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A STUDY ON THE MILD SOLUTION OF SPECIAL RANDOM IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we deal with mild solution of special random impulsive fractional differential equations. Initially, we present the existence of the mild solution via Leray-Schauder fixed point method. After that, we establish the exponential stability of the system. Finally, we give examples to illustrate the effectiveness of the theoretical results.

1. Introduction

Impulsive differential Equations are very adaptive Mathematical model that replicate the evolution of large classes of real process. Recently, in the fields of science and technology, we use fractional differential equations and impulsive fractional differential equations as a great mathematical tool in modelling. The stabilities like continuous dependence Mittag – Leffler Stability, Hyers Ulam stability and Hyers-Ulam-Rassins stability for fractional differential equations and impulsive fractional differential equations made curiosity in the mind of many researchers in the field of mathematics [10, 8, 14].

For impulsive differential systems, most researchers concentrate on the problems related to fixed time impulses [5, 21, 29]. But the actual jumps happen mostly at random points. The solutions of the random impulsive differential equations are a stochastic process. Now a day, the characteristics of solutions to some integer order differential equations with random impulses have been analysed [25, 2, 15, 28]. Anguraj et al. [2] established the existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness. Yong and Wu [28] investigated the solutions of stochastic differential equations with Random impulse using Lipschitz condition. Wu et al. [26, 27] discussed the exponential stability and boundedness of differential equations with random impulses. Sayooj et al.[17] have studied some characteristics of random integro differential equations with non local conditions. In [16, 18, 19], the author found sufficient conditions for the existence as well as stability of special random impulsive differential system with non local conditions using contraction mapping principle and continuous dependence on initial conditions.

Now a days, fractional calculus has a lot of advanced research work has been done. And also it have proved to be valuable tools in the modeling on many phenomina in various field of science and engineering [4, 12, 13, 20, 22, 30]. The study about impulsive fractional differential equations have a great attention, Akram Ben Alissa et al. [1] study the impulsive wave equation and analysis of this problem from different angles to prove some results about impulsive controlability and obervability without any geometrical condition on space Ω. In Many researchers studies about the existence, stability and uniqueness of fractional differential equations without random impulses [5, 9]. In this paper we make

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a first attempt to study the existence and exponential stability results for special random impulsive fractional differential systems by make use of the Leray − Schauder alternative fixed point theorem.

The main contributions of this work are given below:

- ↪ We substantiate sufficient conditions for the existence of solutions for special random impulsive fractional differential equations entangling the Caputo fractional derivative.
- ↪ We prove the results on existence of solutions of special random impulsive fractional differential equations by the use of the Leray − Schauder alternative fixed point theorem. This problem helps to solve many complicated random impulsive fractional systems.
- ↪ We find exponential stability in the quadratic mean of special random impulsive fractional differential equations.
- ↪ We provide examples of special random impulsive fractional differential systems as well as random impulsive fractional differential systems. It helps to interpret the effectiveness of the proposed results.

And the remaining work is constructed as follows: this paper consist of 4 sections. In Section 1 we present few preliminaries, hypotheses results about fractional derivatives. Section 2 will be concerned with existence and followed by exponential stability in the quadratic mean of special random impulsive fractional differential equations in Section 3. The last section is allocated to examples illustrating the applicability of the imposed conditions.

2. Preliminaries

Consider a real separable Hilbert space X and a non empty set Ω. Let ϱ_k be a random variable and ϱ_k maps Ω to D_k, where $D_k = (0, d_k)$ for every $k \in \mathbb{N}$ (collection of natural numbers) and $0 < d_k < +\infty$. Also for $i, j = 1, 2, \dots$ assume that if $i \neq j$ then ϱ_i and ϱ_j are independent with each other. Also assume ϱ_k follow Erlang distribution. Let ϱ be a real constant, denote $\Re_\varrho = [\varrho, +\infty)$, $\Re^+ = [0, +\infty)$.

Consider the semilinear functional special random impulsive differential equations of the form

$$\begin{aligned} {}^cD_t^a x(t) &= Ax(t) + f(t, x(t), Ux(t), Vx(t)) \qquad\qquad t \neq \xi_k, t \geq t_0, \\ x(\xi_k) &= b_k(\varrho_k)x(\xi_k^-), k = 1, 2, 3, \dots, \\ x(t_0) &= x_{t_0} \end{aligned} \tag{2.1}$$

${}^cD_t^a$ is the Caputo fractional derivative of order $0 < a < 1$. A is the infinitesimal generator of a strongly continuous semi group of bounded linear operators $\mathbb{T}(t)$, $\mathbb{T} \in X$. The function $f : \Re_\varrho \times X \times X \times X \to X, b_k : D_k \to X$ for each $k \in \mathbb{N}$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \varrho_k$ for each $k \in \mathbb{N}$. Obviously $0 < t_0 = \xi_0 < \xi_1 < \xi_2 < \xi_3 \cdots < \xi_k < \dots ; x(\xi_k -) = \lim_{t \uparrow \xi_k} x(t)$ according to their path with the norm $\|x\| = \sup_{t_0 \leq u \leq t} \mid x(u) \mid$ for every t satisfying $t \in [t_0, T]$.

$$\begin{aligned} Ux(t) &= \int_{t_0}^{t} \mathscr{K}(t, r)x(r)dr, \mathscr{K} \in C[\mathscr{D}, \Re^+], \\ Vx(t) &= \int_{t_0}^{T} \mathscr{H}(t, r)x(r)dr, \mathscr{H} \in C[\mathscr{D}_0, \Re^+], \end{aligned}$$

where $\mathscr{D} = \{(t, r) \in \Re^2 : t_0 \leq r \leq t \leq T\}, \mathscr{D}_0 = \{(t, r) \in \Re^2 : t_0 \leq t, r \leq T\}$. Let $\{B_t, t \geq 0\}$ be the simple counting process generated by $\{\xi_n\}$, this implies $\{B_t \geq t\} = \{\xi_n \leq t\}$, also $\mathscr{F}_t$ is the notation for the $\sigma-$ algebra generated by $\{B_t, t \geq 0\}$. The probability space denoted as $(\Omega, P, \{\mathscr{F}_t\})$. And the Hilbert space of all $\{\mathscr{F}_t\}-$ measurable square integrable random variables with values in X is denoted as $\mathscr{L}_2 = \mathscr{L}_2(\Omega, \{\mathscr{F}_t\}, X)$.

Let $\mathscr{B}$ represent Banach space $\mathscr{B}([t_0,T],\mathscr{L}_2)$, the family of all $\{\mathscr{F}_t\}$- measurable random variable ψ with the norm

$$\|\psi\|^2 = \sup_{t\in[t_0,T]} E\|\psi(t)\|^2$$

Definition 2.1. The fractional integral of the order a with the lower limit 0 for a function f is defined as

$$I^a f(t) = \frac{1}{\Gamma(a)}\int_0^t \frac{f(r)}{(t-r)^{1-a}}dr, \quad t>0, a>0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is a gamma function.

