

DYNAMICAL STUDY OF THE THETA-LOGISTIC PREDATOR-PREY MODEL INCORPORATING GREGARIOUS BEHAVIOR OF PREY

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ABSTRACT. Relation between species and their livelihood environment in ecological systems is very complex. For that reason, in order to study predator-prey relations, modeling is essential in biomathematics. The vital components of predator-prey models are the growth function of the prey species in the absence of a predator and the functional response. In this article, we proposed a predator-prey model with gregarious prey. In the existing literature, the square-root functional response incorporates the gregarious behavior of prey. This study considers the generalized square root functional response and theta-logistic growth of prey in the absence of a predator. The effect of functional response parameters on stability, limit cycle, and Hopf bifurcation on the proposed model has been discussed. Numerical analysis is performed on the basis of some hypothetical parameter values to analyze the model numerically.

1. INTRODUCTION

Theoretical ecology is developed on the basis of the principle of mass action, in which uncoordinated populations of prey and predators move in a random and well-mixed fashion across a featureless environment. The conceptual core of the theory is the functional response. For this reason, the selection of the functional response is important in building a practical model. The useful functional responses are Holling type ([24], [2], [20], [18], [23], [32]), Monod-Haldane, Beddington-DeAngelis, Hassell-Varley, ratio-dependent, square root, Crowley-Martin, etc. Holling type I functional response [5] is a linear relationship between the density of the prey and the maximum number of prey killed, while in Holling type II ([24], [8]) the proportion of prey consumed decreases monotonically with the density of the prey. Holling type III ([17], [18], [39]) is described by a sigmoid relation in which the proportion of prey consumed is positively density dependent over some regions of prey density. Monod-Haldane ([19], [9]) functional response involves prey group defense theory. Crowley-Martin functional response ([14], [36], [33]) depends on both the predator and the prey. It takes into account the interaction between predators and is similar to the classical Beddington-DeAngelis functional response. Beddington-DeAngelis ([22], [35], [26], [13]) is similar to Holling type II, which contained an additional term that describes mutual interference by predators. Hassell and Varley functional response [37] depends on the density of the predators differently, where the predators form groups. In our existing literature, we also observe the study of monotonic and non-monotonic functional response ([11], [16]), sigmoid functional response [29], Ivlev-type functional response [38], ratio-dependent functional response ([34], [30]). In the square root functional response ([25], [27], [7], [4], [6], [31], [12], [1], [40], [28]), the predator interacts with prey along the outer corridor of the prey herd. In this paper, we generalize the square-root functional response as x^b where $0 < b \leq 1$.

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Although the logistic model includes more population growth factors, the basic logistic model is still insufficient. As a large population size continues to grow, the individual growth rate should slow down, and this finding is missing in the classical model. To better fit the data and address the limitations of the classic logistic model, Gilpin and Ayala ([10], [15]) presented a new version of the logistic model called the theta-logistic model by adding a term to the classic logistic model. Astrom ([3]) studied the paradox of biological control using theta-logistic model with Holling II functional response. The linear density dependence held by the classical logistic model can be altered to curvilinear. It provides the additional generality and flexibility to explain the impact of the change in individual growth rate parameter r for population density x . In our study, we consider this theta-logistic growth for prey in the absence of predator. The positivity and boundedness of the model are explored along with the local stability analysis at equilibrium points. The limit cycle and global stability are discussed, and Hopf bifurcation is studied. The effects of parameters b and θ in the dynamics of the system are explored. On the basis of some hypothetical numerical values of parameters of the model, we study the model numerically, and finally, the results of the study sum up in conclusion.

2. FORMULATION OF A THETA-LOGISTIC PREDATOR-PREY MODEL

In this section, we propose a predator-prey model that depends on some suitable biological assumptions. The assumptions for formulating the proposed prey-predator model are (i) in the absence of predators, the prey grows theta-logistically $[rx(1 - (\frac{x}{k})^\theta)]$, (ii) in the absence of prey, the predators die exponentially, (iii) the prey form group incorporates this gregarious behavior of prey in a better way, and (iv) the generalized square root functional response as to x^b where $0 < b \leq 1$. The effect of θ on theta-logistic growth is shown in Figure 1. The effect of b on the generalized square root functional response is shown in Figure 2.

Table 1: Description of non-negative system parameters

Parameters	Ecological description
r	Intrinsic/natural growth rate of prey
k	Environmental carrying capacity for prey
θ	Dimensionless parameter
α	Maximal prey consumption rate of predator
β	Conversion coefficient of food biomass to predator
b	The exponent of the functional response
d	The mortality rate of predator

The proposed model depending on these assumptions is given by:

$$\begin{cases} \frac{dx}{dt} = rx(1 - (\frac{x}{k})^\theta) - \alpha x^b y, \\ \frac{dy}{dt} = \beta x^b y - dy, \end{cases} \tag{2.1}$$

with initial conditions

$$x(t) \geq 0, y(t) \geq 0. \tag{2.2}$$

Here x, y denote the density of the prey and predator population, respectively, at the time t .

3. MATHEMATICAL APPROACHES FOR DYNAMICS OF THE MODEL

3.1. Positivity.

Theorem 3.1. *Every solution of system (2.1) with initial conditions (2.2) exists in the interval $[0, \infty)$ and $x(t) \geq 0, y(t) \geq 0$ for all $t \geq 0$.*

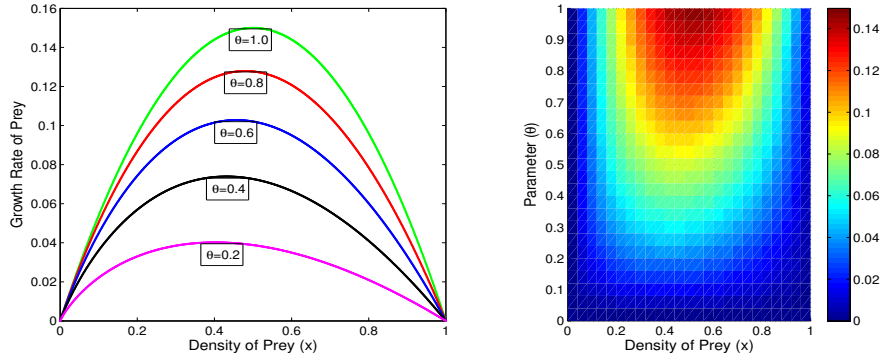


FIGURE 1. Left: graphical representation of the theta-logistic growth function for θ . Right: surface plot where $r = 0.6$, $k = 1$.

