in D_1 \wedge D_3 \Leftrightarrow x \in D_1$ and $x \in D_2, y \in D_1$ and $y \in D_3$

$\Leftrightarrow x \in D_1, y \in D_1$ and $x \in D_2, y \in D_3 \Leftrightarrow x \vee y \in D_1$ and $x \vee y \in D_2 \vee D_3$

$\Leftrightarrow x \vee y \in D_1 \wedge (D_2 \vee D_3)$.

Therefore $D_1 \wedge (D_2 \vee D_3) = (D_1 \wedge D_2) \vee (D_1 \wedge D_3)$ and I(S) is a distributive lattice.

Conversely, suppose that S is directed below and I(S) is distributive lattice.

Let $w \leq a \vee b$ where $a, b, w \in S$.

Now $(w] = (w] \wedge ((a] \vee (b]) = ((w] \wedge (a]) \vee ((w] \wedge (b]) = a_0 \vee a_1$,where $a_0 \in (a]$, $a_1 \in (b]$.

Hence there exists a_0, a_1 in S such that $a_0 \leq a$; $a_1 \leq b$ and $(w] = a_0 \vee a_1$.

Therefore S is distributive semilattice.

3.14. Definition A binary relation θ on a lattice L is called congruence relation if

(i) θ is reflexive : $x \equiv x\ (\theta)$ for all x in L

(ii) θ is symmetric : $x \equiv y\ (\theta) \Rightarrow y \equiv x\ (\theta)$ for all x, y in L

(iii) θ is transitive : $x \equiv y(\theta)$ and $y \equiv z(\theta)$

$\Rightarrow x \equiv z(\theta)$ for all x, y, z in L

(iv) θ satisfies substitution Property : $x \equiv x_1(\theta)$ and $y \equiv y_1(\theta)$

$\Rightarrow x \vee y \equiv x_1 \vee y_1(\theta)$ and $x \wedge y \equiv x_1 \wedge y_1\ (\theta)$ for all x, y, x_1, y_1 in L.

3.15. Theorem

Let D be an ideal of semilattice S. Then the following conditions are equivalent.

(i) D is distributive.

(ii) The map $\varphi : X \rightarrow D \vee X$ is a homomorphism of I(S) onto

$[D) = \{X \text{ in } I(S) \,/\, X \geq D\}$.

(iii) The binary relation θ_D on I(S) is defined by $X \equiv Y\ (\theta_D)$ if and only if

$D \vee X = D \vee Y$,, where X, Y in I(S) is a congruence relation.

3.15.1. Proof: Let D be an ideal of semilattice S.

<u>To prove that (i) $\Rightarrow$ (ii):</u>

Suppose (i) holds. Then $D\vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all X, Y in I(S)

Define a map $\varphi : X \rightarrow D \vee X$ by $\varphi (X) = D \vee X$. $\rightarrow (1)$

For X, Y in I(S), $\varphi (X \vee Y) = D \vee (X \vee Y) = (D \vee D) \vee (X \vee Y) = D \vee [D \vee (X \vee Y)]$

$= D \vee [D \vee X \vee Y] = D \vee (D \vee X) \vee Y)] = (D \vee X) \vee (D \vee Y) = \varphi (X) \vee \varphi (Y)$.

Similarly, $\varphi (X \wedge Y) = D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y) = \varphi (X) \wedge \varphi (Y)$.

Therefore φ is homomorphism.

Next let X in [D). Then $X \geq D$ so that $\varphi (X) = D \vee X = X$.

Therefore for any X in [D), there exists X in I(S) such that $\varphi (X) = X$ so that φ is homomorphism of I(S) onto [D).

<u>To prove (ii) $\Rightarrow$ (iii):</u>

Suppose the map $\varphi: X \rightarrow D \vee X$ is a homomorphism of I(S) onto

$[D) = \{ X \text{ in } I(S) \,/\, X \geq D\}$.Define the binary relation θ_D in I(S) as $X \equiv Y\ (\theta_D)$ if and only if $D \vee X = D \vee Y$ where X, Y in I(S). We shall show that the relation is congruence:

(a) For any X in I(S), $D \vee X = D \vee X$ trivially so that $X \equiv X\ (\theta_D)$ for all X in I(S). Therefore θ_D is reflexive.

