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ON APPROXIMATING INITIAL DATA IN SOME LINEAR EVOLUTION EQUATIONS INVOLVING FRACTIONAL LAPLACIAN

RAMESH KARKI

Abstract. We study an inverse problem of recovering initial datum in a one-dimensional linear evolution equation with the Dirichlet boundary conditions when the solution to the equation is known only at a suitably fixed space location and suitably chosen finitely many later time instances. To be more explicit, we consider a one-dimensional linear evolutionary equation involving a Dirichlet fractional Laplacian and an unknown initial datum f which is assumed to be in a suitable subset of a Sobolev space, and then construct n future times so that from the known values of the solution at a suitably fixed space location and at these n future times, we recover f with a desired accuracy.

1. Introduction

Consider the initial-boundary value problem

$$u_t = -a(t)(-\Delta)^{1/2}u,\ u(0,t) = u(\pi,t) = 0,\ u(x,0) = f(x) \tag{1.1}$$

where a is a positive continuous function of $t > 0$ such that $\int_0^t a(s)ds$ exists, $\Delta = \partial^2/\partial x^2$, and at the moment, f is in $L^2[0,\pi]$. The problem of finding a solution $u(x,t)$ to (1.1) is quite common when the initial datum f in a suitable function space is known. However, we are interested in studying an inverse type problem, meaning a problem of recovering the initial datum f with a desired accuracy if the solution $u(x,t)$ is known only at a fixed point x_0 in $[0,\pi]$ and suitably selected n later time instances $t_j,\ j = 1,2,3,\dots,n$.

This type of inverse problem is not well-posed in general. For the well-posed, we further assume that f is in the closed subset $\mathcal{B}_r$ of the Sobolev space $H^r[0,\pi],\ r > 0$, given by

$$\mathcal{B}_r := \left\{ f \in H^r[0,\pi] : ||f||_{H^r[0,\pi]} \le 1 \right\}. \tag{1.2}$$

We are indeed motivated to study this problem from similar problems studied in [1, 9]. In [9], the authors have considered the temperature distribution of a thin uniform one-dimensional body of finite length represented by the one-dimensional heat equation with Dirichlet boundary conditions and an initial condition. Then they have studied the recovery of the initial temperature measurement with a near optimal rate when temperature measurements taken at a fixed point of the body and at finitely many later times are known. Also, they have asked some questions, one of which is whether their method can be extended to the case of involving a diffusion equation with a diffusion coefficient depending continuously on time. This question has been addressed in [1]. Moreover, the authors in [1] have also studied the problem of recovering initial data in an initial-boundary value problem involving a parabolic PDE with constant coefficients and even order partial differential coefficients with respect to

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spacial variable. In this paper, we study the case, given in (1.1), that involves a fractional diffusion equation with a diffusion coefficient depending continuously on time. It could be a base for generalizing the method in other factional evolution equations, some of which are mentioned in Section 5. These problems are of particular interest as they arise in different areas such as stochastic control theory and mathematical finance, classical mechanics in the context of heat propagation, population dynamics, the theory of water waves, quantum mechanics and phase transition problems [2, 5, 10, 13].

We know that the Dirichlet Laplacian $-\Delta$ on $L^2[0,\pi]$ has eigenvalues $\lambda_n = n^2,\ n = 1, 2, \ldots$ with the corresponding eigenfunctions $e_n(x) := \sin nx,\ n = 1, 2, \ldots$ that form an orthonormal basis for $L^2[0,\pi]$ when normalized with respect to the inner product $\langle u, v\rangle_{L^2} := \dfrac{2}{\pi}\displaystyle\int_0^\pi u(x)v(x)\, dx$. Then each $u \in L^2[0,\pi]$ has the following Fourier sine series representation

$$u = \sum_{k=1}^{\infty} \hat{u}_k e_k,$$

where the equality has to be understood in the L^2-sense or in the sense of almost every $x \in [0,\pi]$ and $\hat{u}_k := \langle u, e_k\rangle_{L^2}$, the k^{th} Fourier sine coefficient of u. Referring to [3, 6, 12, 15], the fractional power operator $(-\Delta)^{s/2}$, $s > 0$ on the dense subset $D((-\Delta)^{s/2}) := \{u \in L^2[0,\pi] : \sum_{k=1}^{\infty} k^{2s}|\hat{u}_k|^2 < \infty\} = H^r[0,\pi]$ of $L^2[0,\pi]$ is given by

$$(-\Delta)^{s/2}u = \sum_{k=1}^{\infty} k^s \hat{u}_k e_k \tag{1.3}$$

and has the eigenvalues $\lambda_n^{s/2} = n^s,\ n = 1, 2, \ldots$ with the corresponding eigenfunctions $e_n,\ n = 1, 2, \ldots$. Thus we can equip $H^r[0,\pi]$ with the norm $||f||_{H^r} = \sum_{k=1}^{\infty} k^{2r}|\hat{f}_k|^2$.