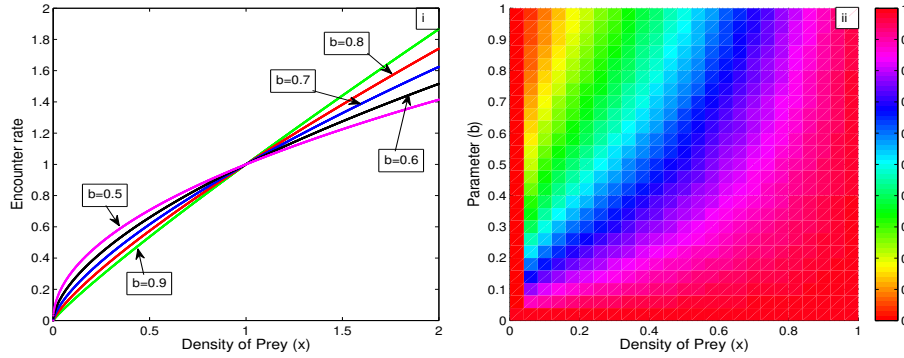


FIGURE 2. (I) Graphical representation of functional responses for b . (II) Surface plot.

Proof. Since the right-hand side of the system is entirely continuous and locally Lipschitzian on C , the solution $(x(t), y(t))$ of (2.1) with initial conditions (2.2) exists and is unique on $[0, \xi]$ where $0 < \xi \leq +\infty$. From the system (2.1) with initial conditions (2.2), we have

$$\begin{aligned} x(t) &= x(0) \exp \left[\int_0^t \left\{ r \left(1 - \left(\frac{x(s)}{k} \right)^\theta \right) - \alpha (x(s))^{(b-1)} y(s) \right\} ds \right] \geq 0, \\ y(t) &= y(0) \exp \left[\int_0^t \left\{ \beta (x(s))^b - d \right\} ds \right] \geq 0, \end{aligned}$$

which completes the proof. \square

3.2. Boundedness.

Theorem 3.2. *All solutions of the system (2.1) are uniformly bounded.*

Proof. Let $(x(t), y(t))$ be any solution of system (2.2). Since

$$\frac{dx}{dt} \leq rx \left(1 - \left(\frac{x}{k} \right)^\theta \right),$$

we have $\lim_{t \rightarrow \infty} \sup x(t) \leq k$. Let $w = x + \frac{\alpha}{\beta}y$, then

$$\frac{dw}{dt} = \frac{dx}{dt} + \frac{\alpha}{\beta} \frac{dy}{dt} = rx(1 - (\frac{x}{k})^\theta) - \frac{\alpha d}{\beta}y \leq 2rk - \rho w, \text{ where } \rho = \min\{r, d\}.$$

Therefore

$$\frac{dw}{dt} + \rho w \leq 2rk.$$

Applying differential inequalities, we obtain

$$0 \leq w(x, y) \leq \frac{2rk}{\rho} + w(x(0), y(0)) \exp(-\rho t).$$

Letting $t \rightarrow \infty$ in the above leads to $0 \leq w(x, y) \leq 2rk/\rho$. Thus, all solutions of (2.1) enter the region

$$B = \left\{ (x, y) : 0 \leq w(x, y) \leq \frac{2rk}{\rho} + \varepsilon, \text{ for any } \varepsilon > 0 \right\}.$$

□

3.3. Equilibria. The proposed system (2.1) has three potential non-negative equilibria, to be specific:

- (i) Trivial equilibrium point $P_0(0, 0)$.
- (ii) Predator free equilibrium point $P_1(k, 0)$.
- (iii) Coexistence equilibrium point $P_2(x^*, y^*)$, where

$$x^* = \left(\frac{d}{\beta}\right)^{\frac{1}{b}} > 0, \quad \text{and} \quad y^* = \frac{r}{\alpha} \left(\frac{d}{\beta}\right)^{\frac{1-b}{b}} \left[1 - \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}}\right].$$

Thus, the critical coexistence point $P_2(x^*, y^*)$ exist if $k^b > \frac{d}{\beta}$. Therefore $P_2(x^*, y^*)$ is positive if $k^b > \frac{d}{\beta}$ and negative if $k^b < \frac{d}{\beta}$. When $k^b = \frac{d}{\beta}$ then P_2 bifurcates into P_1 . Our primary interest is coexistence equilibrium point $P_2(x^*, y^*)$ when $k^b > \frac{d}{\beta}$.

3.4. Local stability analysis. We want to explore the local stability analysis of our proposed model system (2.1) by obtaining the eigenvalue of the community matrix corresponding to respective equilibrium points.

The Community matrix is given by

$$J(x, y) = \begin{bmatrix} r - \frac{r(\theta+1)x^\theta}{k^\theta} - \alpha bx^{(b-1)}y & -\alpha x^b \\ \beta bx^{(b-1)}y & \beta x^b - d \end{bmatrix}.$$

Theorem 3.3. *The system (2.1) in the neighborhood of the predator-free critical point $P_1(k, 0)$ is locally asymptotically stable if $k^b < \frac{d}{\beta}$ and unstable if $k^b > \frac{d}{\beta}$.*

Proof. At $P_1(k, 0)$, the Jacobian matrix of the model is

$$J(P_1(k, 0)) = \begin{bmatrix} -r\theta & -\alpha k^b \\ 0 & \beta k^b - d \end{bmatrix}.$$

Eigenvalues of the Jacobian matrix are $-r\theta$ and $\beta k^b - d$. Therefore, the condition of existence of negative eigenvalues is $k^b < \frac{d}{\beta}$. □

Theorem 3.4. *The model (2.1) in the neighbourhood of the interior critical point $P_2(x^*, y^*)$ is locally asymptotically stable if*

$$\frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} > \frac{1-b}{1-b+\theta}; \tag{3.1}$$

and unstable if (3.1) is reversed.

Proof. At $P_2(x^*, y^*)$ Jacobian matrix of the model system 2.1 is

$$J(P_2(x^*, y^*)) = \begin{bmatrix} r(1-b) - r(1-b+\theta)\frac{1}{k^\theta}\left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} & -\frac{\alpha d}{\beta} \\ \frac{br\beta}{\alpha}\left[1 - \frac{1}{k^\theta}\left(\frac{d}{\beta}\right)^{\frac{\theta}{b}}\right] & 0 \end{bmatrix}.$$

Hence

$$\det J = bdr \left[1 - \frac{1}{k^\theta}\left(\frac{d}{\beta}\right)^{\frac{\theta}{b}}\right] > 0$$

since $k^b > d/\beta$ and

$$\text{tr} J = r(1-b) - r(1-b+\theta)\frac{1}{k^\theta}\left(\frac{d}{\beta}\right)^{\frac{\theta}{b}}.$$

The characteristic equation of J at P_2 is

$$\lambda^2 - \lambda \left[r(1-b) - r(1-b+\theta)\frac{1}{k^\theta}\left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} \right] + bdr \left[1 - \frac{1}{k^\theta}\left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} \right] = 0.$$