(b) For X, Y in I(S), $X \equiv Y\ (\theta_D) \Rightarrow D \vee X = D \vee Y \Rightarrow D \vee Y = D \vee X \Rightarrow Y \equiv X\ (\theta_D)$.

Therefore θ_D is symmetry.

(c) For X, Y, Z in I(S), $X \equiv Y\ (\theta_D)$ and $Y \equiv Z\ (\theta_D) \Rightarrow D \vee X = D \vee Y$ and $D \vee Y = D \vee Z \Rightarrow D \vee X = D \vee Z \Rightarrow X \equiv Z\ (\theta_D)$. Therefore θ_D is Transitive.

(d) Substitution Property:

Suppose $X \equiv X_1\ (\theta_D)$ and $Y \equiv Y_1\ (\theta_D)$ for X, Y, X_1, Y_1 in I(S). Then $D \vee X = D \vee X_1$ and $D \vee Y = D \vee Y_1$. $\rightarrow (2)$

By (1) and (2) and since φ is a homomorphism,

$D \vee (X \vee Y) = \varphi(X \vee Y) = \varphi(X) \vee \varphi(Y)$,

$= (D \vee X) \vee (D \vee Y) = (D \vee X_1) \vee (D \vee Y_1) = \varphi(X_1) \vee \varphi(Y_1) = \varphi(X_1 \vee Y_1)$

$= D \vee (X_1 \vee Y_1)$.

Therefore $X \vee Y \equiv (X_1 \vee Y_1)\theta_D$.

Similarly we can prove that $X \wedge Y \equiv (X_1 \wedge Y_1)\theta_D$.

Therefore θ_D is a congruence relation.

<u>To show that (iii) $\Rightarrow$ (i):</u>

Suppose the binary relation θ_D defined by $X \equiv Y\ (\theta_D)$ if and only if $D \vee X = D \vee Y$ is a congruence relation.

For X, Y in I(S), $D \vee (D \vee X) = (D \vee D) \vee X = D \vee X \Rightarrow D \vee X \equiv X\ (\theta_D)$

$\Rightarrow X \equiv D \vee X\ (\theta_D)$ by symmetry. Similarly we can prove $Y \equiv D \vee Y\ (\theta_D)$.

Then by substitution property $X \wedge Y \equiv [(D \vee X) \wedge (D \vee Y)]\ (\theta_D)$.

Hence $D \vee (X \wedge Y) = D \vee (D \vee X) \wedge (D \vee Y) = (D \vee X) \wedge (D \vee Y)$.

Therefore D is distributive.

3.16. Result

Let D be an ideal of semilattice S. Then by applying the principle of duality to 2.15 we can have the equivalence of the following conditions.

(i) D is dually distributive.

(ii) The map $\varphi : X \rightarrow D \wedge X$ is a homomorphism of

I (S) onto (D] = {X in I(S) / $X \leq D$.}

(iii) The binary relation θ_D on I(S) is defined by $X \equiv Y\ (\theta_D)$ if and only if $D \wedge X = D \wedge Y$, where X, Y in I(S) is a congruence relation.

3.17. Definition

An ideal D of a semilattice S is called standard ideal if

$X \wedge (D \vee Y) = (X \wedge D) \vee (X \wedge Y)$ for all $X, Y \in$ I(s).

The following example shows that every ideal need not be a standard ideal.

3.18. Example

Let S = $\{a_0, a_1, a_2, \ldots a_n, a, b, c, d, 1\}$ be the semilattice as shown in figure 2 and let D = $\{a_0, a_1, a_2, \ldots a_n, a\} \subseteq$ S.

Then for all x, y $\in$ D, $x \vee y = a$ and a $\in$ D. Next let x $\in$ D, t $\in$ S and let $t \leq x$.

Now $t \leq x$ and x $\in$ D implies that $t = a_i$,

$0 \leq i \leq n$ or t = a. In either case t a $\in$ D and D is an ideal of S.

