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THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A NONLINEAR TOXIN-DEPENDENT SIZE-STRUCTURED POPULATION MODEL

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Abstract. In this paper, we study a toxin-mediated size-structured population model with nonlinear reproduction, growth, and mortality rates. By using the characteristic method and the contraction mapping argument, we establish the existence-uniqueness of solutions to the model. We also prove the continuous dependence of solutions on initial conditions.

1. Introduction

How do anthropogenic and natural environmental toxins affect population dynamics and ecological integrity? It is an essential question in environmental toxicology [1, 11]. Mathematical models (including individual-based models, matrix population models, ordinary differential equation models, and so on) have been widely applied to address this question [4, 5, 6, 7, 9]. In terms of the fact that in a population, individuals of different sizes may have different sensitivities to toxins, Huang and Wang [8] developed a size-structured population model for a population living in an aquatic polluted ecosystem, which is given by the following system of nonlinear first-order hyperbolic equations:

$$\begin{cases} u_t + (g(x, P(t))u)_x + \mu(x, P(t), v(x,t))u = 0, & x \in (x_{\min}, x_{\max}), t > 0, \\ v_t + g(x, P(t))v_x + \sigma(x,t)v - a(x,t)E(t) = 0, & x \in (x_{\min}, x_{\max}), t > 0, \\ g(x_{\min}, P(t))u(x_{\min}, t) = \displaystyle\int_{x_{\min}}^{x_{\max}} \beta(x, P(t), v(x,t))u(x,t)\mathrm{d}x, & t > 0, \\ v(x_{\min}, t) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in (x_{\min}, x_{\max}), \\ v(x,0) = v_0(x), & x \in (x_{\min}, x_{\max}) \end{cases} \tag{1.1}$$

where $u(x,t)$ represents the density of individuals of size x at time t; $P(t) = \int_{x_{\min}}^{x_{\max}} u(x,t)dx$ is the total population biomass at time t, where $x_{\min}$ and $x_{\max}$ denote the minimize size and the maximum size of the population, respectively; $v(x,t)$ denotes the size-dependent body burden — concentration of toxin per unit population biomass. The function $g(x, P(t))$ represents the growth rate of an individual of size x which depends on the total population biomass due to competition for resources. The function

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$\mu(x, P(t), v(x,t))$ denotes the mortality rate of an individual of size x which depend on the total population biomass and the body burden. The function $\beta(x, P(t), v(x,t))$ is the reproduction rate of an individual of size x. The function $\sigma(x,t)$ is the toxin elimination rate due to the metabolic process of the population. The function $a(x,t)$ represents the toxin uptake rate by the population from the environment. The function $E(t)$ is the concentration of toxin in the environment at time t. See [8] for detailed model derivation.

In [8], an explicit finite difference approximation to partial differential equation problem (1.1) was developed. The existence and uniqueness of the weak solution — a solution in integral form with two test functions — were established and convergence of the finite difference approximation to this unique weak solution was proved.

The main purpose of this paper is to prove the existence-uniqueness of solutions of problem (1.1) by using the characteristic method and contraction mapping theorem, and the continuous dependence on initial conditions, which are quite different from the method of numerical approximation used in (1.1).

2. Existence and uniqueness results

Throughout the discussion, we let $\Omega_1 = (x_{\min}, x_{\max}) \times (0, \infty)$ and $\Omega_2 = (x_{\min}, x_{\max}) \times (0,\infty) \times (0, \infty)$. We make the following assumptions on the parameters in problem (1.1):

- (H1) $g(x, P)$ is a strictly positive Lipschitz function with respect to x and P in Ω_1 with a common Lipschitz constant L_g.
- (H2) $\mu(x, P, v)$ is a nonnegative Lipschitz function with respect to x, P, and v in Ω_2 with a common Lipschitz constant L_μ.
- (H3) $\beta(x, P, v)$ is a nonnegative Lipschitz function with respect to x, P, v in Ω_2 with a common Lipschitz constant L_β. Furthermore, $\beta(x, P, v)$ is uniformly bounded in Ω_2 with $0 \le \beta \le \beta_M$.
- (H4) $a(x,t)$ is a nonnegative Lipschitz function with respect to x in Ω_1 with a Lipschitz constant L_a. Furthermore, $a(x,t)$ is uniformly bounded in Ω_1 with $0 \le a \le a_M$.
- (H5) $\sigma(x,t)$ is a nonnegative Lipschitz function with respect to x and t in Ω_1 with a Lipschitz constant L_σ.
- (H6) $E(t)$ is a nonnegative continuous function and bounded for $0 < t < \infty$ with $0 \le E(t) \le E_M$.
- (H7) $u_0(x) \in L^1(x_{\min}, x_{\max})$ and $u_0(x) \ge 0$.
- (H8) $v_0(x)$ is a nonnegative Lipschitz function with a Lipschitz constant L_v and bounded for $x_{\min} < x < x_{\max}$ with $0 \le v_0(x) \le v_M$.

