CAUCHY –Jurnal Matematika Murni dan Aplikasi
Volume 7(4) (2023), Pages 597-607
p-ISSN: 2086-0382; e-ISSN: 2477-3344

On the Dominant Local Resolving Set of Vertex Amalgamation Graphs

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ABSTRACT

In graph theory, there is a new topic of the dominant local metric dimension which be symbolized by $Ddim_l(H)$ and it was a combination of local metric dimension and dominating set. There are some terms in this topic that is dominant local resolving set and dominant local basis. An ordered subset W_l is said a dominant local resolving set of G if W_l is dominating set and also local resolving set of G. While dominant local basis is a dominant local resolving set with minimum cardinality. This study uses literature study method by observing the local metric dimension and dominating number before detecting the dominant local metric dimension of the graphs. After obtaining some new results, the purpose of this research is how the dominant local metric dimension of vertex amalgamation product graphs. Some special graphs that be used are star, friendship, complete graph and complete bipartite graph. Based on all observation results, it can be said that the dominant local metric dimension for any vertex amalgamation product graph depends on the dominant local metric dimension of the copied graphs and how the terminal vertex is constructed.

Keywords: dominant local metric dimension; vertex amalgamation; star; friendship; complete bipartite

INTRODUCTION

Metric dimension and dominating set are graph topics with numerous variations. For metric dimensions, there are more than five development concepts, such as partition dimension, local metric dimension, complement metric dimension, central metric dimension, fractional metric dimension, and star metric dimension. More results about metric dimension and its expansion can be seen at [1] about the simultaneous local metric dimension, the local metric dimension of amalgamation [2] and corona product of star and path graph [3] , complement metric dimension [4], fractional metric dimension [5], and the new one is star metrc dimension [6].

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between any two a and b of $V(G)$ is denoted by $d(a, b)$. It be defined as shortest path from a to b. The resolving set of G is an ordered set which can be written as W, where $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$ and $r(v|W) = (d(v, w_1), d(v, w_2), \dots., d(v, w_k))$ is defined as

Submitted: December 19, 2022 Reviewed: December 20, 2022 Accepted: March 16, 2023
DOI: http://dx.doi.org/10.18860/ca.v7i4.18891

representation of $v \subseteq V(G)$ to W by using the concept of distance. The rule to select resolving set of G is every vertex of $V(G)$ should have different representation to W. The minimum cardinality of W is called the basis of G [7]. The number of basis is referred to the metric dimension because the concept of this topic is based on distance. Next, we introduce the differences between metric dimension and local metric dimension. In the metric dimension, all vertices must have different representations to the resolving set, whereas in the local metric dimension, only any two adjacent vertices must be different. It also can be said that a vertex's representation can be the same as another vertices even though they are not adjacent. [8]. Some examples of the local metric dimension have been published at [9], [10] and [11]. More clearly, there is a paper which describe the similarity between metric dimension and the local metric dimension [12]. In 2021, Umilasari *et al* introduced the new concept, which is a combination of dominating set and local metric dimension. They defined that an ordered subset W_l is said a dominant local resolving set of G if W_l is dominating set and also local resolving set of G. For a clearer understanding of this term, you can see the illustration in Figure 1.

All the vertices in the graph of Figure 1 (a) can be dominated by x_4. But the vertices which adjacent $\{V(G)\backslash x_4\}$ have same representation to x_4. While $\{x_2, x_3\}$ in Figure 1 (b) is a local resolving set. As can be seen, each pair of adjacent vertices has a different representation to $\{x_2, x_3\}$. The problem is x_5 and x_6 cannot be dominated by x_2 or x_3. If we take two vertices like in Figure 1(c), $\{x_2, x_4\}$ can dominate all vertices of the graph, the representation of any two vertices to $\{x_2, x_4\}$ is different. Therefore, $\{x_2, x_4\}$ are elements of a dominant local metric dimension of the graph.