Definition 2.2. The Riemann–Liouville derivative of order a with the lower limit 0 for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$$^L D^a f(t) = \frac{1}{\Gamma(n-a)}\frac{d^n}{dt^n}\int_0^t \frac{f(r)}{(t-r)^{a+1-n}}dr, \quad t>0, n-1<a<n.$$

Definition 2.3. The Caputo derivative of order a for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$$^c D^a f(t) =^L D^a\left[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right], \quad t>0, n-1<a<n.$$

Definition 2.4. A semigroup $\{\mathbb{T}(t), t\geq t_0\}$ is said to be uniformly bounded if $\|\mathbb{T}(t)\|\leq \mathcal{W}$ for all $t\geq t_0$, where $\mathcal{W}\geq 1$ is some constant. If $\mathcal{W}=1$, then the semigroup is said to be contraction semigroup.

Definition 2.5. For a given $T\in(t_0,+\infty)$, a stochastic process $\{x(t)\in\mathscr{B}, 0<t_0\leq t\leq T\}$ is called a solution to the equation (2.1) in $(\Omega, P, \{\mathscr{F}_t\})$, if

(i) $x(t)\in\mathscr{B}$ is $\mathscr{F}_t$- adapted;
(ii)

$$\begin{aligned} x(t) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k} b_i(\varrho_i)\mathbb{T}(t-t_0)x_{t_0} \\ &+ \frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr \\ &+ \frac{1}{\Gamma(a)}\int_{\xi_k}^{t}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t), \qquad t\in[t_0,T], \end{aligned}$$

where $T\in(t_0,+\infty)$, $\prod_{j=m}^{n}(\cdot)=1$ as $m>n$, $\prod_{j=i}^{k} b_j(\varrho_j) = b_k(\varrho_k)b_{k-1}(\varrho_{k-1})\ldots b_i(\varrho_i)$, and

$I_{\mathbb{A}}(\cdot)$ is the index function.

Remark: The proof of mild solution similar to [3, 23, 29], so we omit it.

Hypotheses. Some hypotheses which are used for proving the main results are given below;

($\mathscr{H}_1$) There exist a continuous non-decreasing function $H:\Re^+\to(0,\infty)$ and $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3 \in L^1([\varrho,T],\Re^+)$ so that

$$E\|f(t,\zeta_1,\zeta_2,\zeta_3)\|^2 \leq \mathscr{L}_1(t)H\big(E\|\zeta_1\|\big)^2 + \mathscr{L}_2(t)H\big(E\|\zeta_2\|\big)^2 + \mathscr{L}_3(t)H\big(E\|\zeta_3\|\big)^2$$

($\mathscr{H}_2$) $E\Bigg\{\max_{i,k}\Big\{\prod_{j=i}^{k}\|b_j(\varrho_j)\|\Big\}\Bigg\}$ is uniformly bounded if,

$$E\Bigg\{\max_{i,k}\Big\{\prod_{j=i}^{k}\|b_j(\varrho_j)\|\Big\}\Bigg\} \leq \Theta, \qquad \text{for each } \varrho_j \in D_j, j \in \mathbb{N}, \Theta > 0 \text{ a constant}$$

($\mathscr{H}_3$) Define $\mathscr{L}, K^*$ and H^* such that,

$$\begin{aligned}
\mathscr{L} &= \max\{\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3\},\\
K^* &= \sup_{t\in[t_0,T]} \int_{t_0}^{t} |\mathscr{K}(t,r)|^2 dt < \infty, \text{ and}\\
H^* &= \sup_{t\in[t_0,T]} \int_{t_0}^{T} |\mathscr{H}(t,r)|^2 dt < \infty.
\end{aligned}$$

Our existence and exponential stability theorems are based on the succeeding theorem, which is a version of the topological transversal theorem.

Lemma 2.1. *Let E be a convex subset of a Banach space X, and assume that $0 \in E$. Let $F : E \to E$ be a completely continuous operator, and let*

$$U(F) = \{x \in E : x = \lambda Fx \text{ for some } 0 < \lambda < 1\},$$

then either $U(F)$ is unbounded or F has a fixed point.

3. Existence

Here, we presents the results on existence of solutions of special random impulsive fractional differential equations by make use of the Leray – Schauder alternative fixed point theorem.

Theorem 3.1. *Assume ($\mathscr{H}_1$), ($\mathscr{H}_2$), and ($\mathscr{H}_3$) hold, then the system (2.1) has mild solution $x(t)$, defined on $[t_0, T]$, provided the following inequality is satisfied:*

$$\Gamma \int_{t_0}^{T} \mathscr{L}(r)dr < \int_{\gamma_1}^{\infty} \frac{dr}{H(r)}, \tag{3.1}$$

where $\Gamma = 2\mathscr{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{(2a-1)\Gamma(a)}$, $\gamma_1 = 2\mathscr{W}^2\Theta^2 E\|\varphi\|^2$ *and* $\mathscr{W}\Theta \geq \frac{1}{\sqrt{2}}$.

Proof. Let Ψ be an operator from $\mathscr{B}$ to $\mathscr{B}$ and the arbitrary positive number $T \in (t_0, \infty)$:

$$\begin{aligned}
\Psi x(t) = \sum_{k=0}^{+\infty}\Bigg[&\prod_{i=1}^{k} b_i(\varrho_i)\mathbb{T}(t-t_0)x_{t_0}\\
&+ \frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k} b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t-r)^{a-1}\mathbb{T}(t-r)f(r, x(r), Ux(r), Vx(r))dr\\
&+ \frac{1}{\Gamma(a)}\int_{\xi_k}^{t}(t-r)^{a-1}\mathbb{T}(t-r)f(r, x(r), Ux(r), Vx(r))dr\Bigg] I_{[\xi_k,\xi_{k+1})}(t), \qquad t \in [t_0, T]
\end{aligned}$$

First we deduce the solution of the integral equation and assume $\lambda \in (0,1)$:

$$
\begin{aligned}
x(t) = \lambda \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\varrho_i) \mathbb{T}(t-t_0) x_{t_0} \\
&+ \frac{1}{\Gamma(a)} \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\varrho_j) \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1} \mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r)) dr \\
&+ \frac{1}{\Gamma(a)} \int_{\xi_k}^{t} (t-r)^{a-1} \mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r)) dr \Big] I_{[\xi_k, \xi_{k+1})}(t), \qquad t \in [t_0, T]
\end{aligned}
$$

Hence by $(\mathscr{H}_1), (\mathscr{H}_2)$ and $(\mathscr{H}_3)$

$$
\begin{aligned}
\|x(t)\|^2 \le \lambda^2 \Bigg[\sum_{k=0}^{+\infty} \Bigg[\Big\| \prod_{i=1}^{k} b_i(\varrho_i) \Big\| \|\mathbb{T}(t-t_0)\| \|x_{t_0}\| \\
&+ \frac{1}{\Gamma(a)} \sum_{i=1}^{k} \Big\| \prod_{j=i}^{k} b_j(\varrho_j) \Big\| \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1} \|\mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r))\| dr \\
&+ \frac{1}{\Gamma(a)} \int_{\xi_k}^{t} (t-r)^{a-1} \|\mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r))\| dr \Bigg] I_{[\xi_k, \xi_{k+1})}(t) \Bigg]^2 \\
&\le 2 \Bigg[\sum_{k=0}^{+\infty} \Bigg[\Big\| \prod_{i=1}^{k} b_i(\varrho_i) \Big\|^2 \|\mathbb{T}(t-t_0)\|^2 \|x_{t_0}\|^2 \Bigg] \\
&+ 2 \Bigg[\sum_{k=0}^{\infty} \frac{1}{\Gamma(a)} \sum_{i=1}^{k} \Big\| \prod_{j=i}^{k} b_j(\varrho_j) \Big\| \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1} \|\mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r))\| dr \\
&+ \frac{1}{\Gamma(a)} \int_{\xi_k}^{t} (t-r)^{a-1} \|\mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r))\| dr \Bigg] I_{[\xi_k, \xi_{k+1})}(t) \Bigg]^2 \\
&\le 2\mathcal{W}^2 \Theta^2 \Big\| x_{t_0} \Big\|^2 + 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \Big\| f(r, x(r), Ux(r), Vx(r)) \Big\|^2 dr, \\
\|x(t)\|^2 &\le 2\mathcal{W}^2 \Theta^2 \Big\| \varphi \Big\|^2 + 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \Big\| f(r, x(r), Ux(r), Vx(r)) \Big\|^2 dr,
\end{aligned}
$$