Similarly we can show that X = $\{a_0, a_1, a_2, \ldots a_n, b\}$ and

Y = $\{a_0, a_1, a_2, \ldots a_n, c\}$, are ideals of S. Now $X \wedge Y = \{a_0, a_1, \ldots a_n\}$; $D \vee Y = S$,

$X \wedge D = \{a_0, a_1, \ldots a_n\}$, $X \wedge (D \vee Y) = \{a_0, a_1, \ldots a_n, b\} = X$ and $(X \wedge D) \vee (X \wedge Y) = \{a_0, a_1, \ldots a_n\}$.

Therefore $X \wedge (D \vee Y) \neq (X \wedge D) \vee (X \wedge Y)$, shows that X is not a standard ideal.

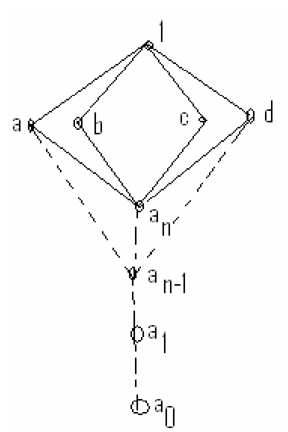

Figure 2. Semilattice ideal need not be a standard ideal.

3.19. Theorem

Let L be lattice and let θ be the binary relation on L defined by: $x \equiv y(\theta)$ if and only if $x \lneq y$. If θ is reflexive and symmetric, then θ is a congruence relation if and only if the following three properties are satisfied for all x, y, z in L.

(i) $x \equiv y(\theta) \Leftrightarrow x \wedge y \equiv (x \vee y)(\theta)$

(ii) $x \leq y \leq z, x \equiv y(\theta)$ and $y \equiv z(\theta) \Rightarrow x = z(\theta)$

(iii) $x \equiv y(\theta)$ and $x \leq y \Rightarrow x \wedge t = (y \wedge t)(\theta)$ and $x \vee t = (y \vee t)(\theta)$ for all $t \in$ L

3.19.1. Proof: Let the binary relation θ defined on a lattice L by:

$x \equiv y(\theta)$ if and only if $x \leq y$ be reflexive and symmetric. Assume that θ is a congruence relation. We prove that θ satisfies the properties (i), (ii) and (iii).

(i) Let $x \equiv y(\theta)$. Then $x \leq y$ and this implies $x \wedge y = x$ and $x \vee y = y$ so that $x \wedge y \equiv (x \vee y)(\theta)$.Conversely, suppose $x \wedge y \equiv (x \vee y)(\theta)$. Then $x \wedge y \leq x \vee y$ and this implies $x \wedge y \leq x$ or $x \wedge y \equiv x(\theta)$. Since θ is symmetric we have $x \equiv x \wedge y(\theta)$. Since θ is a congruence relation, this gives $x \equiv y(\theta)$.

(ii) Let $x \leq y \leq z$, then $x \leq y$ and $y \leq z$ and this implies $x \equiv y(\theta)$ and $y \equiv z(\theta)$.

Since θ is a congruence relation θ is transitive, so that $x \equiv z(\theta)$

(iii) Let $x \equiv y(\theta)$ and $x \leq y$, then for t ∈ L, $x \vee t \leq y \vee t$ implies $x \vee t \equiv (y \vee t)(\theta)$ and similarly $x \wedge t \leq y \wedge t$ for t in L, we have $x \wedge t \equiv y \wedge t(\theta)$.

Conversely, suppose θ satisfies the properties (i), (ii) and (iii).

We shall show that θ is a congruence relation.

Given θ is reflexive and symmetric.

Let $x \equiv y(\theta)$ and $y \equiv z(\theta)$. Then $x \leq y$ and $y \leq z$ and these imply $x \leq y \leq z$.

By property (ii) we have $x \equiv z(\theta)$.Therefore θ is transitive.

Let $x \equiv x_1(\theta)$ and $y \equiv y_1(\theta)$ so that $x \leq x_1$ and $y \leq y_1$.

This together with the property (iii) gives

$x \vee y \equiv x_1 \vee y_1$ and also $x \wedge y \equiv x_1 \wedge y_1(\theta)$ for $x_1, y_1, x, y \in \mathrm{L}$.

Thus θ satisfies substitution property.

Hence, θ is a congruence relation.

3.20. Theorem

Let D be an ideal of a semilattice S. Then, the following conditions are equivalent.

(1) D is standard ideal.