The existence and uniqueness theory guarantees the existence of a (strong or L^2-) solution $u(x,t)$ to (1.1), which then has the Fourier sine series representation

$$u(x,t) = \sum_{k=1}^{\infty} \hat{u}_k(t)e_k(x) \tag{1.4}$$

for almost every $x \in [0,\pi]$. However, solutions to more general evolutionary problems than (1.1) can be expressed in the form of general Fourier series representations, which can be obtained by employing tools from the semigroup theory and the spectral theory of self-adjoint operators on Hilbert spaces, and have been discussed in [12, 14, 15, 16]. These representations can be used when extending our problem to more general evolutionary equations.

Under the assumptions we made above, the regularity theory guarantees that the solution $u(x,t)$ to (1.1) is in $D((-\Delta)^{1/2})$). From (1.1), (1.3) and (1.4), we obtain that each time dependent Fourier sine coefficient $\hat{u}_k(t)$ satisfies the initial value problem

$$\frac{d}{dt}\hat{u}_k(t) = -\, ka(t)\hat{u}_k(t),\ \hat{u}_k(0) = \hat{f}_k$$

whose solution is

$$\hat{u}_k(t) = \hat{f}_k e^{-kT(t)},\ k = 1, 2, \ldots, \tag{1.5}$$

where $T(t) = \displaystyle\int_0^t a(s)ds$. From (1.4) and (1.5), we have

$$u(x,t) = \sum_{k=1}^{\infty} \hat{f}_k e^{-kT(t)} e_k(x) \tag{1.6}$$

for $t \geq 0$ and for almost every $x \in [0,\pi]$. The reason why the last equality holds in the *a. e.* sense is because we have used the strong (not the classical) solution to (1.1). Since we are going to deal with the L^2-error of approximation to f, it will be sufficient to have this type of solution. Throughout the rest of the paper, we will have referred to this solution whenever we call the solution to (1.1).

Now we summarize what we are going to do in the rest of the paper. In Section 2, we will discuss the selection of the space location x_0 and prove that for an increasing sequence $0 < t_1 < t_2 < \dots$ of future times, the values $u(x_0, t_j),\ j = 1, 2, \dots$ are enough to determine f uniquely. This consistency result allows us to approximate f from the first n values $u(x_0, t_j),\ j = 1, 2, \dots, n$, called n samples.

In Section 3, we will discuss the existence of a lower bound for an optimal error of approximation to f. For this, we will use a measurement algorithm discussed in [1, 9] as an encoder or a continuous map a from a subset $\mathcal{B}$ of $L^2[0, \pi]$ into $\mathbb{R}^n$ together with a decoder M or a continuous map from $\mathbb{R}^n$ into L^2. Using this measurement algorithm, we will find an approximation to f as $M(a(f))$ and also discuss the existence of a lower bound for the optimal error of approximation to f in $L^2[0, \pi]$.

Like obtaining a lower bound for the optimal error of approximation to f, one may expect the existence of an upper bound for the optimal error of the approximation. We will not address this in this paper. However, in Section 4, we will prove that there exists a sequence of future times $0 < t_1 < t_2 < \dots$ such that from the first n samples $u(x_0, t_j),\ j = 1, 2, \dots, n,\ f$ can be approximated in $L^2[0, \pi]$ with an accuracy of order n^{-r}.

In Section 5, we will discuss possibilities of extending this method to a few other evolutionary equations and also possibilities of applying it to evolutionary equations with other boundary conditions.

2. Choice of x_0 and consistency of recovery

We need to select $x_0 \in [0, \pi]$ in such a way that the samples $u(x_0, t_j),\ j = 1, 2, \dots,$ determine f uniquely provided the time sequence $0 < t_1 < t_2 < \dots$.

Observing (1.6), we see that the position x_0 in $[0, \pi]$ have to be chosen so that $\sin kx_0 \neq 0$ for all $k = 1, 2, \dots$. So, as in [9], we consider x_0/π to be an algebraic number of second order, that is,

$$d\left(\frac{x_0}{\pi}, \left\{0, \frac{1}{m}, \frac{2}{m} \dots, \frac{m}{m} = 1\right\}\right) \geq \frac{c_0}{m^2},\ m = 1, 2, \dots \tag{2.1}$$

where c_0 is a constant. Then we have

$$d\left(kx_0, \{0, \pi, \dots, k\pi\}\right) \geq c_0 k \pi^{-1},\ k = 1, 2, \dots,$$

and hence

$$|\sin kx_0| \geq d_0 k^{-1},\ k = 1, 2, \dots \tag{2.2}$$

for some constant d_0.