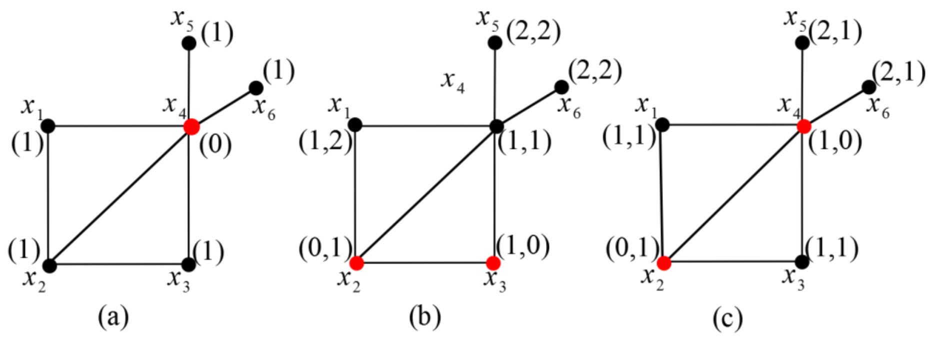


Figure 1. The Illustration of Dominant Local Metric Dimension of Graphs

After obtaining some new results, in this paper we continue to expand on how the dominant local metric dimension of a vertex amalgamation product graphs. The vertex amalgamation product of a graph H, denoted by $amal(H, v, k)$, is defined as creating a new graph by gluing k-copies of H in a terminal vertex v [13]. In this paper, we determine the dominant local metric dimension of the vertex amalgamation for some special graphs, which are star, complete graph, complete bipartite graph, and friendship graph. To make it easier to present each of the theorems produced, several theorems are given below, which can be seen in [14].

Lemma 1. Let G be a connected graph. If there is no local dominant resolving set with cardinality p, then for every $S \subseteq V(G)$ with $|S| < p$ is not a local dominant resolving set.
Lemma 2. Let G be a connected graph and $W_l \subseteq V(G)$. For every $v_i, v_j \in W_l$ then $r(v_i|W_l) \neq r(v_j|W_l)$.

Some new results about the dominant local metric dimension of star, complete graph, complete bipartite graph, and friendship graph are given in Table 1.

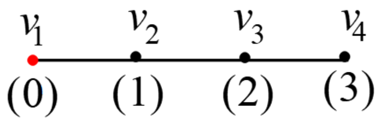
Table 1. Dominant Local Metric Dimension of Special Graphs [14][15][16]

Graphs	**Dominating Number ($\gamma(G)$)**	**Local Metric Dimension ($\dim_l(G)$)**	**Dominan Local Metric Dimension ($\dim_l(G)$)**
Star (S_n)	$\gamma\ (S_n) = 1$	$\dim_l\ (S_n) = 1$	$\mathrm{Ddim}_l(S_n) = 1$
Complete (K_n)	$\gamma\ (K_n) = 1$	$\dim_l\ (K_n) = n - 1$	$\mathrm{Ddim}_l(K_n) = n - 1$
Complete Bipartite ($K_{m,n}$)	$\gamma(K_{m,n}) = 2$	$\dim_l(K_{m,n}) = 2$	$\mathrm{Ddim}_l(K_{m,n}) = 2$
Friendship (F_n)	$\gamma(F_n) = 1$	$\dim_l(F_n) = n$	$\mathrm{Ddim}_l(F_n) = n$

METHODS

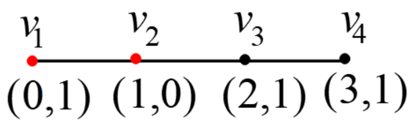
In this research, there are several procedures. We start by determining the special graphs to be operated by the vertex amalgamation product and observing the local metric dimensions and dominating number of the graphs. Then, we construct the vertex amalgamation product graphs from the special graphs that we have chosen. We continue by labeling the vertex and attempting to find the least dominant local basis. This is accomplished by observing and recording the representation of each vertex which can be dominated and has different representation from the local resolving set (two non-neighbouring vertex can have the same representation). The minimum local dominant basis is then determined. In summary, the procedures of the research can be seen in the following flowchart in Figure 2. We also give some examples of each step below.