We begin with the definition of the solutions of problem (1.1).

Definition 2.1. A nonnegative function $(u(x,t), v(x,t))$ on $[x_{\min}, x_{\max}] \times [0, T)$, with $u(x,t)$ and $v(x,t)$ integrable, is a solution of (1.1) if $P(t) = \int_{x_{\min}}^{x_{\max}} u(x,t)\mathrm{d}x$ is a continuous function on $[0,T)$ and $(u(x,t), v(x,t))$ satisfies $(1.1)_{3,4,5,6}$ and the equations

$$Du(x,t) = -\tilde{\mu} u(x,t), \tag{2.1}$$

$$Dv(x,t) = -[\sigma(x,t)v - a(x,t)E(t)] \tag{2.2}$$

with

$$Du(x,t) = \lim_{h \to 0} \frac{u(X(t+h; x,t), t+h) - u(x,t)}{h},$$

$$Dv(x,t) = \lim_{h \to 0} \frac{v(X(t+h; x,t), t+h) - v(x,t)}{h},$$

where T is a positive constant, $\tilde{\mu}(x, P(t), v(x,t)) = g_x(x, P(t)) + \mu(x, P(t), v(x,t))$ and $X(t; x_0, t_0)$ is the solution of the equation for the characteristic curves given by

$$\begin{cases} \dfrac{dx}{dt} = g(x, P(t)), \\ x(t_0) = x_0. \end{cases} \tag{2.3}$$

From (H1), we know that the function $X(t; x_0, t_0)$ is strictly increasing. Thus a unique inverse function $\tau(x; x_0, t_0)$ exists. Let $Z(t) = X(t; x_{\min}, 0)$ be the characteristic through the point $(x_{\min}, 0)$. In what follows, we reduce problem (1.1) to a system of coupled equations for $P(t)$ and $B(t)$ by using the method of characteristics, where

$$B(t) = \int_{x_{\min}}^{x_{\max}} \beta(x, P(t), v(x,t)) u(x,t) \mathrm{d}x.$$

Integrating (2.1) along the characteristics, we have

$$u(x,t) = \begin{cases} u_0(X(0;x,t)) e^{-\int_0^t \tilde{\mu}(X(s;x,t), P(s), v(X(s;x,t),s)) \mathrm{d}s}, & x \geq Z(t), \\ \dfrac{B(\tau(x_{\min}))}{g(x_{\min}; P(\tau(x_{\min})))} e^{-\int_{\tau(x_{\min})}^t \tilde{\mu}(X(s;x_{\min},\tau(x_{\min})), P(s), v(X(s;x_{\min},\tau(x_{\min})),s)) \mathrm{d}s}, & x < Z(t), \end{cases} \tag{2.4}$$

where $\tau(x_{\min}) = \tau(x_{\min}; x, t)$. Similarly, we have

$$v(x,t) = \begin{cases} v_0(X(0;x,t)) e^{-\int_0^t \sigma(X(s;x,t),s) \mathrm{d}s} + \displaystyle\int_0^t a(X(s;x,t),s) E(s) e^{-\int_s^t \sigma(X(\tau;x,t),\tau) \mathrm{d}\tau} \mathrm{d}s, & x \geq Z(t), \\ 0, & x < Z(t). \end{cases} \tag{2.5}$$

Then

$$\begin{aligned} P(t) = \int_{x_{\min}}^{x_{\max}} u(x,t) \mathrm{d}x = & \int_0^t B(\eta) e^{-\int_\eta^t \mu(X(s,x_{\min},\eta), P(s), v(X(s;x_{\min},\eta),s)) \mathrm{d}s} \mathrm{d}\eta \\ & + \int_{x_{\min}}^{x_{\max}} u_0(\xi) e^{-\int_0^t \mu(X(s;\xi,0), P(s), v(X(s;\xi,0))) \mathrm{d}s} \mathrm{d}\xi \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} B(t) = & \int_0^t \beta(X(t;x_{\min},\eta), P(t), v(X(t;x_{\min},\eta),t)) B(\eta) e^{-\int_\eta^t \mu(X(s,x_{\min},\eta), P(s), v(X(s;x_{\min},\eta),s)) \mathrm{d}s} \mathrm{d}\eta \\ & + \int_{x_{\min}}^{x_{\max}} \beta(X(t;x_{\min},\xi), P(t), v(X(t;x_{\min},\xi),t)) u_0(\xi) e^{-\int_0^t \mu(X(s;\xi,0), P(s), v(X(s;\xi,0))) \mathrm{d}s} \mathrm{d}\xi. \end{aligned} \tag{2.7}$$