Therefore, the roots of the characteristic equation of $J(P_2(x^*, y^*))$ has negative real part if and only if $\text{tr} J < 0$, which is equivalent to (3.1). This proof is completed. \square

3.5. Existence of limit cycles. In predator-prey systems existence and stability of a limit cycle is related to the existence and stability of a positive equilibrium. If the equilibrium is asymptotically stable, there may exist limit cycles, the innermost of which must be unstable from the inside and the outermost of which must be stable from the outside. If the limit cycles do not exist in this case, the equilibrium is globally asymptotically stable. If the positive equilibrium exists and is unstable, there must occur at least one limit cycle. Kuang and Freedman [21] presented the following result on the uniqueness of limit cycles for the system:

$$\begin{cases} \frac{dx}{dt} = xg(x) - yp(x), & x(0) \geq 0, \\ \frac{dy}{dt} = y[-\gamma + q(x)], & y(0) \geq 0 \end{cases} \quad (3.2)$$

where $\gamma > 0$ and all the functions are sufficiently smooth on $[0, \infty)$ and satisfy $p(0) = q(0) = 0$ and $\frac{dp}{dx} > 0$, $\frac{dq}{dx} > 0$ for $x > 0$.

Theorem 3.5. *If there exist constants x^* and m with $0 < x^* < m$; such that $q(x^*) = \gamma$ and $(x - m)g(x) < 0$ for $x \neq m$;*

$$\left[\frac{d}{dx} \left(\frac{xg(x)}{p(x)} \right) \right]_{x=x^*} > 0, \quad \text{and} \quad \frac{d}{dx} \left[\frac{xg'(x) + g(x) - xg(x)\frac{p'(x)}{p(x)}}{-\gamma + q(x)} \right] \leq 0 \text{ for } x \neq x^*.$$

Then the system (3.2) has exactly one limit cycle which is globally asymptotically stable for the set $\{(x, y) | x > 0, y > 0\} \setminus \{E^(x^*, y^*)\}$.*

Using the above result, we get the following result of our proposed model system (2.1).

Theorem 3.6. *The proposed system (2.1) has a unique limit cycle if $\theta > b$*

$$\frac{1}{k^\theta} \left(\frac{d}{\beta} \right)^{\frac{\theta}{b}} < \frac{1-b}{1-b+\theta}. \quad (3.3)$$

Proof. In order to apply the results in Theorem 3.5 to (2.1), we let $p(x) = \alpha x^b$, $q(x) = \beta x^b$, $\gamma = d$ and

$$g(x) = r \left(1 - \left(\frac{x}{k} \right)^\theta \right).$$

Denote $x^* = (d/\beta)^{1/b}$ and $m = k$. It is clear that $p(x)$ and $q(x)$ satisfy all assumptions in Theorem 3.5. Now

$$\left[\frac{d}{dx} \left(\frac{xg(x)}{p(x)} \right) \right]_{x=x^*} = \left[\frac{d}{dx} \left(\frac{rx(k^\theta - x^\theta)}{\alpha k^\theta x^b} \right) \right]_{x=x^*} = \frac{1-b}{1-b+\theta} - \frac{1}{k^\theta} \left(\frac{d}{\beta} \right)^{\frac{\theta}{b}}$$

Therefore

$$\left[\frac{d}{dx} \left(\frac{xg(x)}{p(x)} \right) \right]_{x=x^*} > 0 \quad \text{if (3.3) holds.}$$

Also note that

$$xg'(x) + g(x) - xg(x) \frac{p'(x)}{p(x)} = r \left[(1-b) - (1-b+\theta) \left(\frac{x}{k} \right)^\theta \right] \quad \text{and} \quad -\gamma + q(x) = -d + \beta x^b$$

and

$$\begin{aligned} \frac{d}{dx} \left[\frac{xg'(x) + g(x) - xg(x) \frac{p'(x)}{p(x)}}{-\gamma + q(x)} \right] &= \frac{d}{dx} \left[\frac{r \left[(1-b)k^\theta - (1-b+\theta)x^\theta \right]}{k^\theta(-d + \beta x^b)} \right] \\ &= \frac{(1-b+\theta)rb\beta x^{b-1}}{(-d + \beta x^b)^2} \left[\frac{x^{\theta-b} \{ \theta d + \beta(b-\theta)x^b \}}{b\beta k^\theta} - \frac{1-b}{1-b+\theta} \right] \end{aligned}$$

Since maximum value of function

$$\max \frac{x^{\theta-b} \{ \theta d + \beta(b-\theta)x^b \}}{b\beta k^\theta} = \frac{1}{k^\theta} \left(\frac{d}{\beta} \right)^{\frac{\theta}{b}}$$

provided that $\theta > b$. Therefore

$$\frac{d}{dx} \left\{ \frac{xg'(x) + g(x) - xg(x) \frac{p'(x)}{p(x)}}{-\gamma + q(x)} \right\} \leq 0 \quad \text{if} \quad \frac{1}{k^\theta} \left(\frac{d}{\beta} \right)^{\frac{\theta}{b}} \leq \frac{1-b}{1-b+\theta}.$$

Hence, by Theorem 3.5, the system (2.1) has exactly one limit cycle, which is globally asymptotically stable if $\theta > b$ and (3.3) holds, completing the proof. \square

4. HOPF BIFURCATION

Hopf-bifurcation is defined as the appearance or disappearance of a periodic orbit through a local change in the stability properties of an equilibrium point. Here we explore the possibility of occurrence of Hopf-bifurcation around the interior equilibrium point $P_2(x^*, y^*)$ for bifurcating parameter θ .

Theorem 4.1. *The necessary and sufficient conditions for the system (2.1) to undergo Hopf-bifurcation at $\theta = \theta_h$ around the interior equilibrium point $P_2(x^*, y^*)$ are that $\det J(x^*(\theta), y^*(\theta)) > 0$ and*

$$\frac{d(\text{tr}J(x^*(\theta), y^*(\theta)))}{d\theta} \neq 0 \quad \text{at} \quad \theta = \theta_h.$$

Proof. Consider the community matrix evaluated at $P_2(x^*, y^*)$. For Hopf-bifurcation, the matrix $J(P_2(x^*, y^*))$ must have a pair of purely imaginary eigenvalues. At a critical value $\theta = \theta_h$,

$$\text{tr}J(x^*(\theta), y^*(\theta)) = r(1-b) - r(1-b+\theta) \frac{1}{k^\theta} \left(\frac{d}{\beta} \right)^{\frac{\theta}{b}} = 0.$$