(2) The binary relation θ_D on I(S) defined by $X \equiv Y(\theta_D)$ if and only if $(X \wedge Y) \vee D_1 = X \vee Y$ for some $D_1 \leq D$ is a congruence relation.

(3) D is distributive and for all X, Y $\in$ I(S)

$D \wedge X = D \wedge Y$, $D \vee X = D \vee Y$ implies $X = Y$

3.20.1. Proof: Suppose D is an ideal of a semilattice S. Define the binary relation θ_D on I(S) as $X \equiv Y(\theta_D)$ if and only if $(X \wedge Y) \vee D_1 = X \vee Y$ for some $D_1 \leq D$.

(1) => (2) It is sufficient to prove that

(i) θ_D is reflexive

(ii) θ_D is symmetric

(iii) $X \equiv Y(\theta_D) \Leftrightarrow X \wedge Y \equiv (X \vee Y)(\theta)$

(iv) $X \leq Y \leq Z$, $X \equiv Y(\theta_D)$ and $Y \equiv Z(\theta_D) \Rightarrow X \equiv Z(\theta_D)$

(v) $X \leq Y$ and $X \equiv Y(\theta_D) \Rightarrow X \wedge Z \equiv Y \vee Z(\theta_D)$ for all X, Y, Z $\in$ I(s)

(i) Let X, Y $\in$ I(s) be arbitrary.

Then, by the definition of θ_D we have $(X \vee X) \vee D_1 = X \vee X$

for $X=D_1 \leq D \quad \Rightarrow X \equiv X(\theta_D)$ for all X $\in$ I(S). Thus θ_D is reflexive

(ii) For X, Y $\in$ I(S), $X \equiv Y(\theta_D) \Rightarrow (X \wedge Y) \vee D_1 = X \vee Y$. for some $D_1 \leq D$. $\Rightarrow (Y \wedge X) \vee D_1 = Y \vee X$ for some $D_1 \leq D \quad \Rightarrow Y \equiv X(\theta_D)$.

Thus θ_D is symmetric.

(iii) $X \equiv Y(\theta_D) \Leftrightarrow (X \wedge Y) \vee D_1 = X \vee Y$ for some $D_1 \leq D$

$$\Leftrightarrow [(X \wedge Y) \wedge (X \vee Y)] \vee D_1 = (X \wedge Y) \vee (X \vee Y) \text{ for some } D_1 \leq D,$$

and by taking $X=X \wedge Y$ and $Y= X \vee Y$ we have

$(X \wedge Y) \vee D_1 = X \vee Y$ for some $D_1 \leq D \Leftrightarrow X \wedge Y = (X \vee Y)\theta_D$

(iv) Suppose $X \leq Y \leq Z$, $X \equiv Y(\theta_D)$ and $Y \equiv Z(\theta_D)$

$$\Rightarrow (X \wedge Y) \vee D_1 = X \vee Y \text{ and } (Y \wedge Z) \vee D_2 = Y \vee Z \text{ for } D_1, D_2 \leq D.$$

Now, $X \vee D_1 = Y$ and $Y \vee D_2 = Z$.

Since, $X \le Y$ and $Y \le Z$ we have,

$X \vee (D_1 \vee D_2) = (X \vee D_1) \vee D_2 = Y \vee D_2 = Z$ for $D_1 \vee D_2 \le D$.

Then, $(X \wedge Z) \vee (D_1 \vee D_2) = X \vee (D_1 \vee D_2) = Z = X \vee Z$

Therefore $X \equiv Z(\theta_D)$.

(v) Suppose $X \le Y$ and $X \equiv Y(\theta_D) \Rightarrow (X \wedge Y) \vee D_1 = X \vee Y$ for some $D_1 \le D$.

Since $X \le Y \Rightarrow X \vee Z \le Y \vee Z \Rightarrow ((X \vee Z) \wedge (Y \vee Z)) \vee D_1 = (X \vee Z) \vee (Y \vee Z)$

Therefore $(X \vee Z) \equiv (Y \vee Z)(\theta_D)$.

Similarly we can prove that $(X \wedge Z) \equiv (Y \wedge Z)(\theta_D)$.

Therefore θ_D is a congruence relation.