If $P(t)$ and $B(t)$ are nonnegative continuous solutions of (2.6) and (2.7), then $u(x,t)$ and $v(x,t)$ defined by (2.4) and (2.5) respectively are the solutions of (1.1). On the other hand, if $u(x,t)$ and $v(x,t)$ are the solutions of (1.1), then $P(t)$ and $B(t)$ are nonnegative continuous solutions of (2.6) and (2.7). Therefore, in order to obtain the existence and uniqueness results for problem (1.1), we only need to study the solvability of the system consisting of integral equations (2.6) and (2.7).

By using the contraction mapping theorem, we first obtain the local existence and uniqueness results for problem (1.1). To this end, let

$$\begin{aligned} S_{T;K} &= \{f \in C[0,T] | f(0) = \|u_0\|_{L^1}, 0 \leq f(t) \leq K, \ \textit{where } K > \|u_0\|_{L^1}\}, \\ S_{T;H} &= \{f \in C[0,T] | 0 \leq f(t) \leq H, \ \textit{where } H > \beta_M \|u_0\|_{L^1}\}. \end{aligned}$$

For each $P \in S_{T;K}$, the function $X(t; x_0, t_0)$ is well-defined by the characteristic curve (2.3). Thus, there is a unique function $v(x,t)$ determined by (2.5).

Define the operator $Y : S_{T;K} \times S_{T;H} \to C[0,T] \times C[0,T]$ by $Y(P,B) = (\Phi(P,B), \Psi(P,B))$ where $\Phi(P,B)$ and $\Psi(P,B)$ are given by the right-hand sides of (2.6) and (2.7) respectively. Then, a fixed point of the operator Y corresponds to a solution of (2.6) and (2.7). Next lemma establishes the existence and uniqueness of a fixed point of the operator Y.

Lemma 2.1. *Suppose that hypotheses (H1)-(H8) hold. Then there exists a value $T > 0$ for which Y has a unique fixed point in $S_{T;K} \times S_{T;H} \subset C[0,T] \times C[0,T]$.*

Proof. As mentioned above, we just need to show that Y has a unique fixed point in $S_{T;K} \times S_{T;H}$. For any $P, \hat{P} \in S_{T,K}$, $B, \hat{B} \in S_{T,H}$, let $u, \hat{u}$ and $v, \hat{v}$ be given by (2.4) and (2.5) corresponding to $B, \hat{B}$ and $P, \hat{P}$, respectively. We use the following notations to simplify the expressions:

$$\mu(X_{\hat{P}}(s; x_{\min}, \eta), \hat{P}(s), \hat{v}(s,t)) = \mu_{\hat{P}}, \quad \mu(X_p(s; x_{\min}, \eta), P(s), v(s,t)) = \mu_P;$$

$$\beta(X_{\hat{P}}(s; x_{\min}, \eta), \hat{P}, \hat{v}) = \beta_{\hat{P}}, \quad \beta(X_p(s; x_{\min}, \eta), P, v) = \beta_P;$$

$$\mu(X_{\hat{P}}(s; \xi, 0), \hat{P}, \hat{v}) = \bar{\mu}_{\hat{P}}, \quad \mu_p(X(s; \xi, 0), P, v) = \bar{\mu}_P;$$

$$\beta_{\hat{P}}(X(s; \xi, 0), \hat{P}, \hat{v}) = \bar{\beta}_{\hat{P}}, \quad \beta_p(X(s; \xi, 0), P, v) = \bar{\beta}_P.$$

In terms of (2.7), we can conclude that

$$\Psi(P,B)(t) \le \beta_M \int_0^t B(\eta) \mathrm{d}\eta + \beta_M \|u_0\|_{L^1} \le \beta_M HT + \beta_M \|u_0\|_{L^1} \le H.$$