and

$$
\begin{aligned}
E\|x(t)\|^2 &\le 2\mathcal{W}^2 \Theta^2 E\big[\|\varphi\|^2\big] + 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)} \int_{t_0}^{t} E\big[\|f(r, x(r), Ux(r), Vx(r))\|^2\big] dr \\
&\le 2\mathcal{W}^2 \Theta^2 E\big[\|\varphi\|^2\big] + 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \Big[\mathscr{L}_1(r) H\big(E\|x(r)\|^2\big) \\
&+ \mathscr{L}_2(r) H\big(E\|Ux(r)\|^2\big) + \mathscr{L}_3(r) H\big(E\|Vx(r)\|^2\big) \Big] dr, \\
\sup_{t_0 \le \upsilon \le t} E\|x(\upsilon)\|^2 &\le 2\mathcal{W}^2 \Theta^2 E\|\varphi\|^2 \\
&+ 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \mathscr{L}(r)(1 + K^* + H^*) H\Big(\sup_{t_0 \le \upsilon \le r} E\|x(\upsilon)\|^2 \Big) dr, \\
&\le 2\mathcal{W}^2 \Theta^2 E\|\varphi\|^2 \\
&+ 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1 + K^* + H^*)}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \mathscr{L}(r) H(\omega(r)) dr
\end{aligned}
$$

where

$$\omega(t) = \sup_{t_0 \le \upsilon \le t} E\big[\|x(\upsilon)\|^2\big], \qquad t \in [t_0, T].$$

Moreover, for any $t \in [t_0, T]$,

$$\omega(t) \le 2\mathcal{W}^2\Theta^2 E\big[\|\varphi\|^2\big] + 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \mathscr{L}(r)H(\omega(r))dr.$$

Represent by the right hand side of the above inequality as $\mathscr{V}(t)$, then

$$\omega(t) \le \mathscr{V}(t) \quad \text{for} \quad t \in [t_0, T], \qquad \mathscr{V}(t_0) = 2\mathcal{W}^2\Theta^2 E\|\varphi\|^2 = \gamma_1$$

and

$$\begin{aligned}\mathscr{V}'(t) &= 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} \mathscr{L}(t)H(\omega(t)) \\ &\le 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} \mathscr{L}(t)H(\omega(t)), \qquad t \in [t_0, T].\end{aligned}$$

Then

$$\frac{\mathscr{V}'(t)}{H(\mathscr{V}(t))} \le 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} \mathscr{L}(t), \qquad t \in [t_0, T]$$

Apply the change of variable and integrate the previous inequality from t_0 to t, we get

$$\begin{aligned}\int_{\mathscr{V}(t_0)}^{\mathscr{V}(t)} \frac{dr}{H(r)} &\le 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} \int_{t_0}^{t} \mathscr{L}(r)dr \\ &\le 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} \int_{t_0}^{T} \mathscr{L}(r)dr \\ &< \int_{\gamma_1}^{\infty} \frac{dr}{H(r)} = \int_{\mathscr{V}(t_0)}^{\infty} \frac{dr}{H(r)}.\end{aligned}$$

By the mean value theorem and the above inequality, there is a constant Υ such that $\mathscr{V}(t) \le \Upsilon$, and therefore $\omega(t) \le \Upsilon$. Where as $\sup_{t_0 \le \upsilon \le T} E\|x(\upsilon)\|^2 = \omega(t)$ hold for each $t \in [t_0, T]$, we have $\sup_{t_0 \le \upsilon \le T} E\|x(\upsilon)\|^2 \le \Upsilon$, where Υ depends on the function $\mathscr{L}$ and H and on T, therefore

$$E\|x(t)\|^2 = \sup_{t_0 \le \upsilon \le T} E\|x(\upsilon)\|^2 \le \Upsilon$$

In the following steps, we will show that Ψ is continuous and completely continuous.

Step 1: *We show that* Ψ *is continuous.*

For every $t \in [t_0, T]$ and consider $\{x_n\}$ be a convergent sequence of elements of $x \in \mathscr{B}$, then

$$\begin{aligned}\Psi x_n(t) = \sum_{k=0}^{+\infty} \Big[&\prod_{i=1}^{k} b_i(\varrho_i)\mathbb{T}(t-t_0)\varphi(0) \\ &+ \frac{1}{\Gamma(a)} \sum_{i=1}^{k} \prod_{j=i}^{k} b_j(\varrho_j) \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1}\mathbb{T}(t-r)f(r, x_n(r), Ux_n(r), Vx_n(r))dr \\ &+ \frac{1}{\Gamma(a)} \int_{\xi_k}^{t} (t-r)^{a-1}\mathbb{T}(t-r)f(r, x_n(r), Ux_n(r), Vx_n(r))dr \Big] I_{[\xi_k, \xi_{k+1})}(t).\end{aligned}$$

So

$$\Psi x_n(t)-\Psi x(t)=\sum_{k=0}^{+\infty}\Bigg[\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t-r)^{a-1}\mathbb{T}(t-r)\Big[f(r,x_n(r),Ux_n(r),Vx_n(r))$$
$$-f(r,x(r),Ux(r),Vx(r))\Big]dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t}(t-r)^{a-1}\mathbb{T}(t-r)\Big[f(r,x_n(r),Ux(r),Vx(r))$$
$$-f(r,x(r),Ux(r),Vx(r))\Big]dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t),$$

and

$$E\|\Psi x_n-\Psi x\|^2\leq\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}\int_{t_0}^{t}E\|f(r,x_n(r),Ux_n(r),Vx_n(r))$$
$$-f(r,x(r),Ux(r),Vx(r))\|^2dr.$$

So Ψ is continuous.

