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Fixed Point Theorems in Complex-Valued b-Metric Spaces

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ABSTRACT

Fixed point theorems in complex-valued metric spaces have been extensively studied following the extension of the definition of a metric to a complex-valued metric. The definition of b-metric, which is considered as an extension of a metric, is also extended to complex-valued b-metric. This extension causes the study of existence and uniqueness of fixed point in complex-valued b-metric spaces. Fixed point theorems that have been studied in b-metric spaces are now of interest whether the theorems are still satisfied in complex-valued b-metric spaces.
In this paper, we study the existence and uniqueness of fixed point in complex valued b-metric spaces. We consider some fixed point theorems in b-metric spaces to be extended to fixed point theorems in complex valued b-metric spaces. By using the definition of complex-valued b-metric spaces and the properties of partial order on complex numbers, we have obtained some fixed point theorems in complex-valued b-metric spaces which are derived from the corresponding theorems in b-metric spaces.

Keywords: b-metric spaces, complex-valued b-metric spaces, fixed point theorems

INTRODUCTION

Fixed point theorems have been the main topic of many researches in various spaces including metric spaces. Fixed point theorems in metric spaces have been studied extensively by many researchers as in [1], [2], and [3]. After Bakhtin [4] introduced the notion of b-metric spaces in 1989, researchers started to develop fixed point theorems in the spaces, for example in [5], [6], and [7]. In 2011, Azzam et. al [8] introduced a new metric space called complex-valued metric space as an extension of the classical metric space in terms of the defined metric. Fixed point theorems are still of interest to study in the new space. This is shown by many articles talked about fixed point theorems in complex valued-metric spaces as in [9], [10], dan [11]. The extension of the notion of metric into complex-valued metric causes the extension of b-metric spaces into complex-valued b-metric spaces. In this paper, we observe fixed point theorems which have been established in b-metric space whether the theorems are still applicable in the spaces.

METHODS

First, we recall some basic definitions and properties of b-metric spaces given in [5] and of complex-valued b-metric spaces given in [12].

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b-Metric Spaces

A b-metric space is considered as a generalization of a metric space regarding the definition of the metric as defined as follows.

Definition 1 (see [5]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to [0, \infty)$ is called a b-metric space if for all $x, y, z \in X$ the following conditions are satisfied.

$d(x, y) = 0$ if and only if $x = y$;
$d(x, y) = d(y, x)$;
$d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Example 2. Let (X, d) be a metric space. The function $\rho(x, y)$ is defined by $\rho(x, y) = \left(d(x, y)\right)^2$. Then (X, ρ) is a b-metric space with coefficient $s = 2$. This can be seen from the nonnegativity property and triangle inequality of metric to prove the property (iii).

Complex-Valued b-Metric Spaces

Unlike in real numbers which has completeness property, order is not well-defined in complex numbers. Before giving the definition of complex-valued b-metric spaces, we define partial order in complex numbers (see [12]). Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define partial order $\preccurlyeq$ on $\mathbb{C}$ as follows. We have $z_1 \preccurlyeq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. This means that we would have $z_1 \preccurlyeq z_2$ if and only if one of the following conditions holds:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

If one of the conditions (i), (ii), and (iii) holds, then we will write $z_1 \precnsim z_2$. Particularly, we have $z_1 \prec z_2$ if the condition (iv) is satisfied.

From the definition of partial order on complex numbers, partial order has the following properties:

(i) If $0 \preccurlyeq z_1 \precnsim z_2$, then $|z_1| < |z_2|$;

(ii) $z_1 \preccurlyeq z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$;

(iii) If $z \in \mathbb{C}, a, b \in \mathbb{R}$ and $a \leq b$, then $az \preccurlyeq bz$.

A b-metric on a b-metric space is a function having real value. Based on the definition of partial order on complex numbers, real-valued b-metric can be generalized into complex-valued b-metric as follows.

Definition 3 (see [13]). Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{C}$ is called a complex-valued b-metric on X if, for all $x, y, z \in \mathbb{C}$, the following conditions are satisfied:

(i) $0 \preccurlyeq \mathrm{d(x,y)}$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \preccurlyeq \mathrm{s[d(x,z) + d(z,y)]}$.