By a series of computations, we have

$$\begin{aligned} |\Psi(P,B)(t) - \Psi(\hat{P},\hat{B})(t)| \le T\beta_M \|B - \hat{B}\|_\infty + (\beta_M H L_\mu + H L_\beta) T (|X_P - X_{\hat{P}}| \\ + |P - \hat{P}| + |v - \hat{v}|). \end{aligned}$$

Since $X_P(t; x_{\min}, 0)$ and $X_{\hat{P}}(t; x_{\min}, 0)$ are the solutions of

$$\begin{cases} \dfrac{dx}{dt} = g(x, P(t)) \\ x(0) = x_{\min} \end{cases}$$

and

$$\begin{cases} \dfrac{dx}{dt} = g(x, \hat{P}(t)) \\ x(0) = x_{\min} \end{cases}$$

respectively, we have that

$$|X_P - X_{\hat{P}}| \le L_g \int_0^t (|X_P - X_{\hat{P}}| + |P - \hat{P}|) \mathrm{d}s. \tag{2.8}$$

Gronwall's inequality tells us that

$$|X_P - X_{\hat{P}}| \le L_g T e^{L_g T} \|P - \hat{P}\|_\infty. \tag{2.9}$$

Similarly, we get

$$|\beta_P - \beta_{\hat{P}}| \le L_\beta (|X_P - X_{\hat{P}}| + |P - \hat{P}| + |v - \hat{v}|) \mathrm{d}s, \tag{2.10}$$

$$|\mu_P - \mu_{\hat{P}}| \le L_\mu (|X_P - X_{\hat{P}}| + |P - \hat{P}| + |v - \hat{v}|) \mathrm{d}s, \tag{2.11}$$

$$
\begin{aligned}
|v-\hat{v}| \leq v_M \int_0^t |\sigma(X_P)-\sigma(X_{\hat{P}})|\mathrm{d}s + E_M \int_0^t a(X_P,s)\int_s^t |\sigma(X_P)-\sigma(X_{\hat{P}})|\mathrm{d}s\mathrm{d}s \\
+ |v_0(X_P)-v_0(X_{\hat{P}})| + E_M \int_0^t |a(X_P,s)-a(X_{\hat{P}},s)|\mathrm{d}s \\
\leq (v_M T L_\sigma + L_v + E_M a_M T L_\sigma + E_M L_a T)|X_P - X_{\hat{P}}| \\
\leq (v_M T L_\sigma + L_v + E_M a_M T L_\sigma + E_M L_a T) L_g T e^{L_g T}\|P-\hat{P}\|_\infty.
\end{aligned}
\tag{2.12}
$$

Thus,

$$
|\Psi(P,B)(t)-\Psi(\hat{P},\hat{B})(t)| \leq T\beta_M\|B-\hat{B}\|_\infty + h_1(T)T\|P-\hat{P}\|_\infty, \tag{2.13}
$$

where

$$
\begin{aligned}
h_1(T) = (\beta_M H L_\mu + H L_\beta)\, T \big[L_g T e^{L_g T} + 1 \\
+(v_M T L_\sigma + L_v + E_M a_M T L_\sigma + E_M L_a T) L_g T e^{L_g T} \big].
\end{aligned}
$$

For the Φ component, note that

$$
\begin{aligned}
\Phi(P,B)(t)-\Phi(\hat{P},\hat{B})(t)) = &\int_0^t (B(\eta)-\hat{B}(\eta)) e^{-\int_\eta^t \mu_P \mathrm{d}s}\mathrm{d}\eta \\
&+ \int_0^t \hat{B}(\eta)(e^{-\int_\eta^t \mu_P \mathrm{d}s} - e^{-\int_\eta^t \mu_{\hat{P}} \mathrm{d}s})\mathrm{d}\eta \\
&+ \int_{x_{\min}}^{x_{\max}} u_0(\xi)(e^{-\int_0^t \bar{\mu}_P \mathrm{d}s} - e^{-\int_0^t \bar{\mu}_{\hat{P}} \mathrm{d}s})\mathrm{d}\xi \\
&\leq \int_0^t |B(\eta)-\hat{B}(\eta)|\mathrm{d}\eta + \int_0^t \hat{B}(\eta)\int_\eta^t |\mu_P - \mu_{\hat{P}}|\mathrm{d}s\mathrm{d}\eta \\
&+ \int_{x_{\min}}^{x_{\max}} u_0(\xi)\int_0^t |\bar{\mu}_P - \bar{\mu}_{\hat{P}}|\mathrm{d}s\mathrm{d}\xi.
\end{aligned}
\tag{2.14}
$$