Step 2: *We show that Ψ is completely continuous operator.*
Represent

$$\Theta_m=\{x\in\mathscr{B}\quad|\quad\|x\|^2\leq m\}$$

where $m\geq 0$.
Step 2.1: *We prove that Ψ maps to Θ_m into an equicontinuous family.*
Let $t_1,t_2\in[t_0,T]$ and $x\in\Theta_m$. Whenever $t_0<t_1<t_2<T$, by $(\mathscr{H}_1),(\mathscr{H}_2),(\mathscr{H}_3)$ and condition (3.1), we obtain

$$\Psi x(t_2)-\Psi x(t_1)=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\varrho_i)\mathbb{T}(t_2-t_0)x_{t_0}$$
$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_2}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t_2)$$
$$-\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\varrho_i)\mathbb{T}(t_1-t_0)x_{t_0}$$
$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t_1-r)^{a-1}\mathbb{T}(t_1-r)f(r,x(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_1}(t_1-r)^{a-1}\mathbb{T}(t_1-r)f(r,x(r),Ux(r),Vx(r))dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t_1)$$
$$=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_i(\varrho_i)\mathbb{T}(t_2-t_0)x_{t_0}$$
$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr$$

$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_2}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr\Big]\big[I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\big]$$
$$+\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\varrho_i)\big[\mathbb{T}(t_2-t_0)-\mathbb{T}(t_1-t_0)\big]x_{t_0}$$
$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}\big[(t_2-r)^{a-1}\mathbb{T}(t_2-r)-(t_1-r)^{a-1}\mathbb{T}(t_1-r)\big]f(r,x(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_1}\big[(t_2-r)^{a-1}\mathbb{T}(t_2-r)-(t_1-r)^{a-1}\mathbb{T}(t_1-r)\big]f(r,x_n(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{t_1}^{t_2}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t_1).$$

Moreover

$$E\|\Psi x(t_2)-\Psi x(t_1)\|^2\le 2E\|I_1\|^2+2E\|I_2\|^2,$$

where

$$I_1=\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\varrho_i)\mathbb{T}(t_2-t_0)x_{t_0}$$
$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr\Big]\big[I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\big]$$

and

$$I_2=\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\varrho_i)\big[\mathbb{T}(t_2-t_0)-\mathbb{T}(t_1-t_0)\big]x_{t_0}$$
$$\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}\big[(t_2-r)^{a-1}\mathbb{T}(t_2-r)-(t_1-r)\mathbb{T}(t_1-r)\big]f(r,x(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_1}(t_2-r)^{a-1}\mathbb{T}[(t_2-r)-(t_1-r)^{a-1}\mathbb{T}(t_1-r)]f(r,x(r),Ux(r),Vx(r))dr$$
$$+\frac{1}{\Gamma(a)}\int_{t_1}^{t_2}(t_2-r)^{a-1}\mathbb{T}(t_2-r)f(r,x(r),Ux(r),Vx(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t_1)$$

Besides,

$$E\|I_1\|^2\le E\Big[\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\varrho_i)\|\|\mathbb{T}(t_2-t_0)\|\|x_{t_0}\|$$
$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\varrho_j)\|\int_{\xi_{i-1}}^{\xi_i}(t_2-r)^{a-1}\|\mathbb{T}(t_2-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr$$
$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_2}(t_2-r)^{a-1}\|\mathbb{T}(t_2-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr\Big]\big[I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\big]\Big]^2$$
$$\le 2\mathcal{W}^2\Theta^2E\|x_{t_0}|^2E(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1))$$

$$
\begin{aligned}
&+2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}E\int_{t_0}^{t_2}\|\mathbb{T}(t_2-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr\\
&\times E(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1))\\
\leq& 2\mathcal{W}^2\Theta^2E\|x_{t_0}\|^2E\big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\big)\\
&+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t_2}\mathscr{L}(r)H(E\|x(r)\|^2)dr\\
&\times E\big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\big)\\
\leq& 2\mathcal{W}^2\Theta^2E\|x_{t_0}\|^2E\big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_[\xi_k,\xi_{k+1})(t_1)\big)\\
&+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t_2}\mathscr{L}^*H(E(m))drE\big(I_{[\xi_k,\xi_{k+1})}(t_2)-I_{[\xi_k,\xi_{k+1})}(t_1)\big)\\
&\to 0,\quad as\quad t_1\to t_2.
\end{aligned}
$$

where $\mathscr{L}^*=\sup\big\{\mathscr{L}(t):t\in[t_0,T]\big\}$.

$$
\begin{aligned}
E\|I_2\|^2\leq& E\Big[\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\|b_i(\varrho_i)\|\|\mathbb{T}(t_2-t_0)-\mathbb{T}(t_1-t_0)\|\|x_{t_0}\|\\
&+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\varrho_j)\|\int_{\xi_{i-1}}^{\xi_i}\|(t_2-r)^{a-1}\mathbb{T}(t_2-r)\\
&-(t_1-r)^{a-1}\mathbb{T}(t_1-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr\\
&+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t_1}\|(t_2-r)^{a-1}\mathbb{T}(t_2-r)-(t_1-r)^{a-1}\mathbb{T}(t_1-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr\\
&+\frac{1}{\Gamma(a)}\int_{t_1}^{t_2}(t_2-r)^{a-1}\|\mathbb{T}(t_2-r)\|\|f(r,x(r),Ux(r),Vx(r))dr\|\Big]I_{[\xi_k,\xi_{k+1})}(t_1)\Big]^2\\
\leq& 3\Theta^2\|\mathbb{T}(t_2-t_0)-\mathbb{T}(t_1-t_0)\|^2E\|x_{t_0}\|^2\\
&+3\max\{1,\Theta^2\}(t_1-t_0)\frac{1}{\Gamma(a)}E\int_{t_0}^{t_1}\|(t_2-r)^{a-1}\mathbb{T}(t_2-r)\\
&-(t_1-r)^{a-1}\mathbb{T}(t_1-r)\|^2\|f(r,x(r),Ux(r),Vx(r))\|^2dr\\
&+3\mathcal{W}^2\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}E\int_{t_1}^{t_2}\|f(r,x(r),Ux(r),Vx(r))\|^2dr\\
\leq& 3\Theta^2\|\mathbb{T}(t_2-t_0)-\mathbb{T}(t_1-t_0)\|^2E\|x_{t_0}\|^2\\
&+3\max\{1,\Theta^2\}(t_1-t_0)\frac{1}{\Gamma(a)}\int_{t_0}^{t_1}\|(t_2-r)^{a-1}\mathbb{T}(t_2-r)\\
&-(t_1-r)^{a-1}\mathbb{T}(t_1-r)\|^2\mathscr{L}^*H(m)dr\\
&+3\mathcal{W}^2\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_1}^{t_2}\mathscr{L}^*H(m)dr\\
&\to 0\qquad as\quad t_1\to t_2.
\end{aligned}
$$

So, Ψ maps Θ_m into an equicontinuous family of functions.

Step 2.2: *We prove that $\Psi\Theta_m$ is uniformly bounded.*

By (3.1), $\|x\|^2 \le m$, $(\mathscr{H}_1),(\mathscr{H}_2)$ and $(\mathscr{H}_3)$, we get

$$\|\Psi x(t)\|^2 \le \Bigg[\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\|b_i(\varrho_i)\|\|\mathbb{T}(t-t_0)\|\|x_{t_0}\|$$

$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}\|b_j(\varrho_j)\|\int_{\xi_{i-1}}^{\xi_i}(t-r)^{a-1}\|\mathbb{T}(t-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr$$

$$+\frac{1}{\Gamma(a)}\int_{\xi_{i-1}}^{t}(t-r)^{a-1}\|\mathbb{T}(t-r)\|\|f(r,x(r),Ux(r),Vx(r))\|dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t)\Bigg]^2.$$

$$\le 2\mathcal{W}^2\Theta^2\|\varphi(0)\|^2+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}\int_{t_0}^{t}\|f(r,x(r),Ux(r),Vx(r))\|^2dr.$$

Thus,

$$E\|\Psi x(t)\|^2 \le 2\mathcal{W}^2\Theta^2E\|\varphi(0)\|^2+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}\int_{t_0}^{t}E\|f(r,x(r),Ux(r),Vx(r))\|^2dr.$$

$$\le 2\mathcal{W}^2\Theta^2E\|\varphi(0)\|^2+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\|b_m\|_L.$$

Therefore $\{(\Psi x(t)),\|x\|^2\le m\}$ is uniformly bounded, so does $\{\Psi\Theta_m\}$. Then by the Arzela – Ascoli theorem, Ψ maps Θ_m into a precompact set in X.

