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A SOLUTION TO A FRACTIONAL ORDER SEMILINEAR EQUATION USING VARIATIONAL METHOD

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Abstract. We will discuss how we obtain a solution to a semilinear pseudo-differential equation involving fractional power of Laplacian by using a method analogous to the direct method of calculus of variations. More precisely, we will discuss the existence of a weak form of solutions to this equation as a minimizer of a suitable energy-type functional whose Euler-Lagrange equation is the semilinear equation, and also discuss the possibility of regularity of such a weak solution so that it will be a solution to the semilinear equation.

1. Introduction

We consider N as a fixed positive integer greater than 1 throughout this paper. We call a function $u : \mathbb{R}^d \to \mathbb{R}$ a $\mathbb{N}\mathbb{Z}^d$-periodic function if and only if $u(x + Ne_j) = u(x)$ for all $j = 1,\ 2,\ \ldots,\ d$, where each $e_j = (0, 0, \ldots, 1, \ldots, 0)$ is the j^{th} standard unit vector in $\mathbb{R}^d$.

We assume that $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is an $N\mathbb{Z}^{d+1}$-periodic continuous function such that

$$\int_0^{y+N} f(x, z)\, dz = \int_0^{y} f(x, z)\, dz \tag{1.1}$$

for all $x \in \mathbb{R}^d,\ y \in \mathbb{R}$. Under these assumptions on f, our main goal is to study the existence of a weak form of solutions to a semilinear pseudo-differential equation (ΨDE) of the form

$$-(-\Delta)^\alpha u(x) = f(x, u(x)), \quad x \in \mathbb{R}^d, \tag{1.2}$$

where Δ is a d-dimensional Laplacian and α is a fixed real number with $0 < \alpha < 1$. Because of nonlocal behaviors of the fractional power of Laplacian, we will consider the problem under the periodic boundary setting. This will allow us to reduce the problem into a variational type problem, or more precisely, a problem of finding a minimizer of an energy type functional I associated with Equation (1.2). Here, the main idea is to use a method that is analogous to a widely used classical method, the direct method of calculus of variations, for studying solutions to nonlinear partial differential equations (see [6, 9, 12] etc.).

Problems similar to the one we mentioned above have been widely studied. Under a stronger assumption that f mentioned above is also infinitely smooth, R. de la Llave and E. Valdinnoci have studied the existence of a special (Birkhoff) class of solutions to a problem similar to ours by using the steepest descent method in [8]. In this method, they consider u as a real-valued function of a spacial variable $x \in \mathbb{R}^d$ and a forward time parameter $t > 0$, then consider a steepest descent equation corresponding to

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the problem, reduce this equation to the abstract Cauchy problem (a type of evolutionary problem in an infinite dimensional space), and find a solution (in the Birkhoff class) to the problem as an equilibrium solution. In [3], T. Blass, R. de la Llave and E. Valdinnoci have studied the existence of this type of solution to the semilinear elliptic equation

$$Au + f(x, u) = 0, \quad x \in \mathbb{R}^d \tag{1.3}$$

involving the positive definite uniformly elliptic self-adjoint operator $A = -\sum_{i,j=1}^{d}(a^{ij}(x)u_{x_i})_{x_j}$ of order 2 with smooth symmetric coefficients a^{ij} by using the Sobolev gradient descent method which is analogous to the steepest descent method but is more specific than the latter one. In [14], R. Karki has generalized this method to the semilinear pseudo-differential equation

$$A^{\alpha}u + f(x, u) = 0, \quad x \in \mathbb{R}^d \tag{1.4}$$

involving the fractional power of the operator A. The advantage of studying these semilinear equations by using the steepest or (Sobolev) gradient descent method is that the solution to the corresponding steepest or gradient descent equation converges to an equilibrium solution at a much faster rate, which can be seen via numerical simulation (see [2]). Other advantages could be exploring the the dynamics of the gradient descent equations corresponding to the problems, which can be applicable in different fields such as mathematical physics, math finance, engineering, mechanics, and Geology etc. Despite having these advantages, we use neither of these methods to solve our problem.

Using the variational approach, J. Moser has thoroughly discussed the existence, the regularity and many other properties of solutions to a problem analogous to ours in [15, 16, 17], especially the case when $(-\Delta)^{\alpha}$ is replaced by $-\Delta$. Now we use this variational approach to find a weak solution to our semilinear fractional Poisson type equation (1.2). By obtaining a weak solution to Equation (1.2), we will devise a powerful tool that is needed to obtain a classical solution to Equation (1.2).

Fractional Poisson type equations (special cases of Equation (1.2)) are studied when exploring anomalous diffusion in the context of geophysical electromagnetics (see [24]) and near-surface geotechical engineering as non-local electromagnetic effects occur due to fractures and stratigraphic layering (see [11, 23]). For the case $\lambda = \frac{1}{2}$, Equation (1.2) and the problem similar to ours appear in many fields, and have been widely studied for last several years. For instance, they appear in the theory of water waves when approximating the Dirichlet problem to the Neumann problem (see [7, 18]). Their applications to the ultrarelativistic limit of quantum mechanics (see [10]) and recently, to phase transition problems involving fractional powers of Laplacian have appeared (see [1, 4, 13]). The operator $(-\Delta)^{\frac{1}{2}}$ also plays a vital role in the thin obstacle problem (see [5, 22]).