a) Let $G = P_4$
b) $dim_l(P_4) = 1$ and $\gamma(P_4) = 2$, it can be seen at [16]
c) Let $|W_l| = 1$, $W_l = \{v_1\}$

Illustration: 

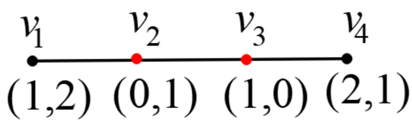
Based on the illustration above, v_1 can't dominate v_3 and v_4. When we choose $W_l = \{v_2\}$, $W_l = \{v_3\}$, $W_l = \{v_4\}$ the condition remains the same. Minimally, there exist one vertex that can't be dominated.

d) Let $|W_l| = 2$, $W_l = \{v_1, v_2\}$

Illustration: 

We can see that v_4 can't be dominated by v_1 or v_2 .

e) Let $|W_l| = 2$, $W_l = \{v_2, v_3\}$

Illustration: 

Since $\forall v_i v_j \in E(P_4)$, $r(v_i|W_l) \neq r(v_j|W_l)$ then W_l is basis local of P_4. All vertices of $V(P_4)$ can be dominated by v_2 and v_3. Therefore, W_l is dominant local basis of P_4 or $Ddim_l(P_4) = 2$.

To more clearly understand this research method, we can see the flowchart in Figure 2.