Step 2.3: *We prove that* $\Psi\Theta_m$ *is compact.* Let $t\in(t_0,T]$ be fixed, and let ϵ be a real number such that $\epsilon\in(0,t-t_0)$ for $x\in\Theta_m$, we establish

$$(\Psi_\epsilon x)(t)=\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}b_j(\varrho_j)\mathbb{T}(t-t_0)x_{t_0}$$

$$+\frac{1}{\Gamma(a)}\sum_{i=1}^{k}\prod_{j=i}^{k}b_j(\varrho_j)\int_{\xi_{i-1}}^{\xi_i}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr$$

$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t-\epsilon}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t),\qquad t\in(t_0,t-\epsilon).$$

Being $\mathbb{T}(t)$ is a compact operator, the set

$$H_\epsilon(t)=\{(\Psi_\epsilon x)(t):x\in\Theta_m\}$$

is precompact in X for each $\epsilon\in(0,t-t_0)$. Furthermore, for each $x\in\Theta_m$, we attain

$$(\Psi x)(t)-(\Psi_\epsilon x)(t)=\sum_{k=0}^{+\infty}\Bigg[\frac{1}{\Gamma(a)}\int_{\xi_k}^{t}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr\Bigg]I_{[\xi_k,\xi_{k+1})(t)}$$

$$-\sum_{k=0}^{+\infty}\Bigg[\frac{1}{\Gamma(a)}\int_{\xi_k}^{t-\epsilon}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr\Bigg]I_{[\xi_k,\xi_{k+1})}(t).$$

By making use of $(\mathscr{H}_1),(\mathscr{H}_2)$, $(\mathscr{H}_3)$, condtion 4.1, and $\|x(\mathscr{B})\|^2\le m$, we obtain

$$E\|(\Psi x)(t)-(\Psi_\epsilon x)(t))\|_t^2\le\mathcal{W}^2\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t-\epsilon}^{t}\mathscr{L}^*H(m)dr.$$

Hence, there exist precompact sets arbitrarily close to the set $\{(\Psi x)(t):x\in\Theta_m\}$ is precompact in X. So, Ψ is completely continuous operator.

Furthermore, the set $U(\Psi) = \{x \in \mathscr{B} : x = \lambda\Psi x \text{ for some } 0 < \lambda < 1\}$ is bounded. Hence, by *lemma 2.1*, the operator Ψ has a fixed point in $\mathscr{B}$. So, system (2.1) has a mild solution. □

4. Exponential stability in the quadratic mean

This section, we establish the exponential stability of a second moment of mild solution of system. For an F_t - adapted process, $\Psi(t) : [0,\infty) \to \mathbb{R}$ is almost continuous in t. In order to attain the stability, we suppose that $f(t,0) \equiv 0$ for any $t \leq t_0$ thus the system (2.1) accept a trivial solution. Furthermore, $E\|\Psi\|_t^2 \to 0$ as $t \to \infty$.

Definition 4.1. System (2.1) is said to be exponentially stable in the quadratic mean if there exist positive constants $K_1 > 0$ and $\nu > 0$ such that

$$E\|x(t)\| \leq K_1 E\|\varphi\|^2 e^{-\nu(t-t_0)}, \qquad t \geq t_0.$$

Now we introduce the following hypothesis used in our discussion:

($\mathscr{H}_4$) $\mu H(\psi) \leq H(\mu\psi)$ for all $\psi \in \mathbb{R}^+$, where $\mu > 1$.

($\mathscr{H}_5$) $\|\mathbb{T}(t)\| \leq \mathcal{W} e^{-\xi(t-t_0)}$, $t \geq 1$.

Theorem 4.1. *Assume that the hypothesis of Theorem 2.1 and ($\mathscr{H}_4$) − ($\mathscr{H}_5$) hold. If the following inequality is satisfied, then the system (2.1) is exponentially stable in the quadratic mean:*

$$\Gamma^* \int_{t_0}^{T} \mathscr{L}(r)dr < \int_{\gamma_2}^{\infty} \frac{dr}{H(r)}, \tag{4.1}$$

where $\Gamma^* = 2\mathcal{W}^2 \max\{1,\Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}$, $\gamma_2 = 2\mathcal{W}^2\Theta^2 E\|\varphi\|^2$, *and* $\mathcal{W}\Theta \geq \frac{1}{\sqrt{2}}$.

Proof. Let Ψ be defined in Theorem 2.1. Making use of hypotheses ($\mathscr{H}_1$) − ($\mathscr{H}_5$), we get

$$\begin{aligned}
\|x(t)\|^2 \leq \lambda^2 \Bigg(\sum_{k=0}^{+\infty} \Bigg[\Big\| \prod_{i=1}^{k} b_i(\varrho_i) \Big\| \|\mathbb{T}(t-t_0)\| \|x_{t_0}\| \\
&+ \frac{1}{\Gamma(a)} \sum_{i=1}^{k} \Big\| \prod_{j=i}^{k} b_j(\varrho_j) \Big\| \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1} \|\mathbb{T}(t-r) f(r,x(r),Ux(r),Vx(r))\| dr \\
&+ \frac{1}{\Gamma(a)} \int_{\xi_k}^{t} (t-r)^{a-1} \|\mathbb{T}(t-r) f(r,x(r),Ux(r),Vx(r))\| dr \Bigg] I_{[\xi_k,\xi_{k+1})}(t) \Bigg)^2 \\
&\leq 2 \sum_{k=0}^{+\infty} \Bigg[\Big\| \prod_{i=1}^{k} b_i(\varrho_i) \Big\|^2 \|\mathcal{W}^2 e^{-2k(t-t_0)}\| \|x_{t_0}\|^2 I_{[\xi_k,\xi_{k+1})}(t) \Bigg] \\
&+ 2 \Bigg(\sum_{k=0}^{\infty} \Bigg[\frac{1}{\Gamma(a)} \sum_{i=1}^{k} \Big\| \prod_{j=i}^{k} b_j(\varrho_j) \Big\| \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1} \|\mathcal{W} e^{-\xi(t-r)} f(r,x(r),Ux(r),Vx(r))\| dr \\
&+ \frac{1}{\Gamma(a)} \int_{\xi_k}^{t} (t-r)^{a-1} \mathcal{W}^2 e^{-\xi(t-r)} \|f(r,x(r),Ux(r),Vx(r))\| dr \Bigg] I_{[\xi_k,\xi_{k+1})}(t) \Bigg)^2
\end{aligned}$$