To show that **(2)** $\Rightarrow$ **(3)**:

Suppose the binary relation θ_D on I(s) defined by $X \equiv Y(\theta_D)$ if and only if $(X \wedge Y) \vee D_1 = X \vee Y$ for some $D_1 \le D$ is a congruence relation.

First we prove that D is a Distributive ideal. . For all $X,Y \in I(s)$ we have $X \le D \vee X$

$\Rightarrow X \wedge (D \vee X) = X \Rightarrow [X \wedge (D \vee X)] \vee D = X \vee D \Rightarrow (*1)$

Also $X \vee (D \vee X) = X \vee D \Rightarrow (*2)$

So from (*1) and (*2) we get $[X \wedge (D \vee X)] \vee D = X \vee (D \vee X)$

This together with the definition of θ_D implies $X \equiv (D \vee X)(\theta_D) \Rightarrow (1)$

Similarly one can show that $Y \equiv (D \vee Y)(\theta_D) \Rightarrow (2)$

Since θ_D is a congruence relation, we have $X \wedge Y \equiv [(D \vee X) \wedge (D \vee Y)](\theta_D)$.

$\Rightarrow (X \wedge Y) \wedge [(D \vee X) \wedge (D \vee Y)] \vee D = (X \wedge Y) \vee (D \vee X) \wedge (D \vee Y)$

(by the def of θ_D)

$\Rightarrow (X \wedge Y) \vee D = (D \vee X) \wedge (D \vee Y)$ since $X \wedge Y \le (D \vee X) \wedge (D \vee Y)$.

Therefore $(X \wedge Y) \vee D = D \vee (X \vee Y) = (D \vee X) \wedge (D \vee Y)$ for all X, Y $\in$ I(S)

Therefore D is a distributive ideal.

Next let us assume that

$D \wedge X = D \wedge Y$ and $D \vee X = D \vee Y$, for all X, Y $\in$ I(S) we shall show that $X = Y$.

From (1) we have $X \wedge Y \equiv \left[(D \vee X) \wedge (D \vee Y)\right](\theta_D)$ (since θ_D is congruence relation)

$\equiv [D \vee X) \wedge (D \vee X)](\theta_D)$ (since $D \vee X = D \vee Y$)

$= (D \vee X)(\theta_D)$

$\equiv X(\theta_D)$ (by (1))

But $X \wedge Y \equiv X(\theta_D) \Rightarrow ((X \wedge Y) \wedge X) \vee D_1 = (X \wedge Y) \vee X$ for some $D_1 \leq D$

$\Rightarrow (X \wedge Y) \vee D_1 = X$. (since $X \in I(S)$ and I(S) is a lattice) $\Rightarrow$(3)

Also $D_1 \leq (X \wedge Y) \vee D_1 = X$, $D_1 \leq D \Rightarrow D_1 \leq D \wedge X = D \wedge Y \Rightarrow D_1 \leq D \wedge Y \leq Y \Rightarrow D_1 \leq Y$

and $D_1 \leq X, D_1 \leq Y \Rightarrow D_1 \leq X \wedge Y$.

So $(X \wedge Y) \vee D_1 = X \wedge Y$. But by (3), $(X \wedge Y) \vee D_1 = X$.

Therefore and im $X = X \wedge Y$ plies $X \leq Y$ ……..(4)

Similarly, we can show that $Y \leq X$ ……………(5)

Therefore from (4) and (5) we have X = Y.

To show that **(3) $\Rightarrow$ (1):**

Suppose D is distributive and for all $X,Y \in I(s)$

$D \wedge X = D \wedge Y$, $D \vee X = D \vee Y$ implies X = Y.

We shall show that D is standard ideal or

$X \wedge (D \vee Y) = (X \wedge D) \vee (X \wedge Y)$ for all $X,Y \in I(s)$.

For X, Y $\in$ I(S), let B = $X \wedge (D \vee Y)$ and C = $(X \wedge D) \vee (X \wedge Y)$.

We have $(X \wedge D) \vee (X \wedge Y) \leq X$ and $(X \wedge D) \vee (X \wedge Y) \leq D \vee Y$ so that

$(X \wedge D) \vee (X \wedge Y) \leq X \wedge (D \vee Y)$ or $C \leq B$.

This gives $D \wedge C \leq D \wedge B$ …….(1)

Now $D \wedge X \leq D$ and $D \wedge X \leq (D \wedge X) \vee (X \vee Y) = C$.