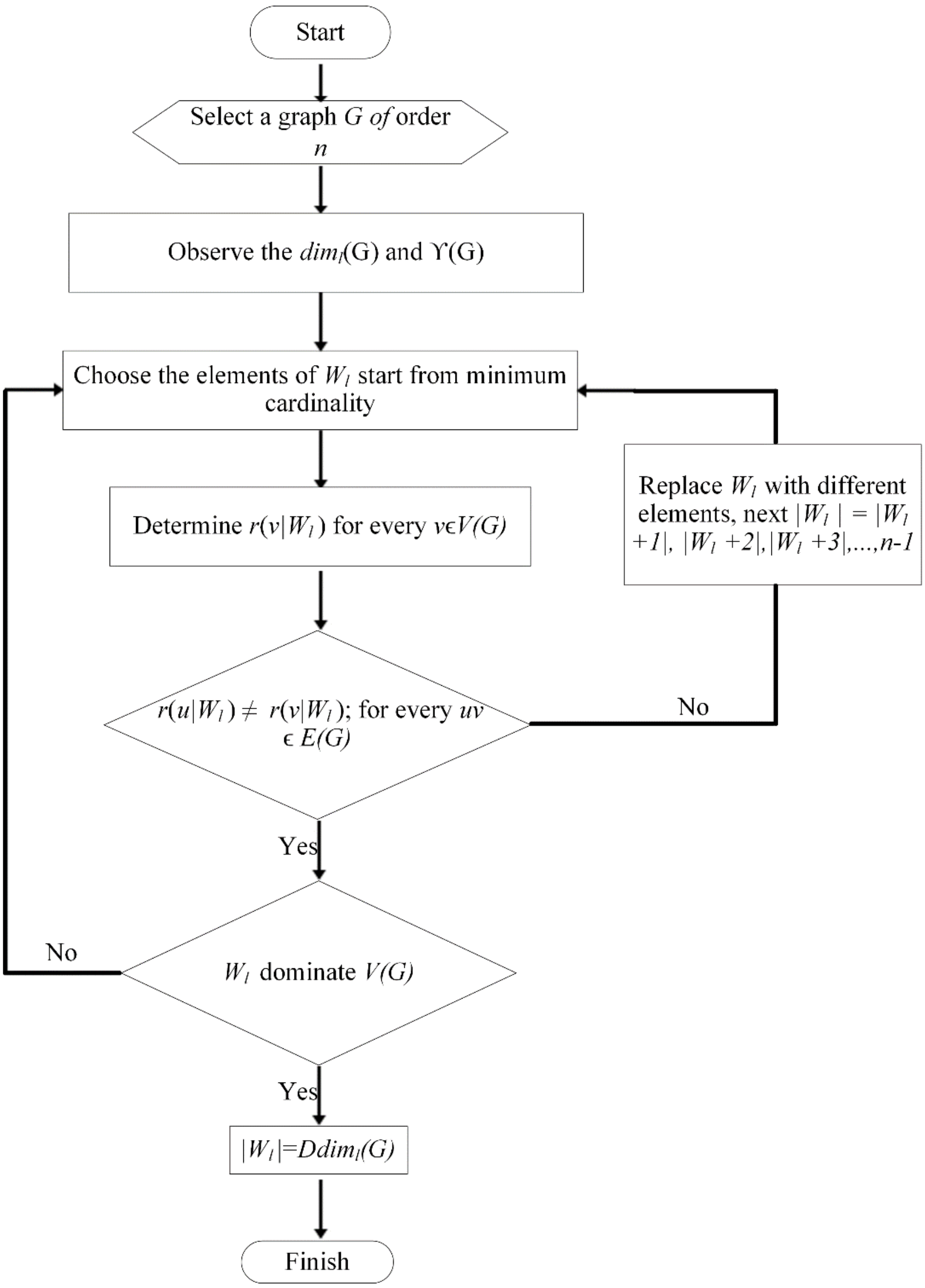


Figure 2. Flowchart for Determining the Minimum Dominant Local Resolving Set of Graphs

RESULTS AND DISCUSSION

In this section, we determine the dominant local metric dimension of the vertex amalgamation product for some special graphs, which are star, complete graph, complete bipartite graph, and friendship graph.

Theorem 1. Let $amal(S_n, v, k)$ is a vertex amalgamation of star with the order of star is $n \geq 3$, then

$$Ddim_l\ (amal(S_n, v, k)) = \begin{cases} 1, & v\ is\ center\ vertex\ of\ S_n \\ k, & v\ is\ pendant\ of\ S_n \end{cases}$$

Proof.
Case 1. v is center vertex of star
It is very clearly to see that $amal(S_n, v, k) \cong S_n$, then by the Table 1 we can conclude that $Ddim_l\left(amal(S_n, v, k)\right) = 1$. ■
Case 2. v is pendant vertex of star
Let the vertex set of $amal(S_n, v, k)$ is $V\left(amal(S_n, v, k)\right) = \{v, v_j, u_{1j}, u_{2j}, \dots, u_{ij} | v, u_i \in V(S_n), i = 1,2, \dots, n-2, j = 1,2, \dots, k\}$ and the edge set is $E\left(amal(S_n, v, k)\right) = \{vv_j, v_j u_{ij} | i = 1,2,3, \dots, n-2, j = 1,2,3, \dots, k\}$. Choose $W_l = \{v_j\}$ is the local basis of $amal(S_n, v, k)$ for every $j = 1,2,3, \dots, k$, $|W_l| = k$. We can show below that the representation of every two adjacent vertices of $V(amal(S_n, v, k))$ is different.

i. For $u_{ij}v_j \in E\left(amal(S_n, v, k)\right)$
Since v_j is element of W_l, then there exist 0 on i^{th} element in $r\left(v_j | W_l\right)$, while for $r\left(u_{ij} | W_l\right)$ there are no zero elements, hence $r\left(v_j | W_l\right) \neq r\left(u_{ij} | W_l\right)$.

ii. For $vv_j \in E\left(amal(S_n, v, k)\right)$
Since v_j is element of W_l, then there exist 0 on i^{th} element in $r\left(v_j | W_l\right)$, while for $r(v | W_l)$ there are no zero elements, hence $r\left(v_j | W_l\right) \neq r(v | W_l)$.