The pair (X, d) is called a complex valued b-metric space.

Example 4. Let $X = \mathbb{C}$. Define the mapping $d: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in X$. Then $(\mathbb{C}, d)$ is complex-valued b-metric space with $s = 2$.

Definition 5 (see [13]). Let (X, d) be a complex-valued b-metric space.

(i) A point $x \in X$ is called interior point of set $A \subseteq X$ if there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in Y: d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called limit point of a set A if for every $0 \prec r \in \mathbb{C}$, $B(x,r) \cap (A - \{x\}) \neq \emptyset$.
(iii) A subset $A \subseteq X$ is open if each element of A is an interior point of A.
(iv) A subset $A \subseteq X$ is closed if each limit point of A is contained in A.

Definition 6 (see [13])**.** Let (X,d) be a complex valued b-metric space, (x_n) be a sequence in X and $x \in X$.

(i) (x_n) is convergent to $x \in X$ if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x) \prec r$. Thus x is the limit of (x_n) and we write $\lim(x_n) = x$ or $(x_n) \to x$ as $n \to \infty$.
(ii) (x_n) is said to be Cauchy sequence if for every $0 \prec r \in \mathbb{C}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $d(x_n, x_{n+m}) \prec r$, where $m \in \mathbb{N}$.
(iii) If for every Cauchy sequence in X is convergent, then (X,d) is said to be a complete complex valued b-metric space.

Lemma 7 (see [13])**.** Let (X,d) be a complex valued b-metric space ad let (x_n) be a sequence in X. Then (x_n) converges to x if and only if $|d(x,y)| \to 0$ as $n \to \infty$.

Lemma 8 (see [13]). Let Let (X,d) be a complex valued b-metric space ad let (x_n) be a sequence in X. Then (x_n) is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

RESULTS AND DISCUSSIONS

The following fixed point theorem is studied by Mishra,et.al [5] in b-metric spaces:

Theorem 9. Let (X,ρ) be a complete b-metric space and let $T: X \to X$ be a function with the following

$$\rho(Tx,Ty) \leq a\rho(x,Tx) + b\rho(y,Ty) + c\rho(x,y), \forall x,y \in X$$

where $a, b,$ and c are non-negative real numbers and satisfy $a + s(b+c) < 1$ for $s \geq 1$ then T has a uniqe fixed point.

This theorem is extended to a fixed point theorem in complex valued b-metric spaces as follows.

Theorem 10. Let (X,d) be a complete complex valued b-metric space and let $T: X \to X$ be a function with the following

$$d(Tx,Ty) \preccurlyeq ad(x,Tx) + bd(y,Ty) + cd(x,y), \forall x,y \in X$$

where $a, b,$ and c are non-negative real numbers and satisfy $a + s(b+c) < 1$ for $s \geq 1$ then T has a uniqe fixed point.

Proof. Let $x_0 \in X$ and (x_n) be a sequence in X such that

$$x_n = Tx_{n-1} = T^n x_0.$$

Note that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
d(x_{n+2}, x_{n+1}) &= d(Tx_{n+1}, Tx_n) \\
&\preccurlyeq ad(x_{n+1}, Tx_{n+1}) + bd(x_n, Tx_n) + cd(x_{n+1}, x_n) \\
&\preccurlyeq ad(x_{n+1}, x_{n+2}) + bd(x_n, x_{n+1}) + cd(x_{n+1}, x_n) \\
&\preccurlyeq ad(x_{n+2}, x_{n+1}) + (b+c)d(x_{n+1}, x_n)
\end{aligned}$$

$$(1-a)d(x_{n+2}, x_{n+1}) \preccurlyeq (b+c)d(x_{n+1}, x_n)$$

$$d(x_{n+2}, x_{n+1}) \preccurlyeq \left(\frac{b+c}{1-a}\right) d(x_{n+1}, x_n).$$