$\Rightarrow D \wedge X \leq D \wedge C \leq D \wedge B = D \wedge \left[X \wedge (D \vee Y)\right]$

$= \left[D \wedge (D \vee Y)\right] \wedge X = D \wedge X$

Therefore $D \wedge B = D \wedge C$.

Also since D is distributive $D \vee B = D \vee (X \wedge (D \vee Y)) = (D \vee X) \wedge (D \vee (D \vee Y))$

$= (D \vee X) \wedge (D \vee Y)$

$$= D \vee (X \wedge Y)$$

$$= (D \vee (D \wedge X)) \vee (X \wedge Y) \text{(by absorption property)}$$

$$= (D \vee (X \wedge D)) \vee (X \wedge Y)$$

$$= D \vee (X \wedge D) \vee (X \wedge Y)$$

$$= D \vee C$$

Therefore $D \vee B = D \vee C$.

Hence, $D \wedge B = D \wedge C$ and $D \vee B = D \vee C$.

So by (**3**) B = C and D is a standard ideal.

3.21. Theorem

Every standard ideal in a semilattice S is a distributive ideal but converse is not true.

3.21.1. Proof: By the theorem, 2.20 every standard ideal in a semilattice S is a distributive ideal. In the semilattice $S = \{a_0, a_1, a_2, \ldots a_n, a, b, c, d, 1\}$ as shown in figure 3 the ideal $D = \{a_0, a_1, a_2, \ldots a_n, 1\}$ is a distributive ideal but not a standard ideal.

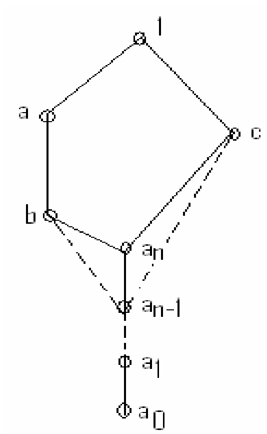

Figure 3. Semilattice distributive ideal is not a standard ideal.

3.22. Theorem

The necessary and sufficient condition for a distributive ideal D to be standard in a semilattice S is that $D \wedge X = D \wedge Y$ and $D \vee X = D \vee Y$ for all $X, Y \in I(S)$ implies X = Y.

3.22.1. Proof: Immediate from the theorem 2.20.

3.23 Statement

Suppose φ is a homomorphism of a semilattice S on to a semilattice S_1 and D is a standard ideal of S. The binary relation θ_D defined by $x \equiv y(\theta_D)$ if and only if $\varphi(x) = \varphi(y)$ where $x, y \in S$, is such that

(i) θ_D is a congruence relation on S

(ii) S/θ_D is a semilattice

(iii) $S/\theta_D \cong S_1$

3.23.1. Proof: (i) First let us show that θ_D is a congruence relation on S.

Since $\varphi(x) = \varphi(x)$ for $x \in S$, by definition of θ_D, we have $x \equiv x\ (\theta_D)$. Thus θ_D is a reflexive.

Suppose $x \equiv y(\theta_D)$ for $x, y \in S$.Then $\varphi(x) = \varphi(y)$ or $\varphi(y) = \varphi(x)$, which implies that $y \equiv x\ (\theta_D)$. Thus θ_D is symmetric.

Suppose $x \equiv y(\theta_D)$ and $y \equiv z(\theta_D)$ for $x, y, z \in S$.

Then $\varphi\ (x) = \varphi(y)$ and $\varphi(y) = \varphi(z)$ so that $\varphi(x) = \varphi(z)$ which implies $x \equiv z(\theta_D)$. Thus θ_D is transitive.

Suppose $x \equiv x_1\ (\theta)$ and $y \equiv y_1(\theta)$ .Then we have $\varphi\ (x) = \varphi(x_1)$ and $\varphi(y) = \varphi(y_1)$

Now $\varphi\ (x \vee y) = \varphi(x) \vee \varphi(y)$ (as φ is homomorphism)

$= \varphi(x_1) \vee \varphi(y_1) = \varphi(x_1 \vee y_1)$ (as φ is homomorphism).

This implies $x \vee y \equiv (x_1 \vee y_1)\ (\theta_D)$.