We will prove the existence of a weak solution to Equation (1.2) in two main steps. First, we will show that a regular enough minimizer (i.e., a member of a suitable Sobolev space) of the energy-type functional I is a weak solution to Equation (1.2), and then show that the functional I does indeed have a minimizer. Schematically, we will first discuss some function spaces with respective norms, and some operators such as Laplacian and fractional Laplacian defined on these spaces in Section 2.1. In Section 2.2, we will introduce a few key terms such as an energy-type functional I whose Euler-Lagrange equation is Equation (1.2), a minimizer of I and a weak form of solution to Equation (1.2), and show that a minimizer with sufficient regularity is indeed such a solution. In Section 2.3, we will prove the existence of weak solutions to Equation (1.2) in the form of minimizer of I.

Finally, we will comment on possibilities of proving regularity results related to a weak solution to Equation (1.2), and also briefly discuss other stronger forms of solutions to Equation (1.2) in Section 3.

2. Existence of weak solutions in the form of minimizers

2.1. **Some preliminaries.** In this subsection, we will characterize some function spaces and their norms using the ideas adopted in [14]. Such characterizations will provide us some tools which will be really handy while working with these spaces and their norms later.

We use the notation $N\mathbb{T}^d$ to denote quotient space $\mathbb{R}^d/N\mathbb{Z}^d$, which is the same as the d-dimensional square $[0,N]^d$ after identifying its opposite sides. In other words, $N\mathbb{T}^d$ is a d-dimensional torus.

We use $C^\infty(N\mathbb{T}^d)$ to denote the space of all smooth functions $u:\mathbb{R}^d\to\mathbb{R}$ that are $N\mathbb{Z}^d$-periodic and equip $C^\infty(N\mathbb{T}^d)$ with the uniform norm. Any u in $C^\infty(N\mathbb{T}^d)$ has a Fourier series representation

$$u(x)=\sum_{j\in\mathbb{Z}^d} e^{\frac{2\pi}{N}i\langle x,j\rangle}\hat{u}_j,\quad x\in\mathbb{R}^d \tag{2.1}$$

where each $\hat{u}_j$ is the j^{th} Fourier coefficient of u and is given by

$$\hat{u}_j=\frac{1}{N^d}\int_{N\mathbb{T}^d} e^{-\frac{2\pi}{N}i\langle x,j\rangle}u(x)\,dx. \tag{2.2}$$

We use $L^2(N\mathbb{T}^d)$ to denote the space of all square integrable functions $u:\mathbb{R}^d\to\mathbb{R}$ that are $N\mathbb{Z}^d$-periodic. Then $L^2(N\mathbb{T}^d)$ is the completion of $C^\infty(N\mathbb{T}^d)$ with respect to the norm

$$\|u\|_{L^2(N\mathbb{T}^d)}=\sum_{j\in\mathbb{Z}^d}|\hat{u}_j|^2. \tag{2.3}$$

Using (2.1) together with the convergence in $L^2(N\mathbb{T}^d)$, each u in $L^2(N\mathbb{T}^d)$ can be expressed as

$$u(x)=\sum_{j\in\mathbb{Z}^d} e^{\frac{2\pi}{N}i\langle x,j\rangle}\hat{u}_j,\quad x\in\mathbb{R}^d \tag{2.4}$$

where the equality needs to be understood in almost everywhere sense. We consider the Sobolev space $H^s(N\mathbb{T}^d)$ with $s>0$ as the completion of $C^\infty(N\mathbb{T}^d)$ in $L^2(N\mathbb{T}^d)$ under the norm

$$\|u\|_{H^s(N\mathbb{T}^d)}=\left\|(I+(-\Delta)^s)^{\frac{1}{2}}u\right\|_{L^2(N\mathbb{T}^d)}. \tag{2.5}$$

Throughout the rest of this subsection, we will be referring to $s>0$. As we can express the operator $-\Delta: D(-\Delta)\subseteq L^2(N\mathbb{T}^d)\to L^2(N\mathbb{T}^d)$ as

$$(-\Delta)u(x)=\left(\frac{2\pi}{N}\right)^2\sum_{j\in\mathbb{Z}^d}|j|^2\hat{u}_j e^{\frac{2\pi}{N}i\langle x,j\rangle}, \tag{2.6}$$

we can use the spectral integral from the spectral theory of unbounded self-adjoint operators on Hilbert spaces (see [19, 20, 21]) to express the operator $(-\Delta)^s$ on $L^2(N\mathbb{T}^d)$ as

$$(-\Delta)^s u(x)=\left(\frac{2\pi}{N}\right)^{2s}\sum_{j\in\mathbb{Z}^d}|j|^{2s}\hat{u}_j e^{\frac{2\pi}{N}i\langle x,j\rangle}. \tag{2.7}$$

Therefore, by the virtue of Equations (2.3) - (2.7), we have

$$\|u\|^2_{H^s(N\mathbb{T}^d)}=\sum_{j\in\mathbb{Z}^d}\left[1+\left(\frac{2\pi}{N}\right)^{2s}|j|^{2s}\right]|\hat{u}_j|^2. \tag{2.8}$$