Now the following consistency result.

Lemma 2.1. *For a sequence $\{t_j\}_{j=1}^{\infty}$ of later time instances satisfying $t_1 < t_2 < t_3 < \dots$ and the choice of $x_0 \in [0, \pi]$ described by (2.2), suppose $u(x_0, t_j),\ j = 1, 2, 3, \dots$ are known. Then f can be determined uniquely.*

Proof: Consider the function

$$F_0(z) := \sum_{k=1}^{\infty} c_k z^k, \tag{2.3}$$

where

$$c_k := \hat{f}_k e_k(x_0),\ k = 1, 2, 3, \dots \tag{2.4}$$

Since the sequence $\{c_k\}_{k=1}^{\infty}$ is in l^2, F_0 is holomorphic in the unit complex disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Since the sequence of points $z_j = e^{-T(t_j)},\ j = 1, 2, 3, \dots$ in $\mathbb{D}$ converges in $\mathbb{D}$ and $F_0(z_j) = u(x_0, t_j),\ j = 1, 2, 3, \dots,$ the Identity Principle of one-complex variable implies that F_0 can be determined uniquely.

This together with (2.3) and (2.4) implies that $c_k,\ k = 1, 2, 3, \ldots$ can be determined uniquely and hence $\hat{f}_k,\ k = 1, 2, 3, \ldots$. In this way, f can be determined uniquely. ■

3. Lower bound on optimal error

By following the techniques of [7, 9] (one may also see [8, 11]), we obtain a measurement algorithm to determine a lower bound for the optimal error of recovery of f. First, we consider two continuous mappings a and M, where a maps each f in a compact subspace $\mathcal{B}$ of $L^2 := L^2[0, \pi]$ into a point in $\mathbb{R}^n$ and M maps each point $y \in \mathbb{R}^n$ into a function $M(y)$ in L^2. We view the map a as an encoder or sensor, whereas the map M as a decoder. The set $\{M(y) \in L^2 : y \in \mathbb{R}^n\}$ can be viewed as an n-dimensional manifold. An encoder a together with a decoder M forms our measurement algorithm. Using this algorithm, we obtain point $M(a(f)) =: \bar{f}$ in this manifold, which we consider as an approximation to f, and define the manifold width $\delta_n(\mathcal{B}, L^2)$ as the best of optimal L^2-errors

$$\delta_{a,M}(\mathcal{B}, L^2) := \sup_{f \in \mathcal{B}} ||f - \bar{f}||_{L^2}, \tag{3.1}$$

or more precisely,

$$\delta_n(\mathcal{B}, L^2) := \inf_{a,M} \delta_{a,M,n}(\mathcal{B}, L^2) \tag{3.2}$$

where n is fixed and the infimum is taken over all continuous maps a and M as described above.

In particular, for our approximation problem, we consider an encoder a as a map $f \mapsto (u_1, u_2, \ldots, u_n)$ mapping $\mathcal{B}_r$ into $\mathbb{R}^n$, which extracts n samples $u_j := u(x_0, t_j),\ j = 1, 2, 3, \ldots$ using the information about f, and denote this map by a_n. This map is indeed continuous.

Lemma 3.1. *In our measurement algorithm, the map $a_n : f \mapsto (u_1, u_2, \ldots, u_n)$ mapping $\mathcal{B}_r$ into $\mathbb{R}^n$ is continuous.*

Proof: If $\bar{f} \in \mathcal{B}_r$ with the Fourier sine coefficients $\hat{\bar{f}}_k$, then for each $f \in \mathcal{B}_r$ with the Fourier sine coefficients $\hat{f}_k$ and for each $j = 1, 2, 3\ldots, n$, we have

$$\begin{aligned}
|u_j - \bar{u}_j| =& |\sum_{k=1}^{\infty} \hat{f}_k e^{-kT(t_j)} e_k(x_0) - \sum_{k=1}^{\infty} \hat{\bar{f}}_k e^{-kT(t_j)} e_k(x_0)| \\
\leq& \sum_{k=1}^{\infty} |\hat{f}_k - \hat{\bar{f}}_k| e^{-kT(t_j)} \\
\leq& \left(\sum_{k=1}^{\infty} |\hat{f}_k - \hat{\bar{f}}_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |e^{-kT(t_j)}|^2 \right)^{1/2} \\
\leq& ||f - \bar{f}||_{L^2} ||\{e^{-kT(t_1)}\}_{k=1}^{\infty}||_{l^2},
\end{aligned}$$

from which the proof of the lemma follows. ■

In particular, for our approximation problem, we consider a decoder M as a map $(u_1, u_2, \ldots, u_n) \mapsto \bar{f}_n$ mapping each n-tupple of n samples into an approximation $\bar{f}_n \in L^2$ to f, and denote this map by M_n. Thus $\delta_{a_n,M_n,n}(\mathcal{B}_r, L^2) = \sup_{f \in \mathcal{B}_r} ||f - \bar{f}_n||_{L^2}$, where $\bar{f}_n = M_n(a_n(f))$. Now we deduce the following.