If we take $\gamma = \frac{b+c}{1-a}$ and continuing this process, then we have

$$d(x_{n+2}, x_{n+1}) \preccurlyeq \gamma^{n+1} d(x_1, x_0).$$

For $m \in \mathbb{N}$,

$$\begin{aligned}
d(x_n, x_{n+m}) &\preccurlyeq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\
&\preccurlyeq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+m}) \\
&\preccurlyeq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \cdots + s^m d(x_{n+m-1}, x_{n+m}) \\
&\preccurlyeq s\gamma^n d(x_0, x_1) + s^2\gamma^{n+1} d(x_0, x_1) + s^3\gamma^{n+2} d(x_0, x_1) + \cdots + s^m \gamma^{n+m-1} d(x_0, x_1) \\
&\preccurlyeq s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]
\end{aligned}$$

Now,
$|d(x_n, x_{n+m})| \leq |s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]|$
$= s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]$
Since $a + s(b + c) < 1$ for $s \geq 1$ then $\gamma < 1$ and $s\gamma < 1$. Taking limit $n \to \infty$, we have $\gamma^n \to 0$. This implies $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. By Lemma 8, the sequence (x_n) is Cauchy sequence in X. Since X is a complete complex valued b-metric space then (x_n) is a convergent sequence.

Let x^* be the limit of (x_n). We show that x^* is a fixed point of T.
We have
$d(x^*, Tx^*) \preccurlyeq s[d(x^*, x_n) + d(x_n, Tx^*)]$
$= s[d(x^*, x_n) + d(Tx_{n-1}, Tx^*)]$
$\preccurlyeq s[d(x^*, x_n) + ad(x_{n-1}, Tx_{n-1}) + bd(x^*, Tx^*) + cd(x_{n-1}, x^*)]$
$(1 - bs)d(x^*, Tx^*) \preccurlyeq s[d(x^*, x_n) + ad(x_{n-1}, Tx_{n-1}) + cd(x_{n-1}, x^*)]$
$d(x^*, Tx^*) \preccurlyeq \dfrac{s}{1 - bs}[d(x^*, x_n) + ad(x_{n-1}, Tx_{n-1}) + cd(x_{n-1}, x^*)]$
$d(x^*, Tx^*) \preccurlyeq \dfrac{s}{1 - bs}[d(x^*, x_n) + ad(x_{n-1}, x_n) + cd(x_{n-1}, x^*)]$
Taking the absolute value of both sides, we get
$$\begin{aligned}|d(x^*, Tx^*)| &\leq \frac{s}{1 - bs}|d(x^*, x_n) + ad(x_{n-1}, x_n) + cd(x_{n-1}, x^*)| \\ &\leq \frac{s}{1 - bs}(|d(x^*, x_n)| + |ad(x_{n-1}, x_n)| + |cd(x_{n-1}, x^*)|) \\ &\leq \frac{s}{1 - bs}(|d(x^*, x_n)| + a\gamma^n|d(x_1, x_0)| + c|d(x_{n-1}, x^*)|)\end{aligned}$$
Since (x_n) converges to x^*, taking $n \to \infty$ implies $|d(x^*, x_n)| \to 0$ and $|d(x_{n-1}, x^*)| \to 0$. Thus for $n \to \infty$, we have
$$\lim|d(x^*, Tx^*)| = 0.$$
This yields
$$x^* = Tx^*.$$
Hence x^* is a fixed point of T.

Now we show the uniqueness of the fixed point of T. We assume that there are two fixed points of T which are $x = Tx$ and $y = Ty$. Thus,
$$\begin{aligned}d(x, y) &= d(Tx, Ty) \\ &\preccurlyeq ad(x, Tx) + bd(y, Ty) + cd(x, y) \\ &\preccurlyeq ad(x, x) + bd(y, y) + cd(x, y) \\ &\preccurlyeq cd(x, y).\end{aligned}$$
Taking the absolute value of both sides, we have
$$|d(x, y)| \leq |cd(x, y)|.$$
This implies $d(x, y) = 0$. Based on the properties of complex valued b-metric, we get $x = y$. This completes the proof.