Also, $u\in D((-\Delta)^s)$ if and only if $u\in L^2(N\mathbb{T}^d)$ and

$$\left(\frac{2\pi}{N}\right)^{2s}\sum_{j\in\mathbb{Z}^d}|j|^{2s}|\hat{u}_j|^2<\infty,$$

which is true if and only if $\|u\|^2_{H^s(N\mathbb{T}^d)} < \infty$, meaning $u \in H^s(N\mathbb{T}^d)$. Moreover, for each $u \in D((-\Delta)^s)$ we can define

$$\|u\|_{\dot{H}^s(N\mathbb{T}^d)} = \left[\left(\frac{2\pi}{N}\right)^{2s} \sum_{j\in\mathbb{Z}^d} |j|^{2s}|\hat{u}_j|^2\right]^{\frac{1}{2}}, \tag{2.9}$$

to obtain a seminorm (not a norm) $\|\cdot\|_{\dot{H}^s(N\mathbb{T}^d)}$ on $D((-\Delta)^s)$. Actually, the mean $\hat{u}_0 = \frac{1}{N^d}\int_{N\mathbb{T}^d} u(x)\,dx$ for any $u \in D((-\Delta)^s)$ over $N\mathbb{T}^d$ has no contribution in $\|u\|_{\dot{H}^s(N\mathbb{T}^d)}$ even if u is nonzero.

2.2. **Minimizers and weak solutions.** In order to study solutions to the semilinear ΨDE (1.2), we first consider the energy-type functional

$$I(u) = \int_{N\mathbb{T}^d} \left\{\frac{1}{2}[(-\Delta)^{\frac{\alpha}{2}}u(x)]^2 + F(x,u(x))\right\}\,dx, \tag{2.10}$$

defined on a subspace of $L^2(N\mathbb{T}^d)$, where

$$F(x,y) = \int_0^y f(x,z)\,dz, \quad x \in \mathbb{R}^d,\ y \in \mathbb{R}, \tag{2.11}$$

and then study the critical values of I. Among those critical values, we are basically interested on minimizers of I as discussed in Section 1. We observe that I is naturally defined for all $u \in H^\alpha(N\mathbb{T}^d)$. Taking this into account, we introduce a minimizer of I and a weak solution to Equation (1.2).

Definition 2.1 (Minimizer)**.** A $u \in H^\alpha(N\mathbb{T}^d)$ is a minimizer of I if $I(u+\phi) \geq I(u)$ for all $\phi \in C^\infty(N\mathbb{T}^d)$.

It follows from Definition 2.1 that if $u \in H^\alpha(N\mathbb{T}^d)$ is a minimizer of I, then

$$\frac{d}{dt}I(u+t\phi)|_{t=0} = 0 \tag{2.12}$$

for all $\phi \in C^\infty(N\mathbb{T}^d)$.

Definition 2.2 (Weak solution)**.** A $u \in H^\alpha(N\mathbb{T}^d)$ is a weak solution to Equation (1.2) if u satisfies

$$\langle(-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}\phi\rangle + \langle f(.,u),\phi\rangle = 0 \tag{2.13}$$

for all $\phi \in C^\infty(N\mathbb{T}^d)$.

Now we establish a fundamental result that relates a minimizer of I to a weak solution to Equation (1.2).

Theorem 2.1. *Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be an $N\mathbb{Z}^{d+1}$-periodic continuous function such that Equation* (1.1) *holds. If $u \in H^\alpha(N\mathbb{T}^d)$ is a minimizer of I given by Equation* (2.10)*, then u is a weak solution to Equation* (1.2)*.*

Proof. Let $u \in H^\alpha(N\mathbb{T}^d)$ be a minimizer of I and let $\phi \in C^\infty(N\mathbb{T}^d)$. Then

$$\begin{aligned} I(u+t\phi) - I(u) &= \int_{N\mathbb{T}^d}\left\{\frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}}(u+t\phi)\right]^2 + F(.,u+t\phi)\right\} \\ &\quad - \int_{N\mathbb{T}^d}\left\{\frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}}u\right]^2 + F(.,u)\right\} \\ &= \int_{N\mathbb{T}^d}\frac{1}{2}\left\{2t(-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}\phi + t^2\left[(-\Delta)^{\frac{\alpha}{2}}\phi\right]^2\right\} \\ &\quad + \int_{N\mathbb{T}^d}\left\{F(.,u+t\phi) - F(.,u)\right\}, \end{aligned}$$

so we have

$$\frac{I(u+t\phi)-I(u)}{t} = \int_{N\mathbb{T}^d} (-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} \phi + \frac{t}{2} \int_{N\mathbb{T}^d} \left[(-\Delta)^{\frac{\alpha}{2}} \phi\right]^2 + \int_{N\mathbb{T}^d} \frac{F(.,u+t\phi)-F(.,u)}{t} \tag{2.14}$$

Since $u \in H^\alpha(N\mathbb{T}^d)$ and $\phi \in C^\infty(N\mathbb{T}^d)$, we have $(-\Delta)^{\frac{\alpha}{2}} u$, $(-\Delta)^{\frac{\alpha}{2}} \phi \in L^2(N\mathbb{T}^d)$, so $(-\Delta)^{\frac{\alpha}{2}} u (-\Delta)^{\frac{\alpha}{2}} \phi$, $\left[(-\Delta)^{\frac{\alpha}{2}} \phi\right]^2 \in L^1(N\mathbb{T}^d)$. Thus the first two integrals on the right side of Equation (2.14) are finite. Moreover, since f is $N\mathbb{Z}^{d+1}$-periodic on $\mathbb{R}^d \times \mathbb{R}$, it is bounded there and, therefore, there exists a real number $M > 0$ such that $|F_y(x,y)| = |f(x,y)| \leq M$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}$. Thus we have

