

ON THE FOCUSING GENERALIZED HARTREE EQUATION

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ABSTRACT. In this paper we give a review of the recent progress on the focusing generalized Hartree equation, which is a nonlinear Schrödinger-type equation with the nonlocal nonlinearity, expressed as a convolution with the Riesz potential. We describe the local well-posedness in H^1 and \dot{H}^s settings, discuss the extension to the global existence and scattering, or finite time blow-up. We point out different techniques used to obtain the above results, and then show the numerical investigations of the stable blow-up in the L^2 -critical setting. We finish by showing known analytical results about the stable blow-up dynamics in the L^2 -critical setting.

1. INTRODUCTION

In this paper we give a review of recent progress on a Schrödinger-type equation with nonlocal potential, the focusing *generalized Hartree* (gHartree) equation,

$$iu_t + \Delta u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2} u = 0, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}. \quad (1.1)$$

Here, $u(x, t)$ is a complex-valued function, $*$ denotes the convolution operator in \mathbb{R}^N , and the convolution with $\frac{1}{|x|^{N-\gamma}}$ is associated to the Riesz potential I_γ of order γ given by

$$I_\gamma(x) = C(N, \gamma) \frac{1}{|x|^{N-\gamma}}, \quad 0 < \gamma < N,$$

where

$$C(N, \gamma) = \frac{\Gamma(\frac{N-\gamma}{2})}{\Gamma(\frac{\gamma}{2}) 2^\gamma \pi^{N/2}}.$$

Typically, $p \geq 2$, however, it is also possible to consider powers $1 < p < 2$. The equation (1.1) is a generalization of the standard Hartree equation with $p = 2$, i.e.,

$$iu_t + \Delta u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^2 \right) u = 0, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}, \quad (1.2)$$

which, for example, can be considered as a model for many-body quantum systems in non-relativistic setting; it also arises in the study of long range interactions between the molecules. The work on the mean-field limit of many-body quantum systems, where the number of bosons is very large, but the interactions between them are weak, goes back to Hepp [30], also see [58], [9], [8], [18]. Lieb and Yau [42] mentioned it in the context of Chandrasekhar theory of stellar collapse, which says that after the death of a star, depending on its mass, the stellar remnants can take one of the three forms: neutron stars, white dwarfs and black hole. Lieb and Thirring [41] conjectured that the collapse for boson stars can be predicted by a Hartree-type equation. A special case of the Riesz potential with $\gamma = 2$ in \mathbb{R}^3 is

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known as the Coulomb potential, which goes back to work of Lieb [39] and has been intensively studied, see reviews [22], [21]. It also appears as a model of a boson star in the pseudo-relativistic setting (for example, see [19], [20]), given by

$$iu_t - \sqrt{-\Delta + m^2} u + \left(\frac{1}{|x|} * |u|^2 \right) u = 0, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \quad (1.3)$$

As mentioned before, the distinct feature of the Hartree equation (1.2) is that it models systems with long-range interactions. Possible experimental realizations of such (repulsive) interactions, where the power in the convolution changes, include the interaction of ultracold Rydberg atoms that have large principal quantum numbers [44]. These interactions between atoms in highly excited Rydberg levels are long range and dominated by dipole-dipole-type forces (the strength of the interaction between Rb atoms is about 10^{12} times stronger than that between Rb atoms in the ground state [57]). The spatial dependence of interactions may be $1/|x|^3$ for small $|x|$ and $1/|x|^6$ for larger $|x|$. Other powers such as $1/|x|^2$ are also possible, see [53].

The equation (1.1) can be written as an electrostatic version of the Maxwell-Schrödinger system, describing the interaction between the electromagnetic field and the wave function related to a quantum non-relativistic charged particle (see, for instance, [11] and [40])

$$\begin{cases} iu_t + \Delta u + V|u|^{p-2}u = 0 \\ -\Delta V = (N-2)|\mathbb{S}^{N-1}| |u|^p, \end{cases} \quad (1.4)$$

which can be viewed as the Schrödinger - Poisson system for the wave function u and the potential V ; here, \mathbb{S}^{N-1} is the sphere in \mathbb{R}^N , and $|\mathbb{S}^{N-1}|$ stands for its volume.