Here θ_h is the positive solution of the equation

$$r(1-b) - r(1-b+\theta) \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} = 0.$$

So the characteristic equation of the community matrix becomes

$$\lambda^2 + \det J(x^*(\theta), y^*(\theta)) = 0.$$

The above equation must have a pair of purely imaginary roots $\lambda_{1,2} = \pm i\rho$ where

$$\rho = \sqrt{\det J(x^*(\theta), y^*(\theta))}$$

if $\det J(x^*(\theta), y^*(\theta)) > 0$ at $\theta = \theta_h$. Now we check the transversality condition which confirms that the eigenvalues cross the imaginary axis of the complex plane with non-zero speed. For this, let at any neighboring point θ of θ_h , the eigenvalues of the community matrix are $\lambda_{1,2} = \chi(\theta) \pm i\rho(\theta)$ where

$$\chi(\theta) = \frac{\operatorname{tr} J(x^*(\theta), y^*(\theta))}{2}$$

and

$$\rho(\theta) = \frac{\sqrt{4 \det J(x^*(\theta), y^*(\theta)) - (\operatorname{tr} J(x^*(\theta), y^*(\theta)))^2}}{2}.$$

Now,

$$\frac{d\chi(\theta)}{d\theta} = \frac{d(\operatorname{tr} J(x^*(\theta), y^*(\theta)))}{d\theta}.$$

Thus, the transversality condition satisfied if

$$\frac{d(\operatorname{tr} J(x^*(\theta), y^*(\theta)))}{d\theta} \neq 0$$

at $\theta = \theta_h$. Therefore, the system (2.1) undergoes Hopf-bifurcation at $\theta = \theta_h$. \square

5. IMPACT OF b AND θ IN THE SYSTEM DYNAMICS

Here, θ is a dimensionless parameter influencing the curvature of the relationship between the exponential rate of growth and prey population density. b is the slope of the linear regression of $\ln(\text{prey group density})$ versus $\ln(\text{prey density}, x)$. Our interest in this section is to study the effect of these two parameters in this system separately and jointly.

5.1. Impact of b in the system. To check the influence of b , let differentiate x^*, y^* with respect to b , we acquire

$$\begin{aligned} \frac{dx^*}{db} &= -\frac{1}{b^2} \left(\frac{d}{\beta}\right)^{\frac{1}{b}} \log\left(\frac{d}{\beta}\right) = \begin{cases} < 0 & \text{if } d > \beta, \\ > 0 & \text{if } d < \beta; \end{cases} \\ \frac{dy^*}{db} &= -\frac{1}{b^2} \frac{r}{\alpha} \left(\frac{d}{\beta}\right)^{\frac{1-b}{b}} \log\left(\frac{d}{\beta}\right) \left[1 - \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right)\right]. \end{aligned}$$

Now,

$$\frac{dy^*}{db} > 0 \text{ for } \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} > \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right)^{-1} \text{ and } d < \beta;$$

$$\frac{dy^*}{db} < 0 \text{ for } \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} < \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right)^{-1} \text{ and } d < \beta;$$

$$\frac{dy^*}{db} > 0 \text{ for } \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} < \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right)^{-1} \text{ and } d > \beta;$$

$$\frac{dy^*}{db} < 0 \text{ for } \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} > \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right)^{-1} \text{ and } d > \beta.$$

Therefore, the parameter (b) has a significant effect on both population densities. The density of prey and predator population increases or decreases under some parametric restrictions.

5.2. Impact of θ in the system. To check the influence of θ , let differentiate x^* , y^* with respect to θ , we acquire

$$\frac{dx^*}{d\theta} = 0, \text{ since } x^* \text{ is independent of } \theta; \text{ and}$$

$$\frac{dy^*}{d\theta} = \frac{r}{\alpha} \left(\frac{d}{\beta}\right)^{\frac{1+\theta-b}{b}} \frac{1}{k^\theta} \left[\log k - \frac{1}{b} \log\left(\frac{d}{\beta}\right) \right].$$

Therefore,

$$\frac{dy^*}{d\theta} > 0, \text{ for } k > \left(\frac{d}{\beta}\right)^{\frac{1}{b}}; \text{ and } \frac{dy^*}{d\theta} < 0 \text{ for } k < \left(\frac{d}{\beta}\right)^{\frac{1}{b}}.$$

Therefore, it is obvious that there is no role of θ to control the density of prey species while predator density decreases with the increase of θ for $k < (d/\beta)^{1/b}$ and predator density increases with the increase of θ for $k > (d/\beta)^{1/b}$.

5.3. Joint impact of b and θ in the system. To explore the joint impact of these parameters, we have computed the following expression

$$\frac{d}{d\theta} \left(\frac{dx^*}{db} \right) = 0;$$

$$\begin{aligned} \frac{d}{d\theta} \left(\frac{dy^*}{db} \right) &= \frac{d}{d\theta} \left[-\frac{1}{b^2} \frac{r}{\alpha} \left(\frac{d}{\beta}\right)^{\frac{1-b}{b}} \log\left(\frac{d}{\beta}\right) \left[1 - \left(\frac{1}{k^\theta}\right) \left(\frac{d}{\beta}\right)^{\frac{\theta}{b}} \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right) \right] \right] \\ &= \frac{1}{b^2} \frac{r}{\alpha} \frac{1}{k^\theta} \left(\frac{d}{\beta}\right)^{\frac{1+\theta-b}{b}} \log\left(\frac{d}{\beta}\right) \left[\left(\frac{1}{b} \log\left(\frac{d}{\beta}\right) - \log k\right) \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right) + \log\left(\frac{d}{\beta}\right) \right]. \end{aligned}$$

Thus,

$$\frac{d}{d\theta} \left(\frac{dy^*}{db} \right) > 0 \text{ for } \left[\left(\frac{1}{b} \log\left(\frac{d}{\beta}\right) - \log k\right) \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right) + \log\left(\frac{d}{\beta}\right) \right] > 0 \text{ and } d > \beta;$$

$$\frac{d}{d\theta} \left(\frac{dy^*}{db} \right) < 0 \text{ for } \left[\left(\frac{1}{b} \log\left(\frac{d}{\beta}\right) - \log k\right) \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right) + \log\left(\frac{d}{\beta}\right) \right] < 0 \text{ and } d > \beta;$$

$$\frac{d}{d\theta} \left(\frac{dy^*}{db} \right) > 0 \text{ for } \left[\left(\frac{1}{b} \log\left(\frac{d}{\beta}\right) - \log k\right) \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right) + \log\left(\frac{d}{\beta}\right) \right] < 0 \text{ and } d < \beta,$$

and

$$\frac{d}{d\theta} \left(\frac{dy^*}{db} \right) < 0 \text{ for } \left[\left(\frac{1}{b} \log\left(\frac{d}{\beta}\right) - \log k\right) \left(1 + \theta \log\left(\frac{d}{\beta}\right)\right) + \log\left(\frac{d}{\beta}\right) \right] > 0 \text{ and } d < \beta.$$

Therefore, θ and b cannot make any impact on the prey population density while the density of the predator population increases under a parametric restriction.