$$\begin{aligned}
\left|\frac{F(.,u+t\phi)-F(.,u)}{t}\right| &\leq \frac{1}{t}\left|F(.,u+t\phi)-F(.,u)\right| \\
&= \frac{1}{t}\left|\int_0^t \frac{d}{ds} F(.,u+s\phi)\, ds\right| \\
&= \frac{1}{t}\left|\int_0^t F_y(.,u+s\phi)\phi\, ds\right| \\
&\leq \frac{1}{t}\int_0^t |F_y(.,u+s\phi)||\phi|\, ds \\
&\leq M|\phi|
\end{aligned}$$

for all $t > 0$. Since $M|\phi| \in L^1(N\mathbb{T}^d)$, the Dominated Convergence Theorem implies that

$$\lim_{t\to 0} \int_{N\mathbb{T}^d} \frac{F(.,u+t\phi)-F(.,u)}{t}$$

exists and equals

$$\int_{N\mathbb{T}^d} F_y(.,u)\phi = \int_{N\mathbb{T}^d} f(.,u)\phi.$$

Therefore, letting $t \to 0$ on the both sides of Equation (2.14), we obtain

$$\frac{d}{dt} I(u+t\phi)|_{t=0} = \langle (-\Delta)^{\frac{\alpha}{2}} u, (-\Delta)^{\frac{\alpha}{2}} \phi \rangle + \langle f(.,u), \phi \rangle. \tag{2.15}$$

Also, Equation (2.12) holds true for a minimizer $u \in H^\alpha(N\mathbb{T}^d)$ of I. From Equation (2.12) and Equation (2.15), we obtain Equation (2.13). Hence u is a weak solutions to Equation (1.2). □

Theorem 2.1 guarantees that in order to find weak solutions to Equation (1.2), it suffices to prove the existence of a minimizer of I in $H^\alpha(N\mathbb{T}^d)$.

Theorem 2.2. *Let $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be an $N\mathbb{Z}^{d+1}$-periodic continuous function such that Equation* (1.1) *holds. Then there exists $u \in H^\alpha(N\mathbb{T}^d)$ such that u is a minimizer of I given by Equation* (2.10)*, and hence a weak solution to Equation* (1.2)*.*

2.3. **Proof of Theorem 2.2.** We will complete the proof of Theorem 2.2 by subsequently proving a few results. The first of them is related to a coercive condition satisfied by I in $H^\alpha(N\mathbb{T}^d)$.

Proposition 2.3 (Coercivity)**.** *Suppose I is given by Equation* (2.10)*. Then there exists a positive constant Λ depending on f, N and d such that*

$$I(u) \geq \frac{1}{2}\left(1 - N^{-d}\right)\left[1 + \left(\frac{2\pi}{N}\right)^{2\alpha}\right]^{-1} \|u\|^2_{H^\alpha(N\mathbb{T}^d)} - \Lambda \tag{2.16}$$

for all $u \in H^\alpha(N\mathbb{T}^d)$.

Proof. Notice that F given by Equation (2.11) is continuous and $N\mathbb{T}^{d+1}$-periodic on $\mathbb{R}^d \times \mathbb{R}$. Let Λ_0 be a positive real number such that $|F(x,y)| \le \Lambda_0$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}$. Then the integrand

$$L((-\Delta)^{\frac{\alpha}{2}} u(x), x, u(x)) := \frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}} u(x)\right]^2 + F(x, u(x)) \tag{2.17}$$

of I satisfies the condition

$$L(p,x,z) := \frac{1}{2}p^2 + F(x,z) \ge \frac{1}{2}p^2 - \Lambda_0 \tag{2.18}$$

for all $(p,x,z) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. Let $u \in H^\alpha(N\mathbb{T}^d)$. First, using (2.18) into (2.10), then using (2.3), (2.7), we get

$$\begin{aligned} I(u) \ge& \frac{1}{2}\left\|(-\Delta)^{\frac{\alpha}{2}} u\right\|^2_{L^2(N\mathbb{T}^d)} - \Lambda_0 N^d \\ =& \frac{1}{2}\sum_{j\in\mathbb{Z}^d}\left(\frac{2\pi}{N}\right)^{2\alpha}|j|^{2\alpha}|\hat{u}_j|^2 - \Lambda_0 N^d \end{aligned}$$

and next using Equation (2.9), we get

$$I(u) \ge \frac{1}{2}\|u\|_{\dot{H}^\alpha(N\mathbb{T}^d)} - \Lambda_0 N^d. \tag{2.19}$$

Taking $j = 0 \in \mathbb{Z}^d$ in Equation (2.2) and using the Cauchy-Schwartz Inequality, we get

$$\begin{aligned} |\hat{u}_0| \le& \frac{1}{N^d}\int_{N\mathbb{T}^d}|u(x)|\,dx \\ \le& N^{-d}\cdot(N^d)^{\frac{1}{2}}\|u\|_{L^2(N\mathbb{T}^d)} \\ =& N^{-\frac{d}{2}}\|u\|_{L^2(N\mathbb{T}^d)} \end{aligned}$$