$$\leq 2\left[\max_{k}\left\{\prod_{j=i}^{k}\|b_j(\varrho_j)\|^2\right\}\right]^2\mathcal{W}^2e^{-2k(t-t_0)}\|x_{t_0}\|^2$$
$$+2\left[\max_{i,k}\left\{1,\prod_{j=i}^{k}\|b_j(\varrho_j)\|\right\}\right]^2\cdot$$
$$\cdot\frac{1}{\Gamma(a)}\sum_{k=0}^{+\infty}\int_{t_0}^{t}(t-r)^{a-1}\mathcal{W}e^{-\xi(t-r)}\|f(r,x(r),Ux(r),Vx(r))\|dr\cdot I^2_{[\xi_k,\xi_{k+1})}(t)$$
$$\leq 2\mathcal{W}^2\Theta^2e^{-2k(t-t_0)}\|x_{t_0}\|^2$$
$$+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}\int_{t_0}^{t}e^{-2\xi(t-r)}\|f(r,x(r),Ux(r),Vx(r))\|^2dr,$$
$$\|x(t)\|^2\leq 2\mathcal{W}^2\Theta^2e^{-2\xi(t-t_0)}\|\varphi\|^2$$
$$+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}\int_{t_0}^{t}e^{-2k(t-r)}\|f(r,x(r),Ux(r),Vx(r))\|^2dr,$$
$$zE\|x(t)\|^2\leq 2\mathcal{W}^2\Theta^2e^{-2k(t-t_0)}E\|\varphi\|^2$$
$$+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t}e^{-2\xi(t-r)}\mathscr{L}(r)H\big(E\|x(r)\|^2\big)dr,$$
$$=2\mathcal{W}^2\Theta^2e^{-2\xi(t-t_0)}E\|\varphi\|^2$$
$$+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}e^{-2k(t-t_0)}\int_{t_0}^{t}e^{2\xi(r-t_0)}\mathscr{L}(r)H\big(E\|x(r)\|^2\big)^2dr.$$

Thus,

$$e^{2\xi(t-t_0)}E\|x(t)\|^2$$
$$\leq 2\mathcal{W}^2\Theta^2E\|\varphi\|^2+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t}e^{2\xi(t-r)}\mathscr{L}(r)H\big(E\|x(r)\|^2\big)^2dr.$$

Furhtermore,

$$\sup_{t_0\leq\upsilon\leq t}e^{2\xi(\upsilon-t_0)}E\|x(t)\|^2$$
$$\leq 2\mathcal{W}^2\Theta^2E\|\varphi\|^2$$
$$+2\mathcal{W}^2\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t}\mathscr{L}(r)H\left(\sup_{t_0\leq\upsilon\leq t}e^{2\xi(\upsilon-t_0)}E\|x(r)\|^2\right)dr.$$

Take

$$\omega_1(t)=\sup_{t_0\leq\upsilon\leq t}e^{2\xi(\upsilon-t_0)}E\|x\|^2,\qquad t\in[t_0,T].$$

Also, for any $t\in[t_0,T]$, we have

$$\omega_1(t)\leq 2\mathcal{W}^2\Theta^2E\|\varphi\|^2+\max\{1,\Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t}\mathscr{L}(r)H\big(\omega_1(r)\big)dr.$$

Denote the right hand side of the above inequality $\mathscr{V}_1(t)$, we obtain

$$\omega_1(t) \leq \mathscr{V}_1(t), \qquad t \in [t_0, T],$$
$$\mathscr{V}_1(t_0) = 2\mathcal{W}^2\Theta^2 E\|\varphi\|^2 = \gamma_2$$

and

$$\begin{aligned}\mathscr{V}_1'(t) &\leq 2\mathcal{W}^2 \max\{1, \Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\mathscr{L}(t)H\big(\omega_1(t)\big)\\ &\leq 2\mathcal{W}^2 \max\{1, \Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\mathscr{L}(t)H\big(\mathscr{V}_1(t)\big), \qquad t \in [t_0, T].\end{aligned}$$

That is,

$$\frac{\mathscr{V}_1'(t)}{H\big(\mathscr{V}_1(t)\big)} \leq 2\mathcal{W}^2 \max\{1, \Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\mathscr{L}(t), \qquad t \in [t_0, T].$$

Apply the change of variable and integrate the previous inequality from t_0 to t, we get

$$\begin{aligned}\int_{\mathscr{V}_1(t_0)}^{\mathscr{V}_1(t)} \frac{dr}{H(r)} &\leq 2\mathcal{W}^2 \max\{1, \Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{t}\mathscr{L}(r)dr\\ &\leq 2\mathcal{W}^2 \max\{1, \Theta^2\}\frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}\int_{t_0}^{T}\mathscr{L}(r)dr\\ &< \int_{\gamma_2}^{\infty}\frac{dr}{H(r)} = \int_{\mathscr{V}_1(t_0)}^{\infty}\frac{dr}{H(r)}, \qquad t \in [t_0, T].\end{aligned}$$

By the mean value theorem and above inequality there exist a constant Υ_1 such that $\mathscr{V}_1(t) \leq \Upsilon_1$, and therefore $\omega_1(t) \leq \Upsilon_1$. Whereas $\sup_{t_0 \leq \upsilon \leq t} e^{2\xi(\upsilon - t_0)}E\|x\|^2 = \omega_1(t)$ holds for each $t \in [t_0, T]$, we have $\sup_{t_0 \leq \upsilon \leq t} e^{2\xi(\upsilon - t_0)}E\|x\|^2 \leq K_1$, where Υ_1 depends on the function $\mathscr{L}$ and H.
Therefore,

$$e^{2\xi(t-t_0)}E\|x\|^2 = \sup_{t_0 \leq \upsilon \leq T} e^{2\xi(\upsilon - t_0)}E\|x\|^2 \leq \Upsilon_1.$$

As in the previous theorem, we will prove that Ψ is completely continuous operator through the following steps.

Step 1: *We show that Ψ is continuous.*

For every $t \in [t_0, T]$ and consider $\{x_n\}$ be a convegent sequence of element of $x \in \mathscr{B}$, we obtain

$$\begin{aligned}&E\|\Psi x_n(t) - \Psi x(t)\|^2\\ &\leq \mathcal{W}^2 \max\{1, \Theta^2\}\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}e^{-2k(t-t_0)}\int_{t_0}^{t} e^{2\xi(r-t_0)}E\|f(r, x_n(r), Ux_n(r), Vx_n(r))\\ &\quad - f(r, x(r), Ux(r), Vx(r))\|^2 dr\\ &\to 0, \qquad as \quad n \to \infty.\end{aligned}$$

So Ψ is continuous.

Step 2: *We show that Ψ is completely continuous operator.*

Represent

$$\Theta_{m_1} = \{x \in \mathscr{B} \quad | \quad \|x\|^2 \leq m_1\}$$

where $m_1 \geq 0$.

Step 2.1: *We prove that* Ψ *maps* Θ_{m_1} *into an equicontinuous family.*

Let $x \in \Theta_{m_1}$ and $t_1, t_2 \in [t_0, T]$. If $t_0 < t_1 < t_2 < T$, then by making use of $(\mathscr{H}_1) - (\mathscr{H}_5)$ and condition (3) and pursuing the similar process of **Step** 2.1 of Theorem 3.1, we obtain

$$E\|\Psi x(t_2) - \Psi x(t_1)\|^2 \to 0 \qquad as \quad t_2 \to t_1.$$

So, Ψ maps to Θ_{m_1} into an equicontinuous family of functions.