6. NUMERICAL SIMULATION

In this section, we analyze the propose predator-prey system numerically base on some hypothetical data as given in Table 2. The equilibrium points with their stability analysis and bifurcation analysis for the vital parameters have been discussed of the proposed system using these values of the parameters in this section.

Table 2: Parameter values used for simulations of the system

Case	r	k	θ	α	β	b	d	Figure
1	0.6	0.95	0.8	0.8	0.75	0.5	0.4	3, 10, 11, 12
2	0.6	0.95	0.8	0.8	0.75	0.55	0.4	4
3	0.6	0.95	0.9	0.8	0.75	0.5	0.4	5
4	0.6	0.95	0.6	0.8	0.75	0.5	0.4	6
5	0.6	0.95	0.8	0.8	0.5	0.5	0.6	7(i)
					0.7		0.5	7(ii)
6	0.6	0.95	[0.4, 1.0]	0.8	0.75	0.5	0.4	8
7	0.6	0.95	0.8	0.8	0.75	[0.5, 0.6]	0.4	9

For $r = 0.6, k = 0.95, \theta = 0.8, \alpha = 0.8, \beta = 0.75, b = 0.5, d = 0.4$, the system (2.1) has three non-negative equilibrium points, trivial singular point $P_0(0, 0)$, predator-free singular point $P_1(0.95, 0)$, coexistence singular point $P_2(0.2844, 0.2476)$ and the nature of these equilibrium points are given in Table 3

Table 3: Nature of each equilibrium point

Equilibrium point	Eigenvalues	Nature
$P_1(0.95, 0)$	$-0.4690, 0.3310$	Unstable
$P_2(0.2844, 0.2476)$	$0.0471 + 0.2684i, 0.0471 - 0.2684i$	Unstable

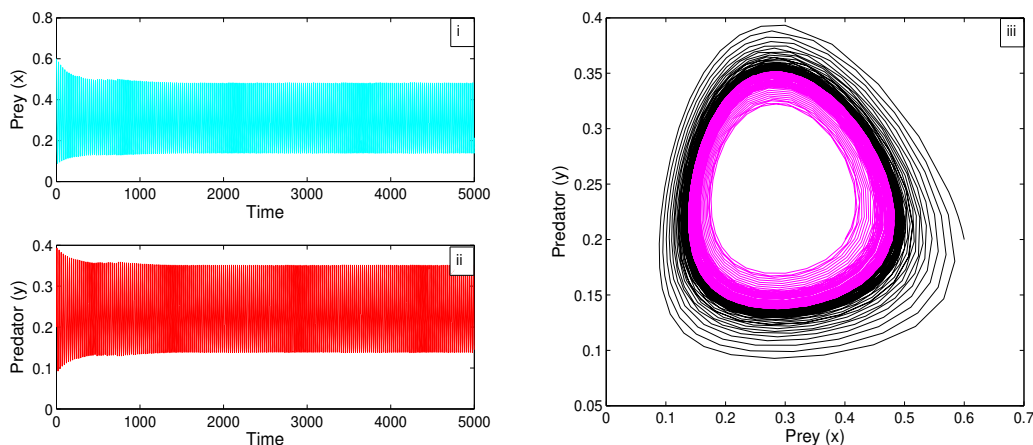


FIGURE 3. Topological nature of system for the parameters as in Table 2. (I) and (II) Exhibit the time series solution while (III) shows the occurrence of a limit cycle.

To see the effect of the parameter b , we draw time series solution for $b = 0.5$ (see Figure 3(i, ii)) exhibits the periodic oscillation and the corresponding phase portrait is given in Figure 3(iii) shows limit cycle oscillation. But when we increase the value b , i.e., $b = 0.55$, the system (2.1) displays stability

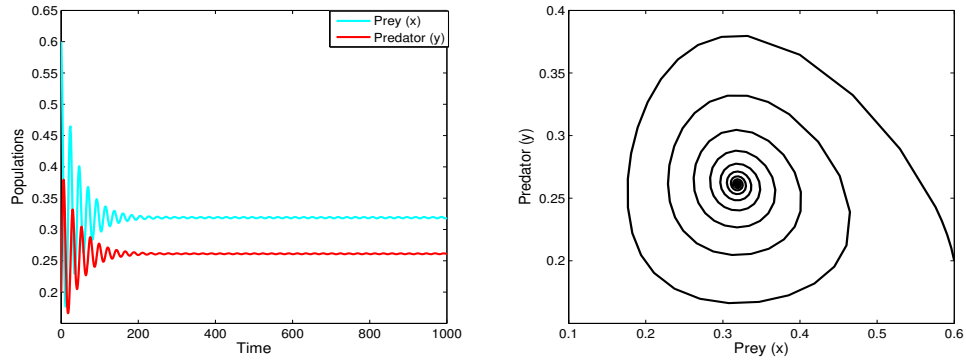


FIGURE 4. Left panel exhibits the time series solution, and right panel demonstrates the corresponding phase diagram.

behavior near the coexistence equilibrium point. Figure 4(i) for time series solution and Figure 4(ii) for corresponding phase portrait for $b = 0.55$. Hence, a Hopf bifurcation occurs to parameter b in the interval $[0.4, 1.0]$ and Figure 9 reflects the bifurcation diagram for b . Figure 9 describes that the system remains unstable until b crosses its threshold value $b = b^* = 0.54$, when b crosses its threshold value the system becomes stable. Now, it is easy to conclude that b has a significant role in controlling the system dynamics.

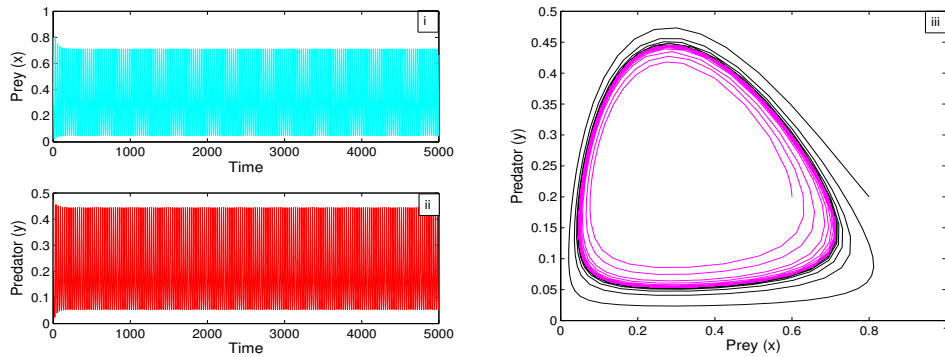


FIGURE 5. Topological nature of system: (I) and (II) Exhibit the time series solution while (III) indicates the occurrence of a limit cycle.