Using the last inequality into Equation (2.8), we have

$$\begin{aligned} \|u\|^2_{H^\alpha(N\mathbb{T}^d)} \le& \sum_{j\in\mathbb{Z}^d-\{0\}}\left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}|j|^{2\alpha}\right]|\hat{u}_j|^2 + N^{-d}\|u\|^2_{L^2(N\mathbb{T}^d)} \\ \le& \sum_{j\in\mathbb{Z}^d-\{0\}}|j|^{2\alpha}\left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}\right]|\hat{u}_j|^2 + N^{-d}\|u\|^2_{L^2(N\mathbb{T}^d)} \\ \le& \left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}\right]\sum_{j\in\mathbb{Z}^d}|j|^{2\alpha}|\hat{u}_j|^2 + N^{-d}\|u\|^2_{H^\alpha(N\mathbb{T}^d)} \\ =& \left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}\right]\|u\|^2_{\dot{H}^\alpha(N\mathbb{T}^d)} + N^{-d}\|u\|^2_{H^\alpha(N\mathbb{T}^d)}, \end{aligned}$$

from which, we obtain

$$\|u\|^2_{\dot{H}^\alpha(N\mathbb{T}^d)} \ge \left(1-N^{-d}\right)\left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}\right]^{-1}\|u\|^2_{H^\alpha(N\mathbb{T}^d)}. \tag{2.20}$$

Combining Inequality (2.19) and Inequality (2.20), we obtain

$$I(u) \ge \frac{1}{2}\left(1-N^{-d}\right)\left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}\right]^{-1}\|u\|^2_{H^\alpha(N\mathbb{T}^d)} - \Lambda_0 N^d$$

and hence

$$I(u) \ge \frac{1}{2}\left(1-N^{-d}\right)\left[1+\left(\frac{2\pi}{N}\right)^{2\alpha}\right]^{-1}\|u\|^2_{H^\alpha(N\mathbb{T}^d)} - \Lambda$$

for some positive constant Λ depending on f, N and d. $\square$

The next important result required to prove Theorem 2.2 is related to weakly lower semicontinuity of I in $H^\alpha(N\mathbb{T}^d)$. Recall that I is weakly lower semicontinous at $u \in H^\alpha(N\mathbb{T}^d)$ if and only if for every sequence $\{u_k\}$ in $H^\alpha(N\mathbb{T}^d)$ converging weakly in $H^\alpha(N\mathbb{T}^d)$ to a limit u,

$$I(u) \leq \liminf_{k\to\infty} I(u_k)$$

and is weakly lower semicontinuous in $H^\alpha(N\mathbb{T}^d)$ if and only if it is weakly lower semicontinuous at each $u \in H^\alpha(N\mathbb{T}^d)$.

Proposition 2.4 (Weakly lower semicontinuity)**.** *I is weakly lower semicontinuous in $H^\alpha(N\mathbb{T}^d)$.*

Proof. Consider any $u \in H^\alpha(N\mathbb{T}^d)$ and any sequence $\{u_k\}$ in $H^\alpha(N\mathbb{T}^d)$ that converges weakly in $H^\alpha(N\mathbb{T}^d)$ to u and set

$$m = \liminf_{k\to\infty} I(u_k).$$

To complete the proof, we need to show that $I(u) \leq m$. Since a weakly convergent sequence is bounded, we have

$$\sup_k \|u_k\|_{H^\alpha(N\mathbb{T}^d)} < \infty.$$

From definition of limit infimum, we may assume, by passing to a subsequence if necessary, that

$$m = \lim_{k\to\infty} I(u_k). \tag{2.21}$$

By the Sobolev Embedding Theorem, the inclusion $H^\alpha(N\mathbb{T}^d) \hookrightarrow L^2(N\mathbb{T}^d)$ is compact. So, we may assume, by passing another subsequence of the last subsequence of our original sequence $\{u_k\}$ if necessary, that $\{u_k\}$ converges strongly in $L^2(N\mathbb{T}^d)$ to u, which then implies that

$$u_k \to u \text{ a. e. in } \Omega,$$

where $\Omega = N\mathbb{T}^d$. Let $\epsilon > 0$ be given. Then, by Egoroff's Theorem, there exists a measurable subset E_ϵ of Ω with

$$|\Omega - E_\epsilon| \leq \epsilon$$

(here $|\Omega - E_\epsilon|$ denotes the Lebesgue measure of $\Omega - E_\epsilon$) such that

$$u_k \to u \text{ uniformly on } E_\epsilon.$$

Define a subset F_ϵ of Ω by

$$F_\epsilon = \left\{x \in \Omega : |u(x)| + |(-\Delta)^{\frac{\alpha}{2}} u(x)| \leq \frac{1}{\epsilon}\right\}.$$

As $\epsilon \to 0$,

$$|\Omega - F_\epsilon| \to 0.$$

Define another subset G_ϵ of Ω by

$$G_\epsilon = E_\epsilon \cap F_\epsilon.$$

As $\epsilon \to 0$,

$$|\Omega - G_\epsilon| \leq |\Omega - E_\epsilon| + |\Omega - F_\epsilon| \to 0.$$