Theorem 3.2. *For a measurement algorithm with an encoder $a_n : f \mapsto (u_1, u_2, \ldots, u_n)$ and a decoder $M_n : (u_1, u_2, \ldots, u_n) \mapsto \bar{f}_n$, we have*

$$\delta_{a_n,M_n,n}(\mathcal{B}_r, L^2) \geq Cn^{-r} \tag{3.3}$$

where C is a constant depending on r only.

Proof: For $\mathcal{B}_r$, the idea discussed in Section 3 of [9] implies $\delta_n(\mathcal{B}_r, L^2) \geq C(r)n^{-r}$ (or see [1, 7, 8]). Therefore, for the measurement algorithm discussed above, we have

$$\delta_{a_n,M_n,n}(\mathcal{B}_r, L^2) \geq \delta_n(\mathcal{B}_r, L^2) \geq C(r)n^{-r},$$

establishing (3.3). ■

Due to some technical challenges, we will not obtain an upper bound for the optimal error of approximation to f. However, we will particularly prove in the next section that we can construct a sequence of future times $0 < t_1 < t_2 < \dots$ such that from the first n samples $u(x_0, t_j),\ j = 1, 2, \dots, n$, f can be approximated in $L^2[0, \pi]$ with an error that has an upper bound of order n^{-r}. This is the main outcome of this paper.

4. Optimal approximation to initial data

As we discussed at the end of the last section, our main goal is to select n future time instances $t_j,\ j = 1, 2, 3, \dots, n$ so that from the known n samples $u(x_0, t_j)$, $j = 1, 2, 3, \dots, n$, we can construct an approximation to f in $L^2[0, \pi]$ with an accuracy of order n^{-r}.

Theorem 4.1. *Let $\mathcal{B}_r$ be as described in (1.2), let $f \in \mathcal{B}_r,\ r > 0$, let a be as described in (1.1) and let $u(x, t)$ denote the solution to the problem (1.1). Fix $x_0 \in [0, \pi]$ such that (2.2) holds. Additionally, fix $t_1 > 0$ and $\rho \geq 2$. There exists a sequence $\{t_j\}_{j=1}^{\infty}$ such that $T(t_{j+1}) = \rho^j T(t_1),\ j = 1, 2, 3, \dots$. If $u(x_0, t_j),\ j = 1, 2, \dots, n$ are known, then there exists $\bar{f}_n$ in $L^2[0, \pi]$ such that*

$$||f - \bar{f}_n||_{L^2} \leq Cn^{-r}, \tag{4.1}$$

where C is a constant that depends on d_0, r, t_1 and ρ.

We begin with considering an increasing sequence $t_1 < t_2 < \dots$ of later times. Set $u(x_0, t_j) := U(t_j)$, $j = 1, 2, 3, \dots$. From (1.6), we have

$$U(t_j) = \sum_{k=1}^{\infty} c_k e^{-kT(t_j)},\ j = 1, 2, 3, \dots, \tag{4.2}$$

where $c_k = \hat{f}_k e_k(x_0),\ k = 1, 2, 3, \dots$. We use $U(t_n)$ to compute c_1 and recursively, $U(t_{n-k+1})$ to compute each c_k, $k = 2, 3, 4, \dots$. So, from (4.2) we obtain

$$c_1 = e^{T(t_n)}U(t_n) - \sum_{j=2}^{\infty} c_j e^{(1-j)T(t_n)} \tag{4.3}$$

and for each $k = 2, 3, 4, \dots$

$$c_k = e^{kT(t_{n-k+1})}U(t_{n-k+1}) - \sum_{j=1}^{k-1} c_j e^{(k-j)T(t_{n-k+1})} - \sum_{j=k+1}^{\infty} c_j e^{(k-j)T(t_{n-k+1})}. \tag{4.4}$$

Suppose we have n samples $U(t_j),\ j = 1, 2, \dots, n$. From these n samples, we construct an approximation $\bar{c}_1$ to c_1 as

$$\bar{c}_1 := e^{T(t_n)}U(t_n) \tag{4.5}$$

and an approximation $\bar{c}_k$ to each $c_k,\ k = 2, 3, 4, \dots, n$ as

$$\bar{c}_k := e^{kT(t_{n-k+1})}U(t_{n-k+1}) - \sum_{j=1}^{k-1} \bar{c}_j e^{(k-j)T(t_{n-k+1})}. \tag{4.6}$$

These c_k's and $\bar{c}_k$'s satisfy an important estimate which is described in the next lemma.