By i and ii therefore W_l is a local resolving set of $amal(S_n, v, k)$. Further, because v_j is adjacent to v and u_{ij}, so we can said that W_l is a dominant local resolving set of $amal(S_n, v, k)$. Next, take any $S \subseteq V(amal(S_n, v, k))$ with $|S| < |W_l|$. Without loss of generality, let $|S| = |W_l| - 1$ with $W_l = \{v_j | j = 1,2,3, \dots, k-1\}$, so u_{ik} are not adjacent to S. So S is not a dominant local resolving set of $amal(S_n, v, k)$. Based on **Lemma 1**, any set T with $|T| < |S|$ is not a dominant local resolving set of G. Therefore, $W_l = \{v_j\}$ is a dominant local basis of $amal(S_n, v, k)$. Then its is proven that $Ddim_l\left(amal(S_n, v, k)\right) = k$ for v is pendant vertex of star of S_n. ■

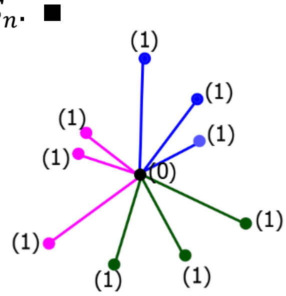


Figure 3. $Ddim_l\left(amal(S_4, v, 3)\right) = 1$.

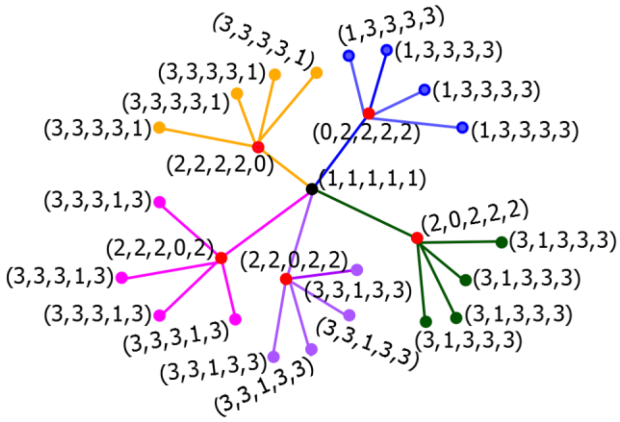


Figure 4. $Ddim_l\left(amal(S_6, v, 5)\right) = 5$

Figure 3 gives the illustration of dominant local metric dimension of $amal(S_n, v, k)$ for v is the center vertex of S_n. While in Figure 4, v is the pendant of Star. The next theorem, we will show the dominan local metric dimension of complete graph. Because the graphs are regular, then we can select any vertex of complete graph as the linkage vertex.

Theorem 2. Let $amal(K_n, v, p)$ is a vertex amalgamation of complete graph with the order of complete graph is $n \geq 3$, then $Ddim_l\left(amal(K_n, v, p)\right) = p \times (Ddim_l(K_n) - 1)$.

Proof. Let $V\left(amal(K_n, v, p)\right) = \left\{v, v_{ij} \middle| i = 1,2,3, \dots, n-1, j = 1,2,3, \dots, p\right\}$ and the edge set of $amal(K_n, v, p)$ is $E\left(amal(K_n, v, p)\right) = \left\{vv_{ij}, v_{xj}v_{yj} \middle| i = 1,2,3, \dots, n-1, j = 1,2,3, \dots, p, v_x v_y \in E(K_n), x \neq y\right\}$. The j-th copy of K_n with $j = 1,2,3, \dots, p$ is called $(K_n)_j$. Let B be a local dominant basis of K_n, B_j is a local dominant basis of $(K_n)_j$, so that for every $j = 1,2,3, \dots, p$, $|B_i| = |B|$. Select $W_l = \bigcup_{j=1}^{m} B_j$, with $B_j = \left\{v_{ij} \middle| j = 1,2,3, \dots, n-2\right\}$ for every $j = 1,2,3, \dots, p$, then $|W_l| = p(n-2)$. By Lemma 2 for every $v_{ij}, v_{kl} \in B_i$ then $r\left(v_{ij} \middle| W_l\right) \neq r(v_{kl}|W_l)$. Next, we take any two adjacent vertices in $V\left(amal(K_n, v, p)\right) \backslash W_l$. Let $x, y \in V\left(amal(K_n, v, p)\right) \backslash W_l$, then for $xy = v, v_{nj} \in V\left(amal(K_n, v, p)\right) \backslash W_l$ with $j = 1,2,3, \dots, p$. Since $amal(K_n, v, p)$ is a connected graph, $d\left(v_{nj}, z\right) = d\left(v_{nj}, v\right) + d(v, z)$for every $z \in B_i$ so that $d\left(z, v_{nj}\right) \neq d(z, v)$ caused $r\left(v_{nj} \middle| B_i\right) \neq r(v|B_i)$. Because of $B_i \subseteq W_l$ then $r\left(v_{nj} \middle| W_l\right) \neq r(z|W_l)$.