Since L given by Equation (2.17) is bounded from below, we may assume without loss of generality that $L \geq 0$ (otherwise, we may consider $\tilde{L} = L + \Lambda_0 \geq 0$ where Λ_0 is as in the proof of Proposition 2.3). Therefore, from

$$I(u_k) = \int_\Omega \frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}} u_k\right]^2 + F(., u_k)$$

for all k, we have

$$I(u_k) \geq \int_{G_\epsilon} \frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}} u_k\right]^2 + F(.,u_k) \tag{2.22}$$

for all k. Recall that if a map $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then

$$\phi(y) \geq \phi(x) + \phi'(x)(y-x)$$

for all $y, x \in \mathbb{R}$. Since the map $p \mapsto \frac{1}{2}p^2$ mapping $\mathbb{R}$ into itself is convex,

$$\frac{1}{2}p_k^2 \geq \frac{1}{2}p^2 + p(p_k - p)$$

for all $p_k, p \in \mathbb{R}$. So, we have

$$\begin{aligned}\frac{1}{2}\left((-\Delta)^{\frac{\alpha}{2}} u_k(x)\right)^2 \geq{}& \frac{1}{2}\left((-\Delta)^{\frac{\alpha}{2}} u(x)\right)^2 \\ &+ (-\Delta)^{\frac{\alpha}{2}} u(x)\left((-\Delta)^{\frac{\alpha}{2}} u_k(x) - (-\Delta)^{\frac{\alpha}{2}} u(x)\right)\end{aligned}$$

for all $x \in \mathbb{R}^d$ and for all k, and thus

$$\begin{aligned}\int_{G_\epsilon} \frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}} u_k\right]^2 \geq{}& \int_{G_\epsilon} \frac{1}{2}\left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2 \\ &+ \int_{G_\epsilon} (-\Delta)^{\frac{\alpha}{2}} u\left((-\Delta)^{\frac{\alpha}{2}} u_k - (-\Delta)^{\frac{\alpha}{2}} u\right).\end{aligned} \tag{2.23}$$

Since $u_k \to u$ uniformly on G_ϵ, for each $x \in \mathbb{R}^d$ we have

$$\begin{aligned}&(-\Delta)^{\frac{\alpha}{2}} u_k(x) - (-\Delta)^{\frac{\alpha}{2}} u(x) \\ &=(-\Delta)^{\frac{\alpha}{2}}(u_k - u)(x) \\ &= \sum_{j\in\mathbb{Z}^d}\left(\frac{2\pi}{N}\right)^{\alpha} |j|^{\alpha} \widehat{(u_k - u)}_j e^{\frac{2\pi}{N} i\langle x,j\rangle} \\ &= \sum_{j\in\mathbb{Z}^d}\left(\frac{2\pi}{N}\right)^{\alpha} |j|^{\alpha} \cdot \frac{1}{N^d}\int_\Omega e^{-\frac{2\pi}{N} i\langle y,j\rangle}(u_k(y) - u(y))\, dy\; e^{\frac{2\pi}{N} i\langle x,j\rangle} \\ &= \sum_{j\in\mathbb{Z}^d}\frac{1}{N^d}\left(\frac{2\pi}{N}\right)^{\alpha} |j|^{\alpha} e^{\frac{2\pi}{N} i\langle x,j\rangle}[\int_{G_\epsilon} e^{-\frac{2\pi}{N} i\langle y,j\rangle}(u_k(y) - u(y))\, dy \\ &\quad + \int_{\Omega - G_\epsilon} e^{-\frac{2\pi}{N} i\langle y,j\rangle}(u_k(y) - u(y))\, dy],\end{aligned}$$

which approaches 0 as $\epsilon \to 0$ and $k \to \infty$ because of the facts that $G_\epsilon \subseteq \Omega$ with $|\Omega - G_\epsilon| \to 0$ as $\epsilon \to 0$ and $u_k \to u$ uniformly on G_ϵ. This limit and the Monotone Convergence Theorem then imply that

$$\int_{G_\epsilon}\left[\left((-\Delta)^{\frac{\alpha}{2}} u_k - (-\Delta)^{\frac{\alpha}{2}} u\right)\right]^2 = \int_\Omega \chi_{G_\epsilon}\left[\left((-\Delta)^{\frac{\alpha}{2}} u_k - (-\Delta)^{\frac{\alpha}{2}} u\right)\right]^2 \tag{2.24}$$

approaches 0 as $\epsilon \to 0$ and $k \to \infty$. By the Cauchy-Schwartz's Inequality,

$$\begin{aligned}\left|\int_{G_\epsilon} (-\Delta)^{\frac{\alpha}{2}} u\left((-\Delta)^{\frac{\alpha}{2}} u_k - (-\Delta)^{\frac{\alpha}{2}} u\right)\right|^2 \leq{}& \int_{G_\epsilon}\left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2 \\ &\cdot \int_{G_\epsilon}\left[\left((-\Delta)^{\frac{\alpha}{2}} u_k - (-\Delta)^{\frac{\alpha}{2}} u\right)\right]^2.\end{aligned} \tag{2.25}$$

From Limit (2.24) and Inequality (2.25), we can obtain that

$$\int_{G_\epsilon} (-\Delta)^{\frac{\alpha}{2}} u\left((-\Delta)^{\frac{\alpha}{2}} u_k - (-\Delta)^{\frac{\alpha}{2}} u\right) \to 0 \tag{2.26}$$

as $\epsilon \to 0$ and $k \to \infty$. Letting $\epsilon \to 0$ and $k \to \infty$ on both sides of Inequality (2.23) and using Limit (2.26), we get

$$\lim_{\epsilon\to 0,\ k\to\infty} \int_{G_\epsilon} \frac{1}{2} \left[(-\Delta)^{\frac{\alpha}{2}} u_k\right]^2 \geq \lim_{\epsilon\to 0} \int_{G_\epsilon} \frac{1}{2} \left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2. \tag{2.27}$$