To explore the impact of θ , first we have considered $\theta = 0.6$ and observe that the system becomes stable as shown in Figure 6. Figure 6(i) shows time series solution and Figure 6(ii) shows the phase portrait of the system dynamics. However, when θ crosses the threshold value $\theta = 0.7$ it shows its periodic oscillation behaviour near the coexistence equilibrium point as displayed in Figure 5. The periodic oscillation for prey, predator and corresponding phase portrait are shown in Figure 5(i, ii, iii), respectively. Therefore, system (2.1) experiences Hopf bifurcation for the parameter θ . The Hopf point occurs at $\theta = 0.7$ and the corresponding bifurcation diagrams are given in Figure 8. Figure 7(i, ii) shows global stability property of the model system.

Figure 8 display that all the populations are in a stable situation until it crosses $\theta = 0.7$ and then become unstable in $(0.7, 1.0]$. Figure 9 display that all the populations are in unstable situation until it

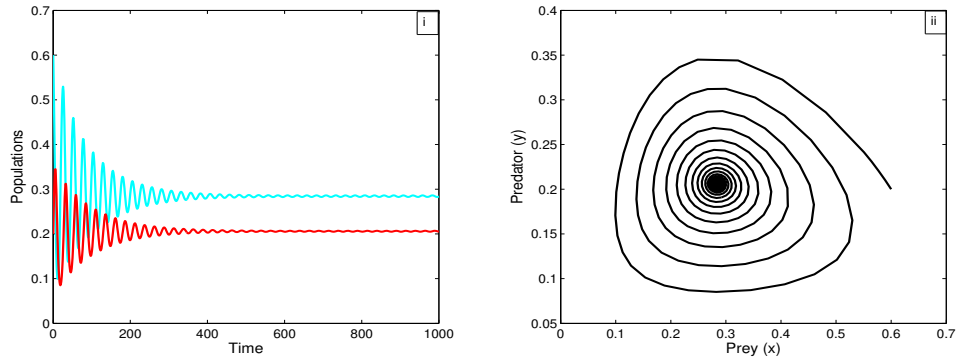


FIGURE 6. Topological nature of system: (I) Exhibits the time series solution while (II) demonstrates the corresponding phase diagram.

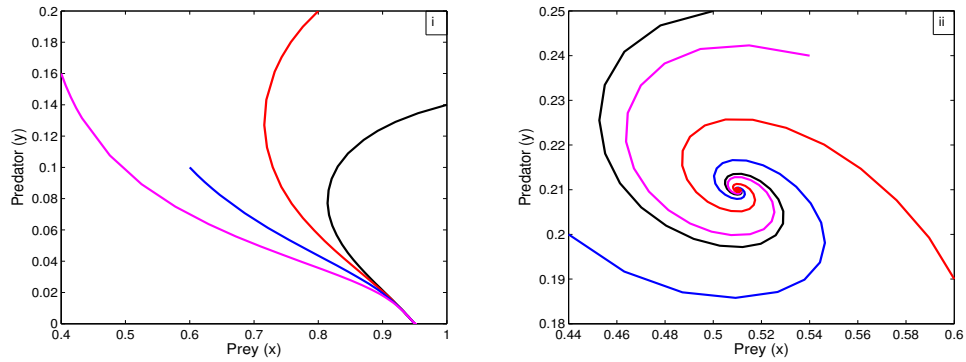


FIGURE 7. Global stability of the system (2.1) in (I) the predator free equilibrium; (II) the coexistence coexistence equilibrium.

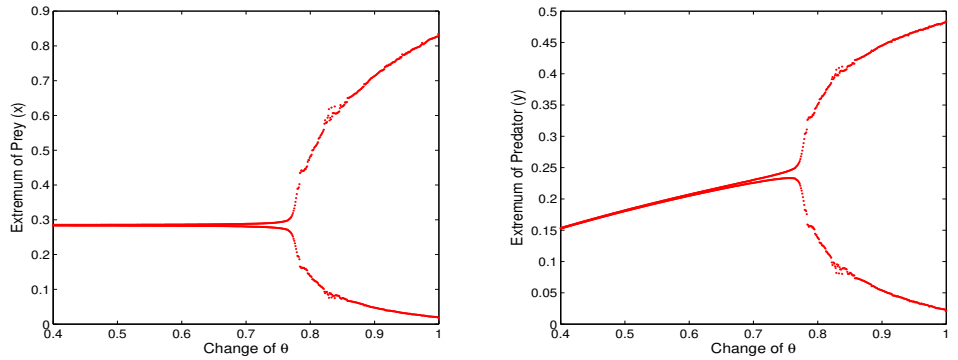


FIGURE 8. Bifurcation diagram for θ in $[0.4, 1]$.

crosses $b = 0.55$ and then become stable in $(0.55, 0.6]$. The coexistent singular point for different value of b and θ shown in Figure 10. The coexistent singular point for different value of b and θ for prey and predator and their corresponding surface plots are presented in Figure 11.

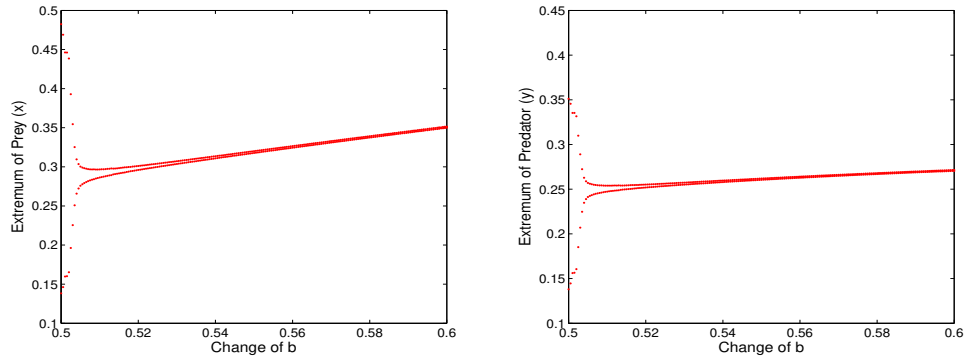


FIGURE 9. Bifurcation diagram for b in $[0.5, 0.6]$.