Corollary 11. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$. Let $T: X \to X$ be a mapping (for some fixed n) satisfying
$$d(T^n x, T^n y) \preccurlyeq ad(x, T^n x) + bd(y, T^n y) + cd(x, y)$$
for all $x, y \in X$ where a, b, c are non-negative real numbers with $a + s(b + c) < 1$. Then T has a unique fixed point in X.
Proof. Suppose $S = T^n$, then by the Theorem 10, there is a fixed point u such that
$$S(u) = u.$$
Thus $T^n(u) = u$.
We have
$$\begin{aligned}d(T(u), u) &= d\big(T(T^n(u)), u\big) \\ &= d\Big(T^n\big(T(u)\big), T^n(u)\Big) \\ &\preccurlyeq ad\Big(T(u), T^n\big(T(u)\big)\Big) + bd\big(u, T^n(u)\big) + cd(T(u), u) \\ &= ad\Big(T(u), T\big(T^n(u)\big)\Big) + bd(u, u) + cd(T(u), u)\end{aligned}$$

$= ad(T(u), T(u)) + bd(u, u) + cd(T(u), u)$
$= cd(T(u), u).$

Taking the absolute value of both sides, we have

$$|d(T(u), u)| \leq c|d(T(u), u)|.$$

Hence

$$(1 - c)|d(T(u), u)| \leq 0.$$

This must be

$$|d(T(u), u)| = 0,$$

which is

$$T(u) = u = T^n(u).$$

So T has a fixed point.
To prove its uniqueness, suppose v is also a fixed point of T. We need to prove that $u = v$.
We see that

$$\begin{aligned} d(u, v) &= d\big(T^n(u), T^n(v)\big) \\ &\preccurlyeq ad\big(u, T^n(u)\big) + bd\big(v, T^n(v)\big) + cd(u, v) \\ &= ad(u, u) + bd(v, v) + cd(u, v) \\ &= cd(u, v). \end{aligned}$$

Taking the absolute value of both sides we get

$$(1 - c)d(u, v) \leq 0.$$

Since $c < 1$, this yields

$$d(u, v) = 0.$$

Thus $u = v$.
This completes the uniqueness of the fixed point.

Theorem 12. Let (X, d) be a complete complex-valued b-metric space with constant $s \geq 1$ and let $T: X \to X$ be a mapping such that

$$d(Tx, Ty) \preccurlyeq \alpha d(x, Ty) + \beta d(y, Tx)$$

for every $x, y \in X$, where $\alpha, \beta \geq 0$ with $\alpha s < \frac{1}{1+s}$ or $\beta s < \frac{1}{1+s}$. Then T has a fixed point in X. Moreover, if $\alpha + \beta < 1$, then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and (x_n) be a sequence in X such that

$$x_n = Tx_{n-1} = T^n x_0.$$

Now, for every $n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d(Tx_{n+1}, Tx_n) \\ &\preccurlyeq \alpha d(x_{n+1}, Tx_n) + \beta d(x_n, Tx_{n+1}) \\ &= \alpha d(x_{n+1}, x_{n+1}) + \beta d(x_n, x_{n+2}) \\ &= \beta d(x_n, x_{n+2}) \\ &\preccurlyeq s\beta d(x_n, x_{n+1}) + s\beta d(x_{n+1}, x_{n+2}) \end{aligned}$$

$$(1 - \beta s)d(x_{n+1}, x_{n+2}) \preccurlyeq s\beta d(x_n, x_{n+1})$$

$$d(x_{n+1}, x_{n+2}) \preccurlyeq \frac{s\beta}{1 - \beta s} d(x_n, x_{n+1}).$$

Taking $\gamma = \frac{s\beta}{1-\beta s}$ and continuing the process yields

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\preccurlyeq \gamma d(x_n, x_{n+1}) \\ &\preccurlyeq \gamma^2 d(x_{n-1}, x_n) \\ &\vdots \\ &\preccurlyeq \gamma^{n+1} d(x_0, x_1). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(x_n, x_{n+m}) &\preccurlyeq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\ &\preccurlyeq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+m}) \\ &\preccurlyeq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \cdots + s^m d(x_{n+m-1}, x_{n+m}) \\ &\preccurlyeq s\gamma^n d(x_0, x_1) + s^2\gamma^{n+1} d(x_0, x_1) + s^3\gamma^{n+2} d(x_0, x_1) + \cdots + s^m\gamma^{n+m-1} d(x_0, x_1) \\ &\preccurlyeq s\gamma^n d(x_0, x_1)[1 + s\gamma + (s\gamma)^2 + \cdots + (s\gamma)^{m-1}]. \end{aligned}$$