Step 2.2: *We prove that* $\Psi\Theta_{m_1}$ *is uniformly bounded.*
By the condition (3.1) and $(\mathscr{H}_1) - (\mathscr{H}_5)$, we get

$$\begin{aligned}\|\Psi x(t)\|^2 \leq\; & 2\left[\max_k \left\{\prod_{j=i}^{k} \|b_j(\varrho_j)\|^2\right\}\right] \mathcal{W}^2 e^{-2k(t-t_0)} \|x_{t_0}\|^2 \\ & + 2\left[\max_{i,k}\left\{1, \prod_{j=i}^{k} \|b_j(\varrho_j)\|\right\}\right]^2 \\ & \times \left(\frac{1}{\Gamma(a)} \sum_{k=0}^{+\infty} \int_{t_0}^{t} (t-r)^{a-1} \mathcal{W} e^{-\xi(t-r)} \|f(r, x(r), Ux(r), Vx(r))\| dr\right) I_{[\xi_k, \xi_{k+1})}(t)^2.\end{aligned}$$

So

$$\begin{aligned}E\|\Psi x(t)\|^2 \leq\; & 2\mathcal{W}^2\Theta^2 e^{-2k(t-t_0)} E\|x_{t_0}\|^2 \\ & + 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)} e^{-2k(t-t_0)} \\ & \times \int_{t_0}^{t} e^{2\xi(r-t_0)} \mathscr{L}^* H(m) dr,\end{aligned}$$

where $\mathscr{L}^* = \sup\{\mathscr{L}(t) : t \in [t_0, T]\}$. Being $e^{-2k(t-t_0)} \to 0$, the right hand side of the previous inequality tends to 0 as $t \to \infty$. ie,

$$\|(\Psi x)\|^2 \to 0 \quad t \to \infty.$$

Therefore $\{(\Psi x(t)), \|x\|^2_{\mathscr{B}} \leq m_1\}$ is uniformly bounded, thus $\{\Psi\Theta_{m_1}\}$ is uniformly bounded.

Step 2.3: *We prove that* $\Psi\Theta_{m_1}$ *is compact.*

Let $t \in (t_0, T]$ be fixed and ϵ be a real number such that $\epsilon \in (0, t-t_0)$, for $x \in \Theta_{m_1}$, we establish

$$\begin{aligned}(\Psi_\epsilon x)(t) =& \sum_{k=0}^{+\infty}\left[\prod_{i=0}^{k} b_j(\varrho_j)\mathbb{T}(t-t_0)x_{t_0}\right. \\ & + \frac{1}{\Gamma(a)} \sum_{i=1}^{k}\prod_{i=i}^{k} b_j(\varrho_j) \int_{\xi_{i-1}}^{\xi_i} (t-r)^{a-1}\mathbb{T}(t-r) f(r, x(r), Ux(r), Vx(r)) dr\end{aligned}$$

$$+\frac{1}{\Gamma(a)}\int_{\xi_k}^{t-\epsilon}(t-r)^{a-1}\mathbb{T}(t-r)f(r,x(r),Ux(r),Vx(r))dr\Big]I_{[\xi_k,\xi_{k+1})}(t), \quad t\in(t_0,t-\epsilon).$$

Being $\mathbb{T}(t)$ is a compact operator, the set

$H_\epsilon(t)=\{(\Psi_\epsilon x)(t): x\in\Theta_{m_1}\}$

is precompact in X for each $\epsilon\in(0,t-t_0)$.

Using $(\mathscr{H}_1)-(\mathscr{H}_5)$, condition (3), and $\|x\|^2\le m_1$, we obtain

$$E\|\Psi x_n(t)-\Psi x(t)\|^2 \le \mathcal{W}^2\frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}e^{-2k(t-t_0)}\int_{t-\epsilon}^{t}e^{2\xi(r-t_0)}\mathscr{L}^*H\big(E\|x(r)\|^2\big)dr.$$

Hence, there exist precompact sets arbitrarily close to the set $\{(\Psi x)(t): x\in\Theta_{m_1}\}$. Thus the set $\{(\Psi x)(t): x\in\Theta_{m_1}\}$ is precompact in X. So, Ψ is a completely continuous operator.

Furthermore, the set $U(\Psi)=\{x\in\mathscr{B}: x=\lambda\Psi x \quad for \quad some \quad 0<\lambda<1\}$ is bounded. Hence, by Lemma 2.1, the operator Ψ has a fixed point in $\mathscr{B}$. So the system 2.1 has a mild solution and $E\|\Psi(t)\|^2\to 0$ as $t\to\infty$. Hence the proof. □

5. Applications

Example 5.1. Consider random impulsive fractional differential equations,

$$\begin{cases} {}^cD_t^a z(t,x) &= z_{xx}(x,t)+F_1(t,z(t,x)) \ t\neq\xi_k,\ t\ge\varrho \\ z(x,\xi_k) &= q(k)\varrho_k z(x,\xi_k^-) \qquad as\ x\in\widehat{\triangle} \\ z(t,0) &= z(t,\pi)=0 \\ z(t_0,x) &= z_0(x),\ x\in\partial\widehat{\triangle} \end{cases} \tag{5.1}$$

Consider $\widehat{\triangle}\subset\Re^n$ be a bounded domain with smooth boundary $\partial\widehat{\triangle}$, $X=L^2(\widehat{\triangle})$, ϱ_k be random variable defined on $D_k\equiv(0,d_k)$ for $k\in\mathbb{N}$, $d_k\in(0,+\infty)$. Also assume that ϱ_k follow Erlang distribution and if $i\neq j$ then ϱ_i and ϱ_j are independent with each other for $i,j=1,2,\dots$. Here q is a function of k, $\xi_k=\xi_{k-1}+\varrho_k$ for $k\in\mathbb{N}$, $t_0\in\Re^+$.

Let A be an operator on X by $Az=\dfrac{\partial^2 z}{\partial x^2}$ with the domain

$$D(A)=\Big\{z\in X \mid z \text{ and } \frac{\partial z}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 z}{\partial x^2}\in X, z=0 \text{ on } \partial\widehat{\triangle}\Big\}$$

Thus A generates a strongly continuous semigroup $S(t)$ which is self adjoint,compact and analytic. Furthermore the operator A can be represented by

$$Az=\sum_{n=1}^{\infty}n^2<z,z_n>z_n,\ z\in D(A)$$

Here $z_n(\zeta)=\sqrt{\frac{2}{\pi}}Sin(n\zeta), n=1,2,\dots,$ forms the orthonormal set of eigenvectors of A. Also for every $z\in X, S(t)z=\sum_{n=1}^{\infty}e^{(-n^2t)}<z,z_n>z_n$, which holds $\|S(t)\|\le e^{(-\pi^2(t-t_0))}, t\ge t_0$. Therefore $S(t)$ is

a semigroup.
Consider that the following assumptions:

(i) $f : \Re_\varrho \times X \to X$, is a continuous function defined by

$$f(t,z)(x) = F_1(t, z(x))\ t_0 \le t \le T, 0 \le x \le \pi$$

and also $\exists$ a continuous non-decreasing function $H : \Re^+ \to (0,\infty)X$ and $\mathscr{L} \in L^1([\varrho, T], \Re^+)$ therefore

$$E\|f(t,z)\|^2 \le \mathscr{L}(t)H\big(E\|z\|^2\big)$$

(ii) $E\Big\{ \max_{i,k} \Big\{\prod_{j=i}^{k} \|b_j(\varrho_j)\|\Big\}\Big\}$ is uniformly bounded if,

$$E\Big\{ \max_{i,k} \Big\{\prod_{j=i}^{k} \|b_j(\varrho_j)\|\Big\}\Big\} \le \theta, \qquad \text{for each } \varrho_j \in D_j, j \in \mathbb{N}, \theta > 0 \text{ a constant}$$

(iii)

$$\Gamma \int_{t_0}^{T} \mathscr{L}(r)dr < \int_{\gamma_1}^{\infty} \frac{dr}{H(r)}, \tag{5.2}$$

where $\Gamma = 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{(2a-1)\Gamma(a)}, \gamma_1 = 2\mathcal{W}^2\Theta^2 E\|\varphi\|^2$ and $\mathcal{W}\Theta \ge \frac{1}{\sqrt{2}}$.