Similarly, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) = \varphi(x_1) \wedge \varphi(y_1)=\varphi(x_1 \wedge y_1)$ implies $x \wedge y \equiv x_1 \wedge y_1(\theta_D)$. Therefore θ_D satisfies substitution property.

Hence θ_D is a congruence relation.

(ii) To prove S/θ_D is a semilattice let $S/\theta_D = \{\ [x]\ \theta_D / x \in S\}$.

Define $\vee$ on S/θ_D by $[x]\ (\theta_D) \vee [y]\ (\theta_D) = (x \vee y)\ (\theta_D)$

where $[x]\ (\theta_D), [y]\ (\theta_D) \in S/\theta_D$. Since $x, y \in S$, $x \vee y \in S$ as S is a semilattice which implies $(x \vee y)\ (\theta_D) \in S/\theta_D$.

Therefore S/θ_D is a semilattice.

(iii) To prove $S/\theta_D \cong S_1$, let us define a map $\psi : S/\theta_D \rightarrow S_1$ by $\psi\ ([x]\ (\theta_D)) = \varphi(x)$ for $[x]\ (\theta_D) \in S/\theta_D$.

For $[x] (\theta_D), [y] (\theta_D) \in S/\theta_D$, $[x] (\theta_D) = [y] (\theta_D) \Rightarrow x \equiv y(\theta_D) \Rightarrow \varphi(x) = \varphi(y)$

$\Rightarrow \psi ([x] (\theta_D)) = \psi ([y] (\theta_D))$.

Therefore ψ is well defined.

Further, for $[x] (\theta_D), [y] (\theta_D) \in S/\theta_D$, $\psi ([x] (\theta_D)) = \psi ([y] (\theta_D)) \Rightarrow \varphi(x) = \varphi(y)$

$\Rightarrow x \equiv y (\theta_D) \Rightarrow [x] (\theta_D) = [y] (\theta_D)$.

This show ψ is one-one.

Let $z_1 \in S_1$. Then there exists $z \in S$ such that $\varphi(z) = z_1$, since φ is onto. So $[z] (\theta_D) \in S/\theta_D$ and $\psi ([z] (\theta_D)) = \varphi(z) = z_1$.

Therefore, for $z_1 \in S_1$, there exists $[z] (\theta_D) \in S/\theta_D$,such that $\psi ([z] (\theta_D)) = z_1$ so that ψ is onto.

Finally, let us show that ψ is homomorphism.

For $[x] (\theta_D), [y] (\theta_D) \in S/\theta_D$ we have

$$\begin{aligned} \psi ([x] (\theta_D)) \vee [y] (\theta_D)) &= \psi ((x \vee y) (\theta_D)) \\ &= \varphi (x \vee y) \\ &= \varphi(x) \vee \varphi(y) \\ &= \psi ([x] (\theta_D)) \vee \psi ([y] (\theta_D)). \end{aligned}$$

Therefore ψ is an onto homomorphism.Hence $S/\theta_D \cong S_1$.

3.24. Theorem

Let S be a semilattice, I is an ideal of S and D is a standard ideal of S such that $D \subseteq I$. Then

(i) I is a standard ideal in S and if and only if I/D is a standard ideal in S/D

(ii) $S/I \cong (S/D) / (I/D)$

3.24.1. Proof: Let S be a semilattice, I is an ideal of S and D a standard ideal of S such that $D \subseteq I$.

(i) Let I be a standard ideal in S.

To prove that I/D is a standard ideal in S/D, it is sufficient to prove that I/D is the homomorphic image of I. Now, define $\varphi : S \rightarrow S/D$ by $\varphi(x) = [x] \theta_D$, where $x \in S$.

As in theorem 2.23 one can see that φ is an onto homomorphism.

If we restrict φ from I to I/D, we have $\varphi(I)$ is an onto homomorphic image of I and $\varphi(I) = I/D$, which implies $\varphi(I) = I/D$ is a standard ideal.

Conversely suppose that I/D is a standard ideal of S/D.

For X, Y $\in$ I(S), let $\overline{X}, \overline{Y}$ be the homomorphic images of X and Y respectively under the map φ : S $\to$ S/D.