Based on the explanation above, $W_l = \bigcup_{i=1}^{m} B_i$ is a local resolving set of $amal(K_n, v, p)$. Since, every $v_{ij} \in W_l$ with $i = 1,2,3, \dots, n-2$ and $j = 1,2,3, \dots, p$ is adjacent to v and v_{nj}, then W_l is a dominating set. So that, $W_l = \bigcup_{j=1}^{p} B_j$ is a local dominant resolving set of $amal(K_n, v, p)$. Take any $S \subseteq V(G)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, then there exists j such as S contains maximal $\left|B_j\right| - 1$ elements of $(K_n)_j$. Since B_j is a local dominant basis of $(K_n)_j$ then there exist two vertices in $(K_n)_j$ that have the same representation toward S, so that S is not a local dominant resolving set of $amal(K_n, v, p)$. Based on **Lemma 1** then $W_l = \bigcup_{j=1}^{p} B_j$ is a local dominant basis of $amal(K_n, v, p)$. By Table 1 we know that $|B_i| = Ddim_l((K_n)_i) - 1$, then it has been proven that $Ddim_l\left(amal(K_n, v, p)\right) = p \times (Ddim_l(K_n) - 1)$. ■

The example of dominant local metric dimension of vertex amalgamation complete graph be given in Figure 5. The graph show that $amal(K_4, v, 3)$ has the dominant local metric dimension equals six.

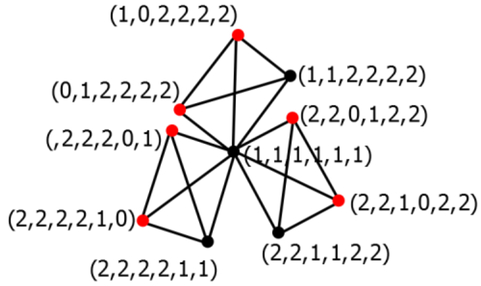


Figure 5. $Ddim_l\left(amal(K_4, v, 3)\right) = 6$.

Theorem 3. Let $amal\left(K_{m,n}, v, p\right)$ is a vertex amalgamation of complete bipartite graph with the order is $m, n \geq 2$, then $Ddim_l\left(amal\left(K_{m,n}, v, p\right)\right) = p + 1$.

Proof. Let the vertex set of $K_{m,n}$ is $V(K_{m,n}) = \{a_i | i = 1,2,\dots,m\} \cup \{b_j | j = 1,2,\dots,n\}$, and the edge set is $E(K_{m,n}) = \{a_i b_j | i = 1,2,\dots,m; j = 1,2,\dots,n\}$. $V\left(amal(K_{m,n}, v, p)\right) = \{v, a_{ik}, b_{jk} | i = 2,3,\dots,m, j = 1,2,\dots,n, k = 1,2,\dots,p\}$ and the edge set is $E\left(amal(K_{m,n}, v, p)\right) = \{vb_{jk}, a_{ik}b_{jk} | i = 2,3,\dots,m, j = 1,2,\dots,n, k = 1,2,\dots,p\}$. Choose, $W_l = \{v, b_{1k}\}$ for every $k = 1,2,3,\dots,p$, then $|W_l| = p+1$. We can show below that the representation of every two adjacent vertices of$V\left(amal(K_{m,n}, v, p)\right)$ is different.

i. For $vb_{jk} \in E\left(amal(K_{m,n}, v, p)\right)$
Since v is element of W_l, then there exist 0 on 1^{st} element in $r(v|W_l)$, while for $r(b_{jk}|W_l)$ there are no zero elements except b_{1k} the representation to W_l is $r(b_{1k}|W_l) = (1,0)$, hence $r(v|W_l) \neq r(b_{jk}|W_l)$.

ii. For $a_{ik}b_{jk} \in E\left(amal(K_{m,n}, v, p)\right)$
Since for $i = 2,3,\dots,m, j = 1,2,\dots,n$, $d(a_{ik}, v) = d(a_{ik}, b_{jk}) + d(b_{jk}, v)$ hence $r(a_{ik}|W_l) \neq r(b_{jk}|W_l)$.