Since $|\Omega - G_\epsilon| \to 0$ as $\epsilon \to 0$, $\chi_{G_\epsilon}(x) \to 1$ for almost every $x \in \Omega$ as $\epsilon \to 0$. By the Monotone Convergence Theorem, we have

$$\lim_{\epsilon\to 0} \int_{G_\epsilon} \left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2 = \lim_{\epsilon\to 0} \int_{\Omega} \chi_{G_\epsilon} \left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2 = \int_{\Omega} \left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2. \tag{2.28}$$

From Inequality (2.27) and Equation (2.28), we have

$$\lim_{\epsilon\to 0,\ k\to\infty} \int_{G_\epsilon} \frac{1}{2} \left[(-\Delta)^{\frac{\alpha}{2}} u_k\right]^2 \geq \int_{\Omega} \frac{1}{2} \left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2. \tag{2.29}$$

Moreover, from the facts that F is continuously differentiable at the functional component y and $F_y(x,y) = f(x,y)$, $x \in R^d$, $y \in \mathbb{R}$, and $u_k \to u$ uniformly on E_ϵ ($\supseteq G_\epsilon$), it follows that

$$\lim_{\epsilon\to 0\ k\to\infty} \int_{G_\epsilon} F(.,u_k) = \int_{\Omega} F(.,u) \tag{2.30}$$

Letting $\epsilon \to 0$ and $k \to \infty$ on the both sides of (2.22), then applying (2.29) and (2.30), we get

$$\begin{aligned}\lim_{k\to\infty} I(u_k) \geq& \int_{\Omega} \frac{1}{2} \left[(-\Delta)^{\frac{\alpha}{2}} u\right]^2 + \int_{\Omega} F(.,u)\\ =& \int_{\Omega} \left[\frac{1}{2}[(-\Delta)^{\frac{\alpha}{2}} u\right]^2 + F(.,u)]\\ =& I(u),\end{aligned}$$

and hence, by Limit (2.21), $I(u) \leq m$ as desired. □

Finally, we are ready to prove the existence of minmizer of I in $H^\alpha(N\mathbb{T}^d)$.

Proposition 2.5 (Existence of minimizer)**.** *I has a minimizer in* $H^\alpha(N\mathbb{T}^d)$.

Proof. Set

$$m_0 = \inf_{u\in H^\alpha(N\mathbb{T}^d)} I(u).$$

Without loss of generality, assume that $m_0 < \infty$ (otherwise, each $u \in H^\alpha(N\mathbb{T}^d)$ becomes a minimizer of I). Let $\{u_k\}$ be a minimizing sequence in $H^\alpha(N\mathbb{T}^d)$, meaning a sequence satisfying

$$\lim_{k\to\infty} I(u_k) = m_0.$$

Then we have

$$\sup_k I(u_k) < \infty.$$

By Proposition 2.3, we have

$$I(u_k) \geq \frac{1}{2}\left(1 - N^{-d}\right)\left[1 + \left(\frac{2\pi}{N}\right)^{2\alpha}\right]^{-1} \|u_k\|_{H^\alpha(N\mathbb{T}^d)} - \Lambda$$

for all $u \in H^\alpha(N\mathbb{T}^d)$, where Λ is a positive constant depending on f, N and d. This implies that

$$\sup_k \|u_k\|_{H^\alpha(N\mathbb{T}^d)} < \infty.$$

Due to the Weak Compactness Theorem, every bounded sequence in a Hilbert space is weakly precompact, that is, every bounded sequence in a Hilbert has a weakly convergent subsequence. Therefore, there exists a subsequence $\{u_{k_j}\}$ of $\{u_k\}$ and an element $u \in H^\alpha(N\mathbb{T}^d)$ such that $u_{k_j} \rightharpoonup u$ in $H^\alpha(N\mathbb{T}^d)$. By lower semicontinuity of I from Proposition 2.4, we have

$$I(u) \leq \liminf_{j\to\infty} I(u_{k_j}) = m_0.$$

On the other hand, it is clear that $m_0 \leq I(u)$. Therefore, $m_0 = I(u)$. This shows that u is a minimizer of I in $H^\alpha(N\mathbb{T}^d)$. □

With Proposition 2.5, we have completed the proof of Theorem 2.2. In this way, we have proved that under the given assumptions on f, Equation (1.2) does have a weak solution in $H^\alpha(N\mathbb{T}^d)$.

3. Some Remarks on Regularity

A weak solution $u \in H^\alpha(N\mathbb{T}^d)$ to Equation (1.2) does not necessarily satisfy Equation (1.2) unless u is sufficiently regular or smooth. But in order to have sufficient regularity or smoothness of u we may need to add further assumptions on the nonlinear functional f. For the moment, suppose we have a suitable f so that $u \in H^s(N\mathbb{T}^d)$ for $s \geq 2\alpha$. Since for the weak solution u, we have

$$\langle(-\Delta)^{\frac{\alpha}{2}}u, (-\Delta)^{\frac{\alpha}{2}}\phi\rangle + \langle f(.,u), \phi\rangle = 0$$

for all $\phi \in C^\infty(N\mathbb{T}^d)$ and $(-\Delta)^{\frac{\alpha}{2}}$ is self-adjoint on $L^2(N\mathbb{T}^d)$ (see [8, 20, 21]), we have

$$\langle(-\Delta)^\alpha u + f(.,u), \phi\rangle = 0$$

for all $\phi \in C^\infty(N\mathbb{T}^d)$ and hence

$$(-\Delta)^\alpha u + f(.,u) = 0 \text{ a. e.} \tag{3.1}$$

This leads us to ask the following.