Let $F(\eta) = B(\eta)-\hat{B}(\eta)$, by (2.7), we get

$$
\begin{aligned}
|F(t)| \leq &\beta_M \int_0^t |F(\eta)|\mathrm{d}\eta + \int_0^t \hat{B}(\eta)|\beta_P - \beta_{\hat{P}}|\mathrm{d}\eta + \beta_M \int_0^t \hat{B}(\eta)|\mu_P - \mu_{\hat{P}}|\mathrm{d}\eta \\
&+ \int_{x_{\min}}^{x_{\max}} u_0(\xi)|\bar{\beta}_P e^{-\int_0^t \bar{\mu}_P \mathrm{d}s} - \bar{\beta}_{\hat{P}} e^{-\int_0^t \bar{\mu}_{\hat{P}} \mathrm{d}s}|\mathrm{d}\xi,
\end{aligned}
\tag{2.15}
$$

which leads to

$$
|F(t)| \leq \beta_M \int_0^t |F(\eta)|\mathrm{d}\eta + \psi(t),
$$

where

$$
\begin{aligned}
\psi(t) = &\int_0^t \hat{B}(\eta)|\beta_P - \beta_{\hat{P}}|\mathrm{d}\eta + \beta_M \int_0^t \hat{B}(\eta)|\mu_P - \mu_{\hat{P}}|\mathrm{d}\eta \\
&+ \int_{x_{\min}}^{x_{\max}} u_0(\xi)|\bar{\beta}_P e^{-\int_0^t \bar{\mu}_P \mathrm{d}s} - \bar{\beta}_{\hat{P}} e^{-\int_0^t \bar{\mu}_{\hat{P}} \mathrm{d}s}|\mathrm{d}\xi.
\end{aligned}
\tag{2.16}
$$

We also find that

$$\begin{aligned}\left|\bar{\beta}_P e^{-\int_0^t \bar{\mu}_P \mathrm{d}s} - \bar{\beta}_{\hat{P}} e^{-\int_0^t \bar{\mu}_{\hat{P}} \mathrm{d}s}\right| =&|(\bar{\beta}_P - \bar{\beta}_{\hat{P}})e^{-\int_0^t \bar{\mu}_P \mathrm{d}s} + \bar{\beta}_{\hat{P}}(e^{-\int_0^t \bar{\mu}_P \mathrm{d}s} - e^{-\int_0^t \bar{\mu}_{\hat{P}} \mathrm{d}s})| \\ &\leq |\bar{\beta}_P - \bar{\beta}_{\hat{P}}| + \beta_M \int_0^t |\bar{\mu}_P - \bar{\mu}_{\hat{P}}| \mathrm{d}s.\end{aligned} \tag{2.17}$$

From the above analysis, we can conclude that

$$\begin{aligned}\psi(t) \leq & [\beta_M \|u_0\|_{L^1} e^{\beta_M T}(L_\beta + \beta_M L_\mu) + \|u_0\|_{L^1}(L_\beta + \beta_M L_\mu T)] \cdot \\ & [L_g e^{L_g T} T + 1 + (v_M T L_\sigma + L_v + E_M a_M T L_\sigma + E_M L_a T) e^{L_g T} T L_g] \|P - \hat{P}\|_\infty \\ =:& J(T)\|P - \hat{P}\|_\infty.\end{aligned}$$

Thus,

$$|F(t)| \leq \beta_M \int_0^t |F(\eta)| \mathrm{d}\eta + \psi(t) \leq \beta_M \int_0^t |F(\eta)| \mathrm{d}\eta + J(T)\|P - \hat{P}\|_\infty.$$

By Gronwall's inequality, we have that

$$|F(t)| \leq J(T)\|P - \hat{P}\|_\infty e^{\int_0^t \beta_M \mathrm{d}\tau} = J(T)\|P - \hat{P}\|_\infty e^{\beta_M t}.$$

Therefore,

$$\begin{aligned}&|\Phi(P, B)(t) - \Phi(\hat{P}, \hat{B})(t)| \\ &\leq T J(T)\|P - \hat{P}\|_\infty e^{\beta_M T} + (\beta_M \|u_0\|_{L^1} e^{\beta_M T} T + \|u_0\|_{L^1}) T^2 L_\mu \|P - \hat{P}\|_\infty \\ &(L_g e^{L_g T} T + 1 + (v_M T L_\sigma + L_v + E_M a_M T L_\sigma + E_M L_a T) T L_g e^{L_g T}) \\ &=: T h_2(T)\|P - \hat{P}\|_\infty.\end{aligned} \tag{2.18}$$