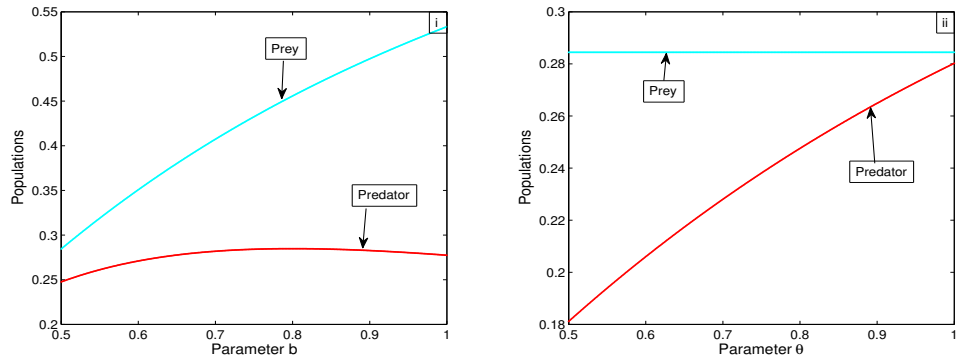


FIGURE 10. Graphical representation of the coexistent singular point for different value of b in (I) and for different value of θ in (II), keeping other parameters the same as in Table 2.

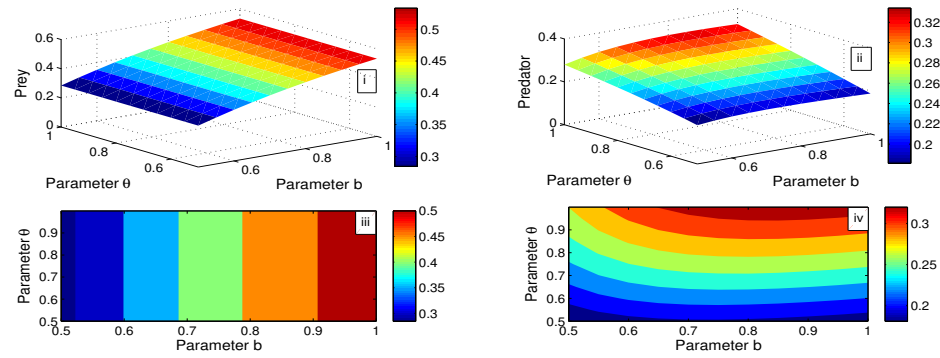


FIGURE 11. Graphical representation of the coexistent singular point for different value of b and θ in (I) for prey and in (II) for predator, corresponding surface plots are (III) and (IV) respectively, keeping other parameters the same as in Table 2.

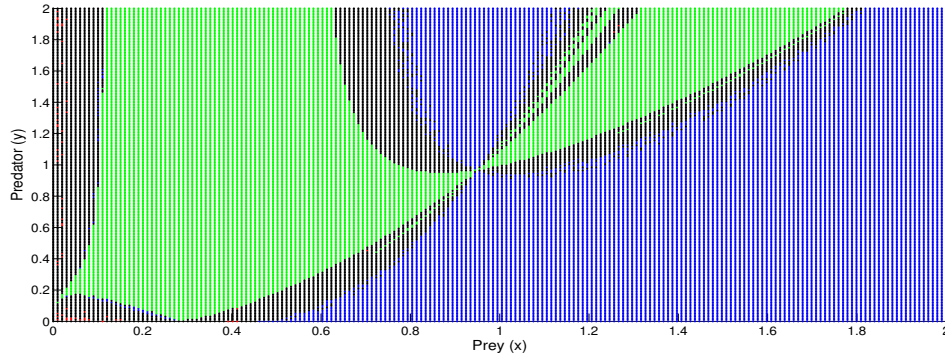


FIGURE 12. Basins of attraction of our model for the values from table 2.

Figure 12 shows the basins of attraction of the proposed model for the values from table 2. The red star is the basin of attraction of the extinction equilibrium $(0, 0)$. The blue star is the basin of attraction of the equilibrium where only prey species survive. The green stars are the basin of the coexistence equilibrium. Finally black star for not above three cases.

7. CONCLUSION

In conclusion, this research paper has investigated the dynamics of a predator-prey model with theta-logistic growth and herd behavior, exploring its topological nature and its impact on biological consequences such as limited food resources and habitat. The proposed system has been shown to exhibit positivity and boundedness, with the existence and stability conditions of its three equilibria derived analytically. Our analysis has identified the occurrence of Hopf bifurcation in the system, with the conversion coefficient of food biomass to predator having a significant effect on both species in the coexisting equilibrium point. The parameter of theta-logistic growth was found to have no impact on prey, but a clear influence on predator. Furthermore, the combined effect of θ and b on prey and predator in the coexisting equilibrium point was explored, along with the attractive basin of attraction of the proposed system. Based on our findings, we conclude that θ and b have a significant effect on the predator-prey model with theta-logistic growth and herd behavior. This research has contributed to the existing knowledge on the dynamics of predator-prey models and can inform further studies on the subject.

REFERENCES

- [1] V. Ajraldi, M. Pittavino, E. Venturino, *Modeling herd behavior in population systems*, *Nonlinear Anal. Real World Appl.* **12**(2011), 2319–2338.
- [2] K. Antwi-Fordjour, R. D. Parshad, M. A. Beauregard, *Dynamics of a predator-prey model with generalized Holling type functional response and mutual interference*, *Mathematical Biosciences*, **326**(2020): 108407.
- [3] M. Astrom, *The paradox of biological control revisited: per capita non-linearities*, *Oikos* **78**(1997), 618-621.
- [4] M. Banerjee, B. W. Kooi, E. Venturino, *An Ecoepidemic Model with Prey Herd Behavior and Predator Feeding Saturation Response on Both Healthy and Diseased Prey*, *Math. Model. Nat. Phenom.* **12**(2)(2017), 133-161.
- [5] S. Biswas, D. Pal, P. K. Santra, E. Bonyah, and G. S. Mahapatra, *Dynamics of a Three-Patch Prey-Predator System with the Impact of Dispersal Speed Incorporating Strong Allee Effect on Double Prey*, *Discrete Dyn. Nat. Soc.* (2022)(2022): Article ID 7919952
- [6] P. A. Braza, *Predator-prey dynamics with square root functional responses*, *Nonlinear Anal. Real World Appl.* **13**(2012), 1837–1843.