Assume that assumptions (i),(ii) and (iii) are satisfied, then the problem (5.1) becomes a random impulsive fractional differential equation. From all the above facts, in view of Theorem 3.1, we conclude that (5.1) has a mild solution.

Remark 5.2. Let the conditions of Example 5.1 along with $(\mathscr{H}_4) - (\mathscr{H}_5)$ be hold. If the following inequality is satisfied,

$$\Gamma^* \int_{t_0}^{T} \mathscr{L}(r)dr < \int_{\gamma_2}^{\infty} \frac{dr}{H(r)}, \tag{5.3}$$

where $\Gamma^* = 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}}{\Gamma(a)(2a-1)}, \gamma_2 = 2\mathcal{W}^2\Theta^2 E\|\varphi\|^2$, and $\mathcal{W}\Theta \ge \frac{1}{\sqrt{2}}$. Then the mild solution z of the Example 5.1 is exponentially stable in the quadratic mean.

Example 5.3. Consider special random impulsive fractional differential equations,

$$\begin{cases} {}^cD_t^a z_t(t,x) &= z_{xx}(x,t) + F_1(t, z(t,x)) + \int_0^T F_2(\theta, z(tsin\theta, x))d\theta\ t \ne \xi_k,\ t \ge \varrho \\ z(x,\xi_k) &= q(k)\varrho_k z(x, \xi_k^-) \qquad as\ x \in \widehat{\triangle} \\ z(t,0) &= z(t,\pi) = 0 \\ z(t_0,x) &= z_0(x),\ x \in \partial\widehat{\triangle} \end{cases} \tag{5.4}$$

Consider $\widehat{\triangle} \subset \Re^n$ be a bounded domain with smooth boundary $\partial\widehat{\triangle}$, $X = L^2(\widehat{\triangle})$, ϱ_k be random variable defined on $D_k \equiv (0, d_k)$ for $k \in \mathbb{N}$, $d_k \in (0, +\infty)$. Also assume that ϱ_k follow Erlang distribution and if $i \ne j$ then ϱ_i and ϱ_j are independent with each other for $i, j = 1, 2, \ldots$. Here q is a function of k, $\xi_k = \xi_{k-1} + \varrho_k$ for $k \in \mathbb{N}$, $t_0 \in \Re^+$.

Let A be an operator on X by $Az = \dfrac{\partial^2 z}{\partial x^2}$ with the domain

$$D(A) = \Big\{ z \in X \mid z \text{ and } \frac{\partial z}{\partial x} \text{ are absolutely continuous}, \frac{\partial^2 z}{\partial x^2} \in X, z = 0 \text{ on } \partial\widehat{\triangle} \Big\}$$

Thus A generates a strongly continuous semigroup $S(t)$ which is self adjoint, compact and analytic. Furthermore the operator A can be represented as

$$Az = \sum_{n=1}^{\infty} n^2 < z, z_n > z_n, \ z \in D(A)$$

Here $z_n(\zeta) = \sqrt{\frac{2}{\pi}} Sin(n\zeta), n = 1, 2, \dots,$ forms the orthonormal set of eigenvectors of A. Also for every $z \in X, S(t)z = \sum_{n=1}^{\infty} e^{(-n^2 t)} < z, z_n > z_n$, which holds $\|S(t)\| \leq e^{(-\pi^2 (t-t_0))}, t \geq t_0$. Therefore $S(t)$ is a semigroup.

Consider that the following assumptions:

(i) $f : \Re_\varrho \times X \to X, f_1 : \Re_\varrho \times X \to X$ is a continuous function defined by

$$f(t, z)(x) = F_1(t, z(x)) \ t_0 \leq t \leq T, 0 \leq x \leq \pi$$

$$f_1(\theta, x(t+\theta))d\theta = \int_0^T F_2(\theta, z(tsin\theta, x))d\theta$$

and also function f and f_1 satisfies the Lipschitz condition.

(ii) $E\Bigg\{ \max_{i,k} \bigg\{ \prod_{j=i}^{k} \|b_j(\varrho_j)\| \bigg\} \Bigg\}$ is uniformly bounded if,

$$E\Bigg\{ \max_{i,k} \bigg\{ \prod_{j=i}^{k} \|b_j(\varrho_j)\| \bigg\} \Bigg\} \leq \theta, \qquad \text{for each } \varrho_j \in D_j, j \in \mathbb{N}, \theta > 0 \text{ a constant}$$

(iii)

$$\Gamma \int_{t_0}^{T} \mathscr{L}(r)dr < \int_{\gamma_1}^{\infty} \frac{dr}{H(r)}, \tag{5.5}$$

where $\Gamma = 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{(2a-1)\Gamma(a)}, \gamma_1 = 2\mathcal{W}^2\Theta^2 E\|\varphi\|^2$ and $\mathcal{W}\Theta \geq \frac{1}{\sqrt{2}}$.

Assume that assumptions (i), (ii) and (iii) are satisfied, then the problem (5.1) becomes a random impulsive fractional differential equation. From all the above facts, in view of Theorem 3.1, we conclude that 5.4 has a mild solution.

Remark 5.4. Let the conditions of Example 5.3 along with $(\mathscr{H}_4) - (\mathscr{H}_5)$ be hold. If the following inequality is satisfied,

$$\Gamma^* \int_{t_0}^{T} \mathscr{L}(r)dr < \int_{\gamma_2}^{\infty} \frac{dr}{H(r)}, \tag{5.6}$$

where $\Gamma^* = 2\mathcal{W}^2 \max\{1, \Theta^2\} \frac{(T-t_0)^{2a-1}(1+K^*+H^*)}{\Gamma(a)(2a-1)}, \gamma_2 = 2\mathcal{W}^2\Theta^2 E[\|\varphi\|^2]$, and $\mathcal{W}\Theta \geq \frac{1}{\sqrt{2}}$. Then the mild solution z of the Example 5.3 is exponentially stable in the quadratic mean.

6. Conclusion

In this article we mainly focused on the existence and stability of the random impulsive fractional differential equations via Leray-Schauder fixed point method. Firstly, we established the existence of mild solution and continued to prove the exponential stability of the system. Finally, we provided an application to assist of our theory. In future, we will study controllability of random impulsive fractional differential system via fixed point approach.

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