Combining (2.13) and (2.18), we obtain

$$\begin{aligned}||Y(P, B) - Y(P, B)|| &= ||\Psi(P, B) - \Psi(\hat{P}, \hat{B})|| + ||\Phi(P, B) - \Phi(\hat{P}, \hat{B})|| \\ &\leq (T h_1(T) + T h_2(T))||P - \hat{P}||_\infty + T\beta_M ||B - \hat{B}||_\infty \\ &= r(T)(||P - \hat{P}||_\infty + ||B - \hat{B}||_\infty),\end{aligned} \tag{2.19}$$

where

$$r(T) = \max\{(T h_1(T) + T h_2(T)), T\beta_M\}.$$

Note that $r(0) = 0$. Therefore, there exists a sufficiently small constant $T > 0$ such that $r(T) \in (0, 1)$. Hence, for such a small T, the mapping Y is a contractive mapping. By the contracting mapping theorem, Y has a fixed point. The proof is completed. □

Note that the uniqueness of the solution $P(t)$ and $B(t)$ of system (2.6)-(2.7) implies that the uniqueness of the solution to problem (1.1) because each $u(x; t)$, $v(x, t)$ given by (2.4) and (2.5) is uniquely determined by $P(t)$ and $B(t)$. Thus, we have the following result on local existence and uniqueness to (1.1).

Theorem 2.2. *Suppose that hypotheses (H1)-(H8) hold. Then there exists a value $T > 0$ such that problem (1.1) has a unique solution up to time T.*

In order to establish the global existence-uniqueness result for problem (1.1), we conclude the following upper bound on $P(t)$ for $t \in [0, T]$.

Lemma 2.3. *Let $u(x,t)$ and $v(x,t)$ be a solution of (1.1) up to time T. Then for $t \in [0,T]$, $P(t)$ satisfies the following bound*

$$P(t) \leq \|u_0\|_{L^1} e^{\beta_M t}.$$

Proof. $P(t)$ is differentiable since $P(t) = \int_{x_{\min}}^{x_{\max}} u(x,t)\mathrm{d}x$ and $u(x,t)$ is differentiable by Definition 2.1. Differentiating (2.6) with respect to t, we get

$$P'(t) = \int_{x_{\min}}^{x_{\max}} (\beta(x, P(t), , v(x,t)) - \mu(x, P(t), , v(x,t)))u(x,t)\mathrm{d}x \leq \beta_M P(t),$$

Gronwall's inequality tells us that

$$P(t) \leq \|u_0\|_{L^1} e^{\beta_M t}.$$

Using similar arguments as in the proof of Theorem 3 in [2], we are able to derive the following global existence-uniqueness result. □

Theorem 2.4. *Suppose that hypotheses (H1)-(H8) hold. Then problem (1.1) has a unique solution for $t \in [0,\infty)$.*

3. Continuous dependence on initial conditions

The purpose of this section is to establish the continuous dependence of solutions on initial conditions. For this purpose, we first show that the fixed point of the operator Φ associated with an initial condition depends continuously on initial conditions.

Lemma 3.1. *Let $P_1(t)$ and $P_2(t)$ be the fixed points of (2.6) associated with initial conditions (u_{01}, v_{01}) and (u_{02}, v_{02}), respectively, then*

$$|P_1(t) - P_2(t)| \leq \frac{e^{\beta_M t}}{1-L} \|u_{01} - u_{02}\|_{L^1}, \tag{3.1}$$

where L is the contraction constant of the operator Φ.

Proof. It is easy to see that

$$|P_1(t) - P_2(t)| \leq |P_1(t) - P_3(t)| + |P_3(t) - P_2(t)|, \tag{3.2}$$

where

$$\begin{aligned} P_3(t) = &\int_0^t B_3(\eta) e^{-\int_\eta^t \mu(X_2(s,x_{\min},\eta), P_2(s), v_2(X_2(s;x_{\min},\eta),s))\mathrm{d}s} \mathrm{d}\eta \\ &+ \int_{x_{\min}}^{x_{\max}} u_{01}(\xi) e^{-\int_0^t \mu(X_2(s;\xi,0), P_2(s), v_2(X_2(s;\xi,0),s))\mathrm{d}s} \mathrm{d}\xi \end{aligned}$$