Lemma 4.2. *Let f, a, x_0 and t_1 and ρ be as in Theorem 4.1. There exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $T(t_{k+1}) = \rho^k T(t_1)$, $k = 1, 2, 3, \ldots$. For this sequence,*

$$|c_k - \bar{c}_k| \leq S(t_1) 2^k e^{-T(t_{n-k+1})}, \ k = 1, 2, 3, \ldots. \tag{4.7}$$

Proof: Notice that T is a strictly increasing positive continuous function of $t > 0$. For each $k = 1, 2, 3, \ldots$, we have $T(t_1) < \rho^k T(t_1)$ and thus $\rho^k T(t_1)$ is in the range of T. Therefore, for each $k = 1, 2, 3, \ldots$, we can choose $t_{k+1} > 0$ such that $T(t_{k+1}) = \rho^k T(t_1)$.

We use the method of induction to prove the second part of the lemma. Since $f \in \mathcal{B}_r$, we have $|c_j|^2 \leq |\hat{f}(j)|^2 \leq j^{-2r} \sum_{k=j}^{\infty} k^{2r} |\hat{f}(k)|^2 \leq j^{-2r}$ for all $j = 1, 2, 3, \ldots$. Then from 4.4 and 4.6,

$$|c_1 - \bar{c}_1| \leq \sum_{j=2}^{\infty} j^{-r} e^{(1-j)T(t_n)} \leq e^{-T(t_n)} \sum_{j=0}^{\infty} e^{-jT(t_1)} \leq 2S(t_1) e^{-T(t_n)}, \tag{4.8}$$

where $S(t_1) := \sum_{j=0}^{\infty} e^{-jT(t_1)} = \dfrac{1}{1 - e^{-T(t_1)}}$. Thus we have proved that (4.7) holds true for $k = 1$. Assume that (4.7) holds true for each $j \in \{1, 2, \ldots, k-1\}$, where $k \geq 2$. For each $k \geq 2$, we obtain from (4.4) and (4.6) that

$$|c_k - \bar{c}_k| \leq \sum_{j=1}^{k-1} e^{(k-j)T(t_{n-k+1})} |c_j - \bar{c}_j| + \sum_{j=k+1}^{\infty} j^{-r} e^{(k-j)T(t_{n-k+1})}. \tag{4.9}$$

Using the induction hypothesis and the formula for $T(t_j)$, $j = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
\sum_{j=1}^{k-1} e^{(k-j)T(t_{n-k+1})} |c_j - \bar{c}_j| \leq & S(t_1) \sum_{j=1}^{k-1} 2^j e^{(k-j)T(t_{n-k+1}) - T(t_{n-j+1})} \\
= & S(t_1) \sum_{j=1}^{k-1} 2^j e^{(k-j)T(t_{n-k+1}) - \rho^{k-j} T(t_{n-k+1})} \\
= & S(t_1) \sum_{j=1}^{k-1} 2^j e^{(k-j-\rho^{k-j}) T(t_{n-k+1})}.
\end{aligned} \tag{4.10}$$

Also, we have

$$\begin{aligned}
\sum_{j=k+1}^{\infty} j^{-r} e^{(k-j)T(t_{n-k+1})} \leq & (k+1)^{-r} \sum_{j=k+1}^{\infty} e^{(k-j)T(t_{n-k+1})} \\
= & (k+1)^{-r} \sum_{j=0}^{\infty} e^{(-j-1)T(t_{n-k+1})} \\
= & (k+1)^{-r} e^{-T(t_{n-k+1})} \sum_{j=0}^{\infty} e^{-jT(t_{n-k+1})} \\
\leq & (k+1)^{-r} e^{-T(t_{n-k+1})} \sum_{j=0}^{\infty} e^{-jT(t_1)} \\
\leq & S(t_1) (k+1)^{-r} e^{-T(t_{n-k+1})}.
\end{aligned} \tag{4.11}$$

From (4.9), (4.10) and (4.11), we have

$$\begin{aligned}|c_k - \bar{c}_k| \leq & S(t_1)\sum_{j=1}^{k-1} 2^j e^{(k-j-\rho^{k-j})T(t_{n-k+1})} + S(t_1)(k+1)^{-r} e^{-T(t_{n-k+1})} \\ = & S(t_1)e^{-T(t_{n-k+1})}\left[\sum_{j=1}^{k-1} 2^j e^{(k-j-\rho^{k-j}+1)T(t_{n-k+1})} + (k+1)^{-r}\right]\end{aligned}$$