From the two explanations above we know that W_l is the local resolving set of $amal(K_{m,n}, v, p)$. Since v is adjacent to b_{jk} for $j = 1,2,\dots,n$ and $k = 1,2,\dots,p$. The vertex b_{1k} is adjacent to a_{ik} for $i = 2,3,\dots,m$ and $k = 1,2,\dots,p$, thus W_l is dominant local resolving set of $amal(K_{m,n}, v, p)$. Take any $S \subseteq V(G)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$ the two possibilities below:

a. If $v \notin W_l$
$v \notin W_l$, then all vertices b_{jk} with $j = 2,3,\dots,n$ and $k = 1,2,\dots,p$ cannot be dominated by W_l.
b. If $v \in W_l$
$v \in W_l$, then there exist $b_{1k} \notin W_l$ for $k = 1,2,\dots,p$. Without loss of generality suppose that $b_{11} \notin W_l$. It means that a_{i1} cannot be dominated by W_l for $i = 2,3,\dots,m$.

Therefore, from two possibilities above S is not a local dominant resolving set of $amal(K_{m,n}, v, p)$ or we can conclude that $W_l = p+1$ is dominant local basis of $amal(K_{m,n}, v, p)$. Hence, we get $Ddim_l\left(amal(K_{m,n}, v, p)\right) = p+1$. ■

The example of a dominant local basis for vertex amalgamation of a complete bipartite graph is depicted as red vertices in Figure 5, where $Ddim_l\left(amal(K_{3,3}, v, 3)\right) = 4$.

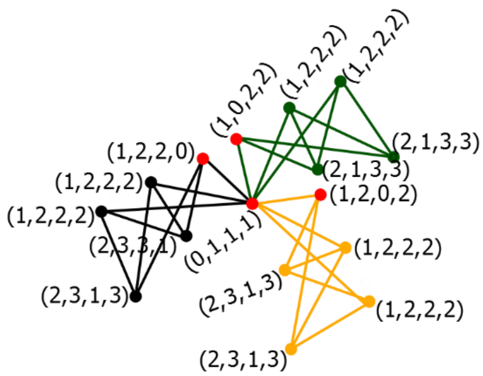


Figure 6. $Ddim_l\left(amal(K_{3,3}, v, 3)\right) = 4$.

Theorem 4. Let $amal(F_n, v, p)$ is a vertex amalgamation of friendship graph with the order is $n \geq 3$ then

$$Ddim_l\left(amal(F_n, v, p)\right) = \begin{cases} Ddim_l(F_n), & v \text{ is a center vertex of } F_n \\ 1 + p(Ddim_l(F_n) - 1), & v \text{ is not a center vertex of } F_n \end{cases}$$

Proof.

Case 1. v is a center vertex of F_n

It is very clearly to see that $amal(F_n, v, p) \cong F_n$, then by the Table 1 we can conclude that $Ddim_l\left(amal(F_n, v, p)\right) = Ddim_l(F_n)$. ■