Since $\beta s < \frac{1}{1+s}$ then $\gamma \in \left(0, \frac{1}{s}\right)$ and $s\gamma < 1$. Taking limit $n \to \infty$, we have $\gamma^n \to 0$. This implies $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$. By Lemma 8, the sequence (x_n) is Cauchy sequence in X. Since X is a complete complex valued b-metric space then (x_n) is a convergent sequence.
Suppose that (x_n) converges to $x^* \in X$. We show that x^* is a fixed point of T.

$$\begin{aligned} d(x^*, Tx^*) &\preccurlyeq s[d(x^*, x_n) + d(x_n, Tx^*)] \\ &= sd(x^*, x_n) + sd(Tx_{n-1}, Tx^*) \\ &= sd(x^*, x_n) + s\alpha d(x_{n-1}, Tx^*) + s\beta d(Tx_{n-1}, x^*) \\ &= sd(x^*, x_n) + s\alpha d(x_{n-1}, Tx^*) + s\beta d(x_n, x^*) \\ &= (s + s\beta) d(x^*, x_n) + s^2\alpha d(x_{n-1}, x^*) + s^2\alpha d(x^*, Tx^*) \end{aligned}$$

$$(1 - \alpha s^2) d(x^*, Tx^*) \preccurlyeq (s + s\beta) d(x^*, x_n) + s^2 \alpha d(x_{n-1}, x^*).$$

Since (x_n) converges to x^* that is $|d(x^*, x_n)| \to 0$ as $n \to \infty$. and taking the absolute value of both sides, we have

$$|d(x^*, Tx^*)| \leq 0.$$

This means that $|d(x^*, Tx^*)| = 0$, i.e. $x^* = Tx^*$. So the limit of the sequence is a fixed point of T.
Now we suppose that $\alpha + \beta < 1$. Assume that there are $u, v \in X$ such that $u = Tu$ and $v = Tv$.

$$\begin{aligned} d(u, v) &= d(Tu, Tv) \\ &\preccurlyeq \alpha d(u, Tv) + \beta d(v, Tu) \\ &= \alpha d(u, v) + \beta d(v, u) \\ &= (\alpha + \beta) d(u, v). \end{aligned}$$

This means that $|d(u, v)| \leq (\alpha + \beta)|d(u, v)|$. Since $\alpha + \beta < 1$, then we must have $|d(u, v)| = 0$, i.e. $u = v$. The proof of the uniqueness of the fixed point of T when $\alpha + \beta < 1$ is complete.

Mir [14] studied the following fixed point theorem in b-metric spaces and we generalize it into complex-valued b-metric spaces.

Theorem 13. Let (X, d) be a complete b-metric space with constant $s \geq 1$ and define the sequence (x_n) by the recursion

$$x_n = Tx_{n-1} = T^n x_0, n = 1,2, \dots$$

Let $T: X \to X$ be a mapping for which there exists $\mu \in \left[o, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \mu[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$. Then there exists $x^* \in X$ such that $x_n \to x^*$ and x^* is unique fixed point of T.

Theorem 14. Let (X, d) be a complete complex-valued b-metric space with constant $s \geq 1$. Let $T: X \to X$ be a mapping for which there exists $\mu \in \left[o, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \mu[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$. Then there exists unique fixed point of T.
Proof. We construct a sequence (x_n) in X by

$$x_n = Tx_{n-1} = T^n x_0, n = 1,2, \dots$$

for any $x_0 \in X$. We show that this sequence (x_n) is Cauchy sequence in X and hence is convergent sequence.
For every $n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d(Tx_{n+1}, Tx_n) \\ &\preccurlyeq \mu\big(d(x_{n+1}, Tx_{n+1}) + d(x_n, Tx_n)\big) \\ &= \mu\big(d(x_{n+1}, x_{n+2}) + d(x_n, Tx_n)\big) \end{aligned}$$

$$(1 - \mu) d(x_{n+2}, x_{n+1}) \preccurlyeq \mu d(x_n, Tx_n)$$

$$d(x_{n+2}, x_{n+1}) \preccurlyeq \frac{\mu}{1 - \mu} d(x_{n+1}, x_n)$$

If this process continues, for all $n \in \mathbb{N}$ we have

$$d(x_{n+2}, x_{n+1}) \preccurlyeq \left(\frac{\mu}{1 - \mu}\right)^{n+1} d(x_1, x_0).$$