and

$$\begin{aligned} &B_3(t) = \\ &\int_0^t \beta(X_2(t;x_{\min},\eta), P_2(t), v_2(X_2(t;x_{\min},\eta),t)) B_3(\eta) e^{-\int_\eta^t \mu(X_2(s,x_{\min},\eta), P_2(s), v_2(X_2(s;x_{\min},\eta),s))\mathrm{d}s} \mathrm{d}\eta \\ &+ \int_{x_{\min}}^{x_{\max}} \beta(X_2(t;x_{\min},\xi), P_2(t), v_2(X_2(t;x_{\min},\xi),t)) u_{01}(\xi) e^{-\int_0^t \mu(X_2(s;\xi,0), P_2(s), v_2(X_2(s;\xi,0),s))\mathrm{d}s} \mathrm{d}\xi. \end{aligned}$$

Direct calculations give

$$
\begin{aligned}
|P_3(t) - P_2(t)| &= \int_0^t (B_3(\eta) - B_2(\eta)) e^{-\int_\eta^t \mu(X_2(s,x_{\min},\eta), P_2(s), v_2(X_2(s;x_{\min},\eta),s))\mathrm{d}s} \mathrm{d}\eta \\
&\quad + \int_{x_{\min}}^{x_{\max}} (u_{01}(\xi) - u_{02}(\xi)) e^{-\int_0^t \mu(X_2(s;\xi,0), P_2(s), v_2(X_2(s;\xi,0)))\mathrm{d}s} \mathrm{d}\xi \\
&\leq \int_0^t |B_3(\eta) - B_2(\eta)| \mathrm{d}\eta + \|u_{01} - u_{02}\|_{L^1}
\end{aligned}
$$

and

$$
|B_3(t) - B_2(t)| \leq \beta_M \int_0^t |B_3(\eta) - B_2(\eta)| \mathrm{d}\eta + \beta_M \|u_{01} - u_{02}\|_{L^1}.
$$

So we can conclude that

$$
|P_3(t) - P_2(t)| \leq e^{\beta_M t} \|u_{01} - u_{02}\|_{L^1}.
$$

From (3.2), by the contraction mapping theorem, we have

$$
\begin{aligned}
|P_1(t) - P_2(t)| &\leq |P_1(t) - P_3(t)| + |P_3(t) - P_2(t)| \leq L|P_1(t) - P_2(t)| + |P_3(t) - P_2(t)| \\
&\leq L|P_1(t) - P_2(t)| + e^{\beta_M t} \|u_{01} - u_{02}\|_{L^1},
\end{aligned}
$$

which implies (3.1). □

In the following, in virtue of the above estimates (3.1), we can show the continuous dependence of solutions on initial conditions.

Theorem 3.2. *Let* (u_1, v_1) *and* (u_2, v_2) *be the solutions of (1.1) with initial conditions* (u_{01}, v_{01}) *and* (u_{02}, v_{02})*, respectively. Then for any* $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon, t, u_{0i}, v_{0i}) > 0$ *such that if* $\|u_{01} - u_{02}\|_{L^1} + \|v_{01} - v_{02}\|_{L^1} < \delta$*, then*

$$
\|u_1 - u_2\|_{L^1} + \|v_1 - v_2\|_{L^1} \leq \varepsilon.
$$

Proof. Firstly we estimate the difference between the two characteristics. By (2.8) and Gronwall's inequality, and combining with (3.1), we find that

$$
|X_{P_1} - X_{P_2}| \leq L_g \int_0^t |P_1(\sigma) - P_2(\sigma)| \mathrm{d}\sigma e^{L_g(t-s)} \leq \frac{L_g e^{(L_g + \beta_M)(t-s)}}{\beta_M(1-L)} \|u_{01} - u_{02}\|_{L^1},
$$

which implies that when $t \geq s$, $|X_{P_1} - X_{P_2}| \to 0$ as $\|u_{01} - u_{02}\|_{L^1} \to 0$. We assume that $Z_1(t) \leq Z_2(t)$. By (2.4), direct calculations show that