Case 2. v is not a center vertex of F_n

Let $V\left(amal(F_n, v, p)\right) = \{v, v_i, x_{ik}, y_{ij} | i = 1,2,3, \dots, p; j = 1,2,3, \dots, n; k = 2,3,4, \dots, n\}$ and $E\left(amal(F_n, v, p)\right) = \{v_i x_{ik}, v_i y_{ij}, x_{ik} y_{ij}, v y_{i1} | i = 1,2, \dots, p; j = 1,2, \dots, n; k = 2,3,4, \dots, n\}$. The i-th copy of F_n with $i = 1,2,3, \dots, p$ is called $(F_n)_i$. Let B be a local dominant basis of F_n, B_i is a local dominant basis of $(F_n)_i$, so that for every $i = 1,2,3, \dots, p$, $|B_i| = |B| = n$. Select $W_l = \{v\} \cup_{i=1}^{p} (B_i - 1)$, suppose $B_i - 1 = \{x_{ik} | k = 2,3, \dots, n\}$ for every $i = 1,2,3, \dots, p$, then $|W_l| = 1 + p(n-1)$. By **Lemma 2** for every $x_{ab}, x_{cd} \in B_i$ then $r(x_{ab}|W_l) \neq r(x_{cd}|W_l)$. Next, we take any two adjacent vertices in $V(G) \backslash W_l$. Let $v_i, y_{ij} \in V(G) \backslash W_l$ with $i = 1,2,3, \dots, p$ and $j = 2,3, \dots, n$. Since $amal(F_n, v, p)$ is a connected graph, for $d\left(y_{ij}, v\right) = d\left(y_{ij}, v_i\right) + d(v_i, v)$ for $j \neq 1, v \in W_l$ so that $d\left(v, y_{ij}\right) \neq d(v, v_i)$ caused $r\left(y_{ij}|W_l\right) \neq r(v_i|W_l)$. Then, W_l is local resolving set of $amal(F_n, v, p)$. Moreover, since v adjacent to v_i and x_{ik} adjacent to y_{ij}, hence W_l is dominant local resolving set of $amal(F_n, v, p)$. Take any $S \subseteq V\left(amal(F_n, v, p)\right)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$ the two possibilities below.

a) If $v \notin W_l$
   $v \notin W_l$, then all vertices y_{i1} with $i = 1,2, \dots, p$ cannot be dominated by W_l.
b) If $v \in W_l$
   $v \in W_l$, then there exist $x_{ik} \notin W_l$ for $k = 2,3 \dots, n$. Without loss of generality suppose that $x_{1n} \notin W_l$. It means that y_{1n} cannot be dominated by W_l.

Therefore, from two possibilities above S is not a local dominant resolving set of $amal(F_n, v, p)$ or we can conclude that $|W_l| = 1 + p(n-1)$ is dominant local basis of $amal(F_n, v, p)$. Hence, we get $Ddim_l\left(amal(F_n, v, p)\right) = 1 + p(Ddim_l(F_n) - 1)$. ■

Figure 6 gives an axample of $amal(F_3, v, 3)$, where v is not the center vertex of friendship. Those graph has dominant local resolving set equals seven.

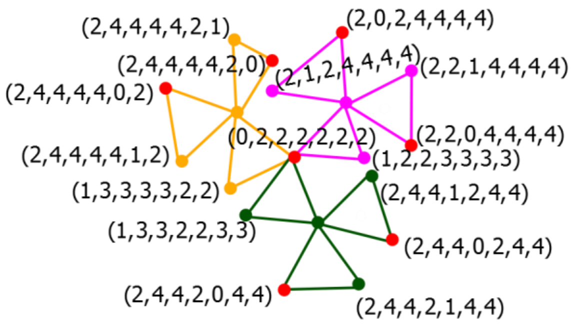


Figure 7. $Ddim_l\ (F_3, v, 3) = 7$

CONCLUSION

Based on the findings of this study, it is possible to conclude that the dominant local metric dimension for any vertex amalgamation product graph is determined by the dominant local metric dimension of the copied graphs and how the terminal vertex is chosen. This topic can be expanded by observing the dominant local metric dimension for the vertex amalgamation product with the special graphs that will be glued are different graphs. Next, we can determined the dominant local metric dimension for another product of graphs. Moreover, the program application of this concept can be generated for any connected graph.

ACKNOWLEDGMENTS

This work was supported by DRPM, KEMENRISTEK of Indonesia, Dcree 672/UN3/2022, Contract No. 75/UN3.15/PT/2022 and 010/E5/PG.02.00.PT/2022, year 2022.

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