- [7] I. M. Bulai, E. Venturino, *Shape effects on herd behavior in ecological interacting population models*, Math. Comput. Simul. **141**(2017), 40–55.
- [8] L. Chen, F. Chen, L. Chen, *Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge*, Nonlinear Anal. Real World Appl. **11**(1)(2010), 246–252.
- [9] R. Chinnathambi, F. A. Rihan, *Stability of fractional-order prey-predator system with time-delay and Monod-Haldane functional response*, Nonlinear Dynamics, **92**(2018), 1637–1648.
- [10] F. Clark, B. W. Brook, S. Delean, H. Resit Akcakaya, and C. J. A. Bradshaw, *The theta-logistic is unreliable for modelling most census data*, Methods Ecol. Evol. **1**(3)(2010), 253–262.
- [11] B. Dai, H. Su, D. Hu, *Periodic Solution of a delayed ratio-dependent predator-prey model with monotonic functional response and impulse*, Nonlinear Anal. Theory Methods Appl. **70**(2009), 126–134.
- [12] S. Djilali, *Impact of prey herd shape on the predator-prey interaction*, Chaos Solit. Fractals, **120**(2019), 139–148.
- [13] Z. Du, Z. Feng, *Periodic solutions of a neutral impulsive predator-prey model with Beddington-DeAngelis functional response with delays*, J. Comput. Appl. Math. **258**(2014), 87–98.
- [14] D. Ghosh, P.K. Santra, O.D. Makinde, G.S. Mahapatra, *Dynamics of predator-dependent nonlinear disease transmission in infected prey with diverse functional response*, Filomat **37**(13)(2023).
- [15] M. E. Gilpin and F. J. Ayala, *Global Models of Growth and Competition*, Proc. Natl. Acad. Sci. **70**(12)(1973), 3590–3593.
- [16] Z. Hu, Z. Teng, L. Zhang, *Stability and bifurcation analysis of a discrete predator-prey model with nonmonotonic functional response*, Nonlinear Anal. Real World Appl. **12**(4)(2011), 2356–2377.
- [17] J. Huang, S. Ruan, J. Song, *Bifurcations in a predator-prey system of Leslie type with generalized Holling type III functional response*, J. Differ. Equ. **257**(2014), 1721–1752.
- [18] Q. Jiang, J. Wang, *Qualitative analysis of a harvested predator-prey system with Holling type III functional response*, Adv. Differ. Equ. **2013**(2013): 249.
- [19] M. Kaur, R. Rani, R. Bhatia, G.N. Verma, S. Ahirwar, *Dynamical study of quadrating harvesting of a predator-prey model with Monod-Haldane functional response*, J. Appl Math. Comput. **1-2**(2021), 397–422.
- [20] M. C. Köhnke, I. Siekmann, H. Seno, H. Malchow, *A type IV functional response with different shapes in a predator-prey model*, J. Theor. Biol. **505**(2020), 110419.
- [21] Y. Kuang, H. I. Freedman, *Uniqueness of limit cycles in Gause-type models of predator-prey systems*, Math. Biosci. **88**(1988), 67–84.
- [22] H. Li, Y. Takeuchi, *Dynamics of the density dependent predator-prey system with Beddington-DeAngelis functional response*, J. Math. Anal. Appl. **374**(2)(2011), 644–654.
- [23] X. Liu, Q. Huang, (2018), *The dynamics of a harvested predator-prey system with Holling type IV functional response*, Biosystems, **169-170**(2018), 26–39.
- [24] G. S. Mahapatra, P. K. Santra, E. Bonyah, *Dynamics on Effect of Prey Refuge Proportional to Predator in Discrete-Time Prey-Predator Model*, Complexity **2021**(2021): Article ID 6209908
- [25] A. Maiti, P. Sen, G. P. Samanta, *Deterministic and stochastic analysis of a prey-predator model with herd behaviour in both*, Syst. Sci. Control. Eng. **4**(2016), 259–269.
- [26] S. Pal, S. Majhi, S. Mandal and N. Pal, *Role of Fear in a Predator-Prey Model with Beddington-DeAngelis Functional Response*, Zeitschrift für Naturforschung A, **74**(7)(2021), 581–595
- [27] D. Pal, P. Santra, G. S. Mahapatra, *Predator-Prey Dynamical Behavior and Stability Analysis with Square Root Functional Response*, Int. J. Appl. Math. Comput. Sci. **3**(2017), 1833–1845.
- [28] Y. Peng, G. Zhang, *Dynamics analysis of a predator-prey model with herd behavior and nonlocal prey competition*, Math. Comput. Simul. **170**(2020), 366–378.
- [29] A. Rojas-Palma, E. González-Olivares, *Optimal harvesting in a predator-prey model with Allee effect and sigmoid functional response*, Appl. Math. Model. **36**(5)(2012), 1864–1874.
- [30] J. Roy, D. Barman, S. Alam, *Role of fear in a predator-prey system with ratio-dependent functional response in deterministic and stochastic environment*, Biosystems, **197**(2020): 104176.
- [31] S. M. Salman, A. M. Yousef, A. A. Elsadany, *Stability, bifurcation analysis and chaos control of a discrete predator-prey system with square root functional response*, Chaos Solit. Fractals, **93**(2016), 20–31.
- [32] P. K. Santra, G. S. Mahapatra, *Dynamical study of discrete-time prey predator model with constant prey refuge under imprecise biological parameters*, J. Biol. Syst. **28**(3)(2020), 681–699.
- [33] P. K. Santra, G. S. Mahapatra, G. R. Phaijoo, *Bifurcation and chaos of a discrete predator-prey model with Crowley-Martin functional response incorporating proportional prey refuge*, Math. Probl. Eng. **2020**(2020), 1–18.
- [34] H.vB. Shi, Y. Li, *Global asymptotic stability of a diffusive predator-prey model with ratio-dependent functional response*, Comput. Appl. Math. **250**(2015), 71–77.

- [35] J. P. Tripathi, S. Abbas, M. Thakur, *Dynamical analysis of a prey–predator model with Beddington–DeAngelis type function response incorporating a prey refuge*, *Nonlinear Dynamics*, **80**(2015), 177–196.
- [36] J. P. Tripathi, S. Tyagi, S. Abbas, *Global analysis of a delayed density dependent predator–prey model with Crowley–Martin functional response*, *Communications in Nonlinear Science and Numerical Simulation*, **30**(2016), 45–69.
- [37] K. Wang, *Periodic solutions to a delayed predator–prey model with Hassell–Varley type functional response*, *Nonlinear Anal. Real World Appl.* **12**(2011), 137–145.
- [38] W. Wang, L. Zhang, H. Wang, Z. Li, (2010), *Pattern formation of a predator–prey system with Ivlev-type functional response*, *Ecol. Modell.* **221**(2)(2010), 131–140.
- [39] Y. Xie, Z. Wang, B. Meng, X. Huang, *Dynamical analysis for a fractional-order prey–predator model with Holling III type functional response and discontinuous harvest*, *Appl. Math. Lett.* **106**(2020): 106342.
- [40] W. Yang, *Analysis on existence of bifurcation solutions for a predator–prey model with herd behavior*, *Appl. Math. Model.* **53**(2018), 433–446.

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