Since $\rho \geq 2$ and $x+1 \leq 2^x$ for $x \geq 1$, we have $k-j+1 \leq 2^{k-j} \leq \rho^{k-j}$ for all $j = 1, 2 \ldots, k-1$. So,

$$\begin{aligned}|c_k - \bar{c}_k| \leq & S(t_1)e^{-T(t_{n-k+1})}\left[\sum_{j=1}^{k-1} 2^j + (k+1)^{-r}\right] \\ \leq & S(t_1)2^k e^{-T(t_{n-k+1})},\end{aligned}$$

proving that (4.7) holds true for $k \geq 2$. This completes the proof of the lemma. ∎

Now we are ready to prove Theorem 4.1.

Under the assumptions of Theorem 4.1 and in the view of $c_k = \hat{f}_k e_k(x_0)$, we use the relation $\bar{c}_k = \hat{\bar{f}}_k e_k(x_0)$ to determine approximate Fourier sine coefficients $\hat{\bar{f}}_k$ to $\hat{f}_k$, $k = 1, 2, \ldots, n$. So, we define $\hat{\bar{f}}_k := \dfrac{\bar{c}_k}{e_k(x_0)}$, $k = 1, 2, \ldots, n$ and then define an approximation to f as

$$\bar{f}_n(x) := \sum_{k=1}^{m} \hat{\bar{f}}_k e_k(x), \text{ where } m = \left\lceil \frac{n}{2} \right\rceil. \tag{4.12}$$

Then the L^2-error of approximation to f satisfies

$$\begin{aligned}||f - \bar{f}_n||_{L^2}^2 \leq & \sum_{k=1}^{m} |\hat{f}_k - \hat{\bar{f}}_k|^2 + \sum_{k=m+1}^{\infty} |\hat{f}_k|^2 \\ \leq & \sum_{k=1}^{m} |\hat{f}_k - \hat{\bar{f}}_k|^2 + \sum_{k=m+1}^{\infty} \left(\frac{k}{m}\right)^{2r} |\hat{f}_k|^2 \\ \leq & \sum_{k=1}^{m} |\hat{f}_k - \hat{\bar{f}}_k|^2 + m^{-2r} \sum_{k=m+1}^{\infty} k^{2r}|\hat{f}_k|^2 \\ \leq & \sum_{k=1}^{m} |\hat{f}_k - \hat{\bar{f}}_k|^2 + m^{-2r}||f||_{H^r}^2 \\ \leq & \sum_{k=1}^{m} |\hat{f}_k - \hat{\bar{f}}_k|^2 + m^{-2r}\end{aligned} \tag{4.13}$$

Using (2.2) and Lemma 4.2, we have

$$|\hat{f}_k - \hat{\bar{f}}_k| = \frac{|c_k - \bar{c}_k|}{|e_k(x_0)|} \leq C_0 k 2^k e^{-T(t_{n-k+1})}, \ k = 1, 2, \ldots, n, \tag{4.14}$$

where $C_0 := S(t_1)/d_0$. Since $0 \leq m - \dfrac{n}{2} < 1$, we have $n - m \leq \dfrac{n}{2} < n - m + 1$ and, therefore, $T(t_{n-k+1}) = \rho^{n-k}T(t_1) \geq \rho^{n/2-1}T(t_1)$ for all $k = 1, 2, 3, \ldots, m$. Also $\ln k \leq k$ for all $k = 1, 2, 3, \ldots, m$.

With these inequalities, we obtain from (4.13) and (4.14) that

$$\begin{aligned}
||f-\bar{f}_n||_{L^2}^2 \leq & \sum_{k=1}^{m} C_0^2 k^2 2^{2k} e^{-2T(t_{n-k+1})} + m^{-2r} \\
\leq & C_0^2 e^{-2T(t_{n-m+1})} \sum_{k=1}^{m} k^2 2^{2k} + m^{-2r} \\
\leq & C_0^2 e^{-2T(t_1)\rho^{n/2-1}} m \cdot m^2 2^{2m} + m^{-2r} \\
\leq & C_0^2 e^{-2T(t_1)\rho^{n/2-1}} n^3 2^{2n} + 2^{2r} n^{-2r}. \qquad (4.15)
\end{aligned}$$