Therefore,

$$\begin{aligned} d(x_n, x_{n+m}) &\preccurlyeq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+m})] \\ &\preccurlyeq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_{n+m}) \end{aligned}$$

$$\begin{aligned}
&\precsim sd(x_n,x_{n+1}) + s^2 d(x_{n+1},x_{n+2}) + s^3 d(x_{n+2},x_{n+3}) + \cdots + s^m d(x_{n+m-1},x_{n+m})\\
&\precsim s\left(\frac{\mu}{1-\mu}\right)^n d(x_0,x_1) + s^2\left(\frac{\mu}{1-\mu}\right)^{n+1} d(x_0,x_1) + s^3\left(\frac{\mu}{1-\mu}\right)^{n+2} d(x_0,x_1) + \cdots\\
&\qquad + s^m\left(\frac{\mu}{1-\mu}\right)^{n+m-1} d(x_0,x_1).\\
&\precsim s\left(\frac{\mu}{1-\mu}\right)^n d(x_0,x_1)\left[1 + s\left(\frac{\mu}{1-\mu}\right) + \left(s\left(\frac{\mu}{1-\mu}\right)\right)^2 + \cdots + \left(s\left(\frac{\mu}{1-\mu}\right)\right)^{m-1}\right].
\end{aligned}$$

Note that $\mu \in \left[o, \frac{1}{2}\right)$ implies $\left(\frac{\mu}{1-\mu}\right) \in [0,1)$. Taking the absolute value of both sides gives

$$|d(x_n,x_{n+m})| \to 0 \text{ as } n \to \infty.$$

Thus (x_n) is Cauchy sequence in X and is therefore a convergent sequence. Suppose $x^* \in X$ is the limit of sequence (x_n). We show that x^* is a fixed point of T.

Using the properties of b-metric and the mapping T, we get

$$\begin{aligned}
d(x^*,Tx^*) &\precsim s[d(x^*,x_n) + d(x_n,Tx^*)]\\
&= sd(x^*,x_n) + sd(Tx_{n-1},Tx^*)\\
&= sd(x^*,x_n) + s\mu d(x_{n-1},Tx_{n-1}) + s\mu d(x^*,Tx^*)\\
&= sd(x^*,x_n) + s\mu d(x_{n-1},x_n) + s\mu d(x^*,Tx^*)\\
(1-s\mu)d(x^*,Tx^*) &\precsim sd(x^*,x_n) + s\mu d(x_{n-1},x_n)\\
&= sd(x^*,x_n) + s\mu\left(\frac{\mu}{1-\mu}\right)^{n-1} d(x_1,x_0)\\
d(x^*,Tx^*) &\precsim \frac{1}{(1-s\mu)}\left(sd(x^*,x_n) + s\mu\left(\frac{\mu}{1-\mu}\right)^{n-1} d(x_1,x_0)\right).
\end{aligned}$$

Taking the absolute value of both sides and taking $n \to \infty$, we have

$$\lim d(x^*,Tx^*) = 0.$$

This implies x^* is a fixed point of T, i.e. $x^* = Tx^*$.

Now we assume that there exists another fixed point v of T, that is $v = Tv$.

We have

$$\begin{aligned}
d(x^*,v) &= d(Tx^*,Tv)\\
&\precsim \mu\big(d(x^*,Tx^*) + d(v,Tv)\big)\\
&= \mu\big(d(x^*,x^*) + d(v,v)\big)\\
&= 0.
\end{aligned}$$

Taking the absolute value of both sides we obtain $|d(x^*,v)| \leq 0$ which is a contradiction to the distinct fixed points. Hence x^* is the unique fixed point of T.

CONCLUSION

Complex valued b-metric spaces are considered as a generalisation of b-metric spaces by changing the definition of real valued metric into complex valued metric. This change is expected to bring wider applications of fixed point theorems. In this paper, we extend fixed point theorems in b-metric spaces into fixed point theorems in complex valued b-metric spaces. By using the properties of partial order on complex numbers, a mapping T on a real valued b-metric space which has unique fixed point still has unique fixed point in a complex valued b-metric space.

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