Remark 3.1. Under what assumptions on f, does a minimizer $u \in H^\alpha(N\mathbb{T}^d)$ of I belong to $H^s(N\mathbb{T}^d),\ s \geq 2\alpha$ so that u satisfies Equation (1.2) in a pointwise almost everywhere sense (meaning u is a strong solution to Equation (1.2))?

Furthermore, suppose we have even a better f so that the weak solution $u \in H^s(N\mathbb{T}^d),\ s \geq 2\alpha$ is smooth enough to become both u and $(-\Delta)^\alpha u$ continuous on $\mathbb{R}^d$. Then Equation (3.1) reduces to

$$(-\Delta)^\alpha u(x) + f(x, u(x)) = 0 \quad \forall\, x \in \mathbb{R}^d.$$

This means u solves Equation (1.2) in a classical sense and thus Equation (1.2) is the Euler-Lagrange equation of $I(u)$. So, it is natural to expect the following.

Remark 3.2. Additionally, how smooth does f need to be so that a weak solution u of Equation (1.2) will satisfy it in a pointwise (everywhere) sense (meaning u is a classical solution to Equation (1.2))?

References

[1] G. Alberti and G. Bellettini, *A nonlocal anisotropic phase transitions I. The optimal profile problem*, Math. Ann. **310**(1998), 527- 560.

[2] T. Blass and R. de la lLave, *Perturbation and Numerical Method for Computing the Minimal Average Energy*, Networks & Heterogeneous Media **6(2)**(2011), 241-255.

[3] T. Blass, R. de la lLave and E. Valdinnoci, *A Comparison Principle for a Sobolev Gradient Semi-flow*, Commun. Pure Appl. Anal. **10**(2011), 69-91.

[4] X. Cabre and J. Sola-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math. 52(12)(2005), 1678-1732

[5] L. Caffarelli, L. Silvetre *An Extension Problem Related to the Fractional Laplacian*, Communications on Partial Differential Equations 32 (2007), 1245-1260.

[6] M. Colombini and S. Modica, *Partial Differential Equations and the Calculus of Variations: Essays in Honor of Ennio De Giorgi*, Vol. II, Birkhauser Basel, 1989.

[7] W. Craig and M. D. Groves, *Hamiltonian long-wave approximations to the water-wave problem*, Wave Motion **19**(1994), 367- 389.

[8] R. de la Llave and E. Valdinnoci, *A Generalization of Aubrey-Mather Theory to Partial Differential Equations an Pseudo Differential Equations*, Annals de l'Institut Henry Poincare (C) Non Linear Analysis 26(2009), 1309-1344.

[9] L. C. Evans, *Partial Differential Equations*, vol. 19, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[10] C. Fefferman and R. de la Llave, *Relativistic stability of matter.* I, Rev. Mat. Iberoamericana **2** (1986), 119-213.

[11] J. Ge, M. E. Everett and C. J. Weiss, *Fractional diffusion analysis of the electromagnetic field in fractured media-Part 2: 3D approach*, Geophysics **80**(2015), 175-185.

[12] M. Giaquinta (Ed.), *Topics in Calculus of Variations*, Lecture Notes in Mathematics 1365, Springer-Verlag, 1987.

[13] A. Garroni and G. Palatucci, *A singular perturbation result with a fraction norm, in: Variational Problems in Material Sience*, Progr. Nonlin. Diff. Eqns. Appl. **68**(2006), 111-126.

[14] R. Karki, *Sobolev Gradients & Application to Nonlinear Pseudo-differential Equations*, Neural, Parallel & Scientific Computations **22**(2014), 359-374.

[15] J. Moser, *Minimal Solutions of Variational Problems on a Torus*, Annals de l'I. H. P. Analyse non linéaire **3**(1986), 229-272.

[16] J. Moser, *A Stability Theorem for Minimal Foliations on a Torus*, Ergodic Theory Dynamical Systems (Charles Conley Memorial Issue), **8**(1988), 251-281.

[17] J. Moser, *Minimal Foliations on a Torus*, in Giaquinta M. (eds): Topics in Calculus of Variations. Lecture Notes in Mathematics, Vol. 1365. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0089178.

[18] P. I. Naumkin and I.A. Shishmarev, *Nonlinear Nonlocal Equations in the Theory of Waves*, American Mathematical Society, Providence, RI, 1994.

[19] L. Roncal and P. R. Stinga, *Fractional Laplacian on the torus*, Communications in Contemporary Mathematics **18(3)**(2016): 1550033.

[20] K. Schmudgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Graduate Text in Mathematics 265, Springer-Verlag, NY, 2012.

[21] M. A. Shubin, *Pseudodifferential Operators and Spectral Theory*, Springer- Verlag, 2nd edition, 2001.

[22] L. Silvestre *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Commun. Pure Appl. Math. **60** (2007), 67-112.

[23] F. Vallianatos *A non-existensive view of electrical resistivity spatial distribution estimated using transient electromagnetic responses in a karstified formation (Keritis basin, Crete, Greece)*, Phys. A **505** (2018), 171-178.

[24] C.J. Weiss, B.G. van Bloemen Waanders and H. Antil, *Fractional operators applied to geophysical electromagnetics*, Geophysical Journal International **220** (2020), 1242-259.

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