Notice that $\dfrac{j^{2r+3}2^{2j}}{e^{2T(t_1)\rho^{j/2-1}}} \leq \dfrac{e^{(2r+3+2\ln 2)j}}{e^{2T(t_1)\rho^{j/2-1}}} \to 0$ as $j \to \infty$. So, for each set of choice of $r>0$, $t_1>0$ and $\rho \geq 2$, there exists a constant C_1 depending on r, t_1 and ρ such that

$$\frac{j^{2r+3}2^{2j}}{e^{2T(t_1)\rho^{j/2-1}}} \leq C_1$$

for all $j=1,2,3,\ldots$. In particular, we have

$$\frac{n^{2r+3}2^{2n}}{e^{2T(t_1)\rho^{n/2-1}}} \leq C_1. \qquad (4.16)$$

Form (4.15) and (4.16), we obtain

$$||f-\bar{f}_n||_{L^2} \leq Cn^{-r},$$

where C is a constant depending on d_0, t_1, r and ρ. In this way, we have established (4.1) and completed the proof of Theorem 4.1.

Finally, we consider the initial-boundary value problem

$$u_t = -a(-\Delta)^{1/2}u,\ u(0,t)=u(\pi,t)=0,\ u(x,0)=f(x), \qquad (4.17)$$

where a is positive real number and $f \in \mathcal{B}_r$. As a special case of Theorem 4.1, we obtain the following.

Corollary 4.3. *Let $\mathcal{B}_r$ be as described in (1.2), let $f \in \mathcal{B}_r$, $r>0$ and let $u(x,t)$ denote the solution to the problem (4.17). Fix $x_0 \in [0,\pi]$ such that it satisfies (2.2). Also, fix $t_1>0$ and let $\rho \geq 2$. Consider a sequence $\{t_j\}_{j=1}^{\infty}$ with $t_{j+1} := \rho^j t_1$, $j=1,2,3,\ldots$. If $u(x_0,t_j)$, $j=1,2,\ldots,n$ are known, then there exists $\bar{f}_n$ in $L^2[0,\pi]$ such that*

$$||f-\bar{f}_n||_{L^2} \leq Cn^{-r}, \qquad (4.18)$$

where C is a constant that depends on d_0, r, t_1 and ρ.

Proof: Set $T(t)=at$. We can see that all the assumptions of Theorem 4.1 are satisfied. Therefore, the proof of the corollary follows from Theorem 4.1. ■

In the next example, we will particularly choose a function f and illustrate the accuracy of an approximation to f versus t_k, ρ and n of Theorem 4.1.

Example 4.4. Consider $r=2$ and define $f:[0,\pi]\to\mathbb{R}$ by

$$f(x)=\frac{1}{4}x(\pi-x).$$

Then a straightforward calculation gives us the k^{th} Fourier since coefficient

$$\hat{f}_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\, dx = \frac{1}{\pi k^3}\left(1+(-1)^{k+1}\right)$$

and also

$$||f||_{H^r}^2 = \sum_{k=1}^{\infty} k^{2r}|\hat{f}_k|^2 \leq \sum_{k=1}^{\infty} k^{2r}\cdot\frac{4}{\pi^2 k^6} = \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{4}{\pi^2}\cdot\frac{\pi^2}{6} \leq 1.$$

Thus $f \in H^r([0,\pi])$. Consider the following problem

$$u_t = -2t(-\Delta)^{\frac{1}{2}}u, \quad u(x,0) = u(x,\pi) = 0, \quad u(x,0) = f(x) \tag{4.19}$$

Then $T(t) = t^2,\ t \geq 0$. Due to (1.6), the solution to this problem is

$$u(x,t) = \sum_{k=1}^{\infty} e^{-kt^2} \hat{f}_k e_k(x). \tag{4.20}$$

Fix x_0, t_1 and ρ as in Theorem 4.1 and consider the time sequence $\{t_j\}_{j=1}^{\infty}$ as $t_{j+1}^2 = \rho^j t_1^2$. Consider n values $u(x_0, t_j)$, $j = 1, 2, 3, \dots, n$. Using these n values, we will determine an approximation $\bar{f}_n$ to f and will demonstrate the desired accuracy of $\bar{f}_n$ versus t_k, ρ and n.

As in the proof of Theorem 4.1, we will use $u(x_0, t_n)$ to obtain an approximation $\bar{c}_1$ to c_1 and $u(x_0, t_{n-k+1})$ to obtain an approximation $\bar{c}_k$ to c_k for $k = 2, 3, \dots, n$. More precisely, from $u(x_0, t_n) = \sum_{k=1}^{\infty} c_k e^{-kt_n^2}$ where $c_k = \hat{f}_k e_k(x_0)$, we have

$$c_1 = e^{t_n^2} u(x_0, t_n) - \sum_{j=2}^{\infty} c_j e^{(1-j)t_j^2}$$

and from $u(x_0, t_{n-k+1}) = \sum_{k=1}^{\infty} c_k e^{-kt_{n-k+1}^2}$, $k = 2, 3, \dots, n$, we have

$$c_k = e^{kt_{n-k+1}^2} u(x_0, t_{n-k+1}) - \sum_{j=1}^{k-1} c_j e^{(k-j)t_{n-k+1}^2} - \sum_{j=k+1}^{\infty} c_j e^{(k-j)t_{n-k+1}^2}.$$