$$
\begin{aligned}
&\int_{x_{\min}}^{x_{\max}} |u_1(x,t) - u_2(x,t)| \mathrm{d}x \\
&\leq \int_{x_{\min}}^{Z_1(t)} \Big| \frac{B(\tau_1(x_{\min}))}{g(x_{\min}, P_1)} - \frac{B(\tau_2(x_{\min}))}{g(x_{\min}, P_2)} \Big| e^{-\int_{\tau_1(x_{\min})}^t \tilde{\mu}(X_1, P_1, v_1)\mathrm{d}s} \mathrm{d}x \\
&\quad + \int_{x_{\min}}^{Z_1(t)} \frac{B_2(\tau(x_{\min}))}{g(x_{\min}, P_2)} (e^{-\int_{\tau_1(x_{\min})}^t \tilde{\mu}(X_1, P_1, v_1)\mathrm{d}s} - e^{\int_{\tau_2(x_{\min})}^t \tilde{\mu}(X_2, P_2, v_2)\mathrm{d}s}) \mathrm{d}x \\
&\quad + \int_{Z_1(t)}^{Z_2(t)} \Big(u_{01}(X_1) e^{-\int_0^t \tilde{\mu}(X_1, P_1, v_1)\mathrm{d}s} - \frac{B(\tau_2(x_{\min}))}{g(x_{\min}, P_2)} e^{\int_{\tau_2(x_{\min})}^t \tilde{\mu}(X_2, P_2, v_2)\mathrm{d}s} \Big) \mathrm{d}x \\
&\quad + \int_{Z_2(t)}^{x_{\max}} u_{01}(X_1) (e^{-\int_0^t \tilde{\mu}(X_1, P_1, v_1)\mathrm{d}s} - e^{\int_0^t \tilde{\mu}(X_2, P_2, v_2)\mathrm{d}s}) \mathrm{d}x \\
&\quad + \textstyle\int_{Z_2(t)}^{x_{\max}} |u_{01}(X_1) - u_{01}(X_2)| e^{-\int_0^t \tilde{\mu}(X_2, P_2, v_2)\mathrm{d}s} \mathrm{d}x \\
&\quad + \int_{Z_2(t)}^{x_{\max}} (u_{01}(X_2) - u_{02}(X_2)) e^{-\int_0^t \tilde{\mu}(X_2, P_2, v_2)\mathrm{d}s} \mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{x_{\min}}^{x_{\max}} |v_1(x,t) - v_2(x,t)| \mathrm{d}x = &\int_{Z_1(t)}^{Z_2(t)} \Big(v_{01}(X_1) e^{-\int_0^t \sigma(X_1, s)\mathrm{d}s} + \int_0^t a(X_1, s) E(s) e^{-\int_s^t \sigma(X_1, \tau)\mathrm{d}\tau} \mathrm{d}s \Big) \mathrm{d}x \\
&+ \int_{Z_2(t)}^{x_{\max}} \int_0^t [a(X_1, s) - a(X_2, s)] E(s) e^{-\int_s^t \sigma(X_1, \tau)\mathrm{d}\tau} \mathrm{d}s\mathrm{d}x \\
&+ \int_{Z_2(t)}^{x_{\max}} \int_0^t a(X_2, s) E(s) [e^{-\int_s^t \sigma(X_1, \tau)\mathrm{d}\tau} - e^{-\int_s^t \sigma(X_2, \tau)\mathrm{d}\tau}] \mathrm{d}s\mathrm{d}x \\
&+ \int_{Z_2(t)}^{x_{\max}} v_{01}(X_1) [e^{-\int_0^t \sigma(X_1, \tau)\mathrm{d}\tau} - e^{-\int_0^t \sigma(X_2, \tau)\mathrm{d}\tau}] \mathrm{d}s\mathrm{d}x \\
&+ \int_{Z_2(t)}^{x_{\max}} (v_{01}(X_1) - v_{01}(X_2)) e^{-\int_0^t \sigma(X_2, \tau)\mathrm{d}\tau} \mathrm{d}x \\
&+ \int_{Z_2(t)}^{x_{\max}} (v_{01}(X_2) - v_{02}(X_2)) e^{-\int_0^t \sigma(X_2, \tau)\mathrm{d}\tau} \mathrm{d}x,
\end{aligned}
$$

where $X_i = X_{P_i}$, $(i = 1, 2)$, $\tilde{\mu}$ is defined in Definition 2.1. The following proof can be completed by using similar arguments as in the proof of Theorem 2 in [2]. □

4. Concluding remarks

In this paper, by using the method of characteristic and contracting mapping theorem, we proved the existence-uniqueness of solutions to problem (1.1). We also derive the continuous dependence on initial conditions of the solutions. In the future, we plan to study the asymptotic behavior of the population under the influence of environmental toxins. In addition, problem (1.1) assumes that the population growth rate $g = g(x, P(t))$. This mortality rate, however, may depend on the body burden v. Including the dependence of the growth rate on the body burden (i.e., $g = g(x, P(t), v(x,t))$) will yield new and challenging problems.

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