Since I/D is a standard ideal in S/D, we have

$\overline{X} \equiv \overline{Y}$ $(\theta_{I/D})$ (from characterization theorem for standard ideal)

$\Rightarrow \left(\overline{X} \wedge \overline{Y}\right) \vee \overline{I}_1 = \overline{X} \vee \overline{Y}$ for some $\overline{I}_1 \leq \overline{I} = I/D$

$\Rightarrow (X \wedge Y) \vee I_1 = X \vee Y$ for some $I_1 \leq I$.

$\Rightarrow X \equiv Y(\theta_I)$ $\Rightarrow$ I is a standard ideal in S.

(ii) To prove that S/I $\cong$ (S/D)/(I/D) define g: S $\to$ (S/D)/(I/D) by g(x) = $[\,\overline{x}\,]\ \theta_{(I/D)}$ where x $\in$ S.

For x = y where x, y $\in$ S. $\Rightarrow [\,\overline{x}\,]\ \theta_{(I/D)} = [\,\overline{y}\,]\ \theta_{(I/D)} \Rightarrow$ g(x) = g(y). Therefore g is well defined.

To show that g is onto, let $[\,\overline{x}\,]\ \theta_{(I/D)} \in$ (S/D)/(I/D). Then $\overline{x} \in$ S/D for some x $\in$ S and g(x) = $[\,\overline{x}\,]\theta_{(I/D)}$. Therefore g is onto.

Finally, for x, y $\in$ S, g(x $\vee$ y) = $[\,\overline{X} \vee \overline{Y}\,]\ \theta_{(I/D)} = [\,\overline{X}\,]\ \theta_{(I/D)} \vee [\,\overline{Y}\,]\ \theta_{(I/D)}$ = g(x) $\vee$ g(y).

This shows that g is a homomorphism

Clearly ker g = I, so that by fundamental theorem of homomorphism S/I $\cong$ (S/D)/(I/D).

3.25. Theorem

A semilattice S is distributive $\Leftrightarrow$ Every ideal D of S is a standard ideal.

3.25.1. Proof: Assume that in a semilattice S every ideal D is a standard ideal.

Then by the theorem 2.20, D is a distributive ideal and $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$ for all X, Y $\in$ I(s). This is true for all D so that I(s) is a distributive lattice.

This implies that S is a distributive semilattce by Theorem 2.13

Conversely, suppose that a semilattice S is a distributive semillatice and D is an ideal of S. Now S is a distributive semilattice of I(s)

$\Rightarrow$ I(s) is a distributive lattice, by Theorem 2.13

$\Rightarrow$ Every element in I(s) is standard, since I(s) does not contain N_5 or M_3

$\Rightarrow$ Every ideal D of S is a standard ideal.

4. CONCLUSION

In this paper, we investigated the notions of distributive (dually) ideal and standard ideal in a semilattice, and established a characterization theorem of standard ideal. We ascertain that set of all ideals of a semilattice is a lattice. We attain the equivalent conditions for a semilattice (ideal of a semilattice) to be distributive (dually distributive). We confirm that every ideal need not be a standard ideal. We define a congruence relation on a lattice and achieve its equivalent conditions. We get hold of the equivalent conditions for an ideal of a semilattice to be a standard ideal. We set up that every standard ideal in a semilattice is a distributive ideal but converse is not true. We take the necessary and sufficient condition for a distributive ideal to be standard in a semilattice. We concluded with the result that a semilattice is distributive if and only if every ideal of it is a standard ideal.

5. REFERENCES

Birkhoff, G. 1967. Lattice theory. Amer. Math. Soc., Callog Publication, XXV Providence, R.I.

Gratzer, G.1962. A characterization of Neutral elements in Lattices. *Magyar Tub Akab. Mat. Kutato Int Kozi*., **7**:191-192.

Gratzer, G. 1978. General Lattice theory. Academic press Inc.

Gratzer, G & Schmidt, E.T.1961.Standard ideals in lattices. *Acta Math. Sci. Hung.,* **12**: 17-86.

Hashimoto, J. 1952. Ideal theory for Lattices. *Math. Japan*, **2**:149-186.

Hossain, M.A & Noor, A. S. A. 2007. Modular and Standard filters of a Directed above Meet Semilattice. *Thammasat. Int.J.Sc.Tech.*, **12(4)**:1-5.