Set $\bar{c}_1 = e^{t_n^2} u(x_0, t_n)$ and

$$\bar{c}_k = e^{kt_{n-k+1}^2} u(x_0, t_{n-k+1}) - \sum_{j=1}^{k-1} \bar{c}_j e^{(k-j)t_{n-k+1}^2},\ k = 2, 3 \dots, n.$$

Using the method of the proof of Lemma 4.2, we get

$$|c_k - \bar{c}_k| \leq S(t_1) 2^k e^{-t_{n-k+1}^2},\ k = 1, 2, 3, \dots, n$$

where $S(t_1) = \sum_{j=1}^{\infty} e^{-jt_1^2}$. Thus the k^{th} Fourier coefficient $\hat{f}_k$ and its approximation $\bar{\hat{f}}_k = \bar{c}_k / e_k(x_0)$ give

$$|\hat{f}_k - \bar{\hat{f}}_k| \leq \frac{|c_k - \bar{c}_k|}{|e_k(x_0)|} \leq \frac{S(t_1)}{d_0} k 2^k e^{-t_{n-k+1}^2}$$

where d_0 as in (2.2). Then an approximation to f defined by $\bar{f}_n = \sum_{k=1}^{m} \bar{\hat{f}}_k e_k$ where $m = \lceil \frac{n}{2} \rceil$ satisfies

$$\begin{aligned}
||f - \bar{f}_n||_{L^2}^2 \leq& \sum_{k=1}^{m} |\hat{f}_k - \bar{\hat{f}}_k|^2 + m^{-2r} \\
\leq& \sum_{k=1}^{m} \frac{S(t_1)^2}{d_0^2} k^2 2^{2k} e^{-2t_{n-k+1}^2} + m^{-2r} \\
\leq& \frac{S(t_1)^2}{d_0^2} e^{-2t_{n-m+1}^2} \sum_{k=1}^{m} k^2 2^{2k} + m^{-2r} \\
\leq& \frac{S(t_1)^2}{d_0^2} e^{-2t_{n-m+1}^2} n^2 2^{2n} \cdot n + \left(\frac{n}{2}\right)^{-2r} \\
=& n^{-4} \left(\frac{S(t_1)^2}{d_0^2} e^{-2\rho^{n/2-1} t_1^2} n^7 2^{2n} + 4 \right).
\end{aligned}$$

But $e^{-2\rho^{j/2-1}t_1^2}j^7 2^{2j} = e^{-2\rho^{j/2-1}t_1^2+7\ln j+2j\ln 2} \le C_0$ for some constant C_0 depending on t_1 and ρ because $\left\{e^{-2\rho^{j/2-1}t_1^2+7\ln j+2j\ln 2}\right\}_{j=1}^{\infty}$ is a convergent sequence. Using this into the last inequality, we get

$$||f-\bar{f}_n||_{L^2} \le Cn^{-2}$$

for some constant C depending on $(r=2,)$ t_1 and ρ, thereby verifying (4.1).

5. Remarks

We may ask whether our approximation method studied in the preceding sections work for more general problems like the following

$$u_t = -a(-\Delta)^{\eta}u,\ u(0,t)=u(\pi,t)=0,\ u(x,0)=f(x), \tag{5.1}$$

where a is a positive real number, $\eta \in (0,1]$ and $f \in \mathcal{B}_r$, and

$$u_t = -a(t)(-\Delta)^{\eta}u,\ u(0,t)=u(\pi,t)=0,\ u(x,0)=f(x), \tag{5.2}$$

where a is a positive continuous function of $t>0$, $\eta \in (0,1]$ and $f \in \mathcal{B}_r$. Even the method has worked for various special cases of these problems such as when $a=1$ and $\eta=1$ (see [9]); when $\eta=1$ (see [1]) and when $\eta=\frac{1}{2}$, the method may require more advanced analytical tools related to spectral properties of unbounded self-adjoint operators on Hilbert spaces (see [12, 15, 16]) to address these general cases.

We may also ask whether it is possible for the current method to be applied for evolutionary equations with other boundary conditions such as the Neumann and Robin boundary conditions.

It would be worth answering any of the above questions.

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Corresponding author, Natural Science and Mathematics, Indiana University East, Richomond, IN 47374, U.S.A.

Email address: `rkarki@iue.edu`