The aim of this paper is to survey the main results from a unified point of view of the generalized Hartree equation and show the current developments in the global existence and finite time blow-up in the gHartree equation.

The paper is organized as follows. In Section 2, we review the necessary background such as invariances of the equation and conserved quantities, then some useful tools such as Strichartz estimates. In Section 3, we state the known results on the local well-posedness in H^1 (available for $0 < s < 1$) and at the critical regularity \dot{H}^s (available for $s > 0$, with certain restrictions in some cases). The local well-posedness is then extended to either global existence or finite-time blow-up in Section 4, depending on initial conditions. In the same section we review results about ground states in the gHartree equation. In Section 5, we discuss initial conditions that predict blow-up in finite time, and can be used in various cases (including energy-supercritical case, $s > 1$). In Section 6 we show the numerical results on the blow-up in the L^2 -critical case, followed by the discussion of spectral property in Section 7.1. Finally, in Section 7 we explain what is known about the stable blow-up dynamics in gHartree equation and open questions.

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2. PRELIMINARIES

We start with the Duhamel formulation (for example, see [62]), where the solution $u : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ to the equation (1.1) is written in the integral form

$$u(t) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-t')\Delta} \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2} u(t') dt' \quad (2.1)$$

for all $t \in I \subset \mathbb{R}$. The interval I is known as the lifespan of u . If $I = \mathbb{R}$, the solution u is said to be global. The first question to understand is whether the equation (1.1), or equivalently, (2.1), can

have local solutions. Before stating the results about the local well-posedness, i.e., existence of a unique local-in time solution satisfying (2.1) that lies in some Sobolev space and continuous dependence on the initial data, we review conserved quantities and other useful properties.

During their lifespans, solutions to (1.1) conserve the mass, energy (Hamiltonian) and momentum, namely, for any $t \in \mathbb{R}$

$$\begin{aligned} M[u(t)] &\stackrel{\text{def}}{=} \int_{\mathbb{R}^N} |u(x, t)|^2 dx = M[u_0], \\ E[u(t)] &\stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(x, t)|^2 dx - \frac{1}{2p} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^{N-\gamma}} * |u(\cdot, t)|^p \right) |u(x, t)|^p dx = E[u_0], \\ P[u(t)] &\stackrel{\text{def}}{=} \text{Im} \int_{\mathbb{R}^N} \bar{u}(x, t) \nabla u(x, t) dx = P[u_0]. \end{aligned}$$

The equation (1.1) enjoys several invariances, among them is the scaling invariance: if $u(x, t)$ solves (1.1), then so does

$$u_\lambda(x, t) = \lambda^{\frac{\gamma+2}{2(p-1)}} u(\lambda x, \lambda^2 t). \tag{2.2}$$

This implies that \dot{H}^{s_c} norm is invariant under the above scaling provided the critical scaling index s_c is

$$s_c = \frac{N}{2} - \frac{\gamma + 2}{2(p - 1)}. \tag{2.3}$$

The equation (1.1) is referred to as the \dot{H}^s -critical if for given N, γ, p in (1.1) the \dot{H}^s norm is invariant under the scaling (2.2) with $s = s_c$, defined by (2.3). In particular,

- if $s_c = 0$, or $p = 1 + \frac{\gamma+2}{N}$, the equation (1.1) is referred to as the *mass-critical* (or L^2 -critical).
- If $s_c = 1$, or $p = 1 + \frac{\gamma+2}{N-2}$, the equation is called the *energy-critical* (or \dot{H}^1 - critical).
- If $0 < s_c < 1$, the equation is *intercritical*.
- If $s_c > 1$, the equation is said to be *energy-supercritical*.

We define the linear Schrödinger evolution from initial data u_0 as follows

$$u(x, t) = e^{it\Delta} u_0(x) = \frac{1}{(4\pi it)^{N/2}} \int_{\mathbb{R}^N} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy.$$

Then by the L^2 -isometry and $L^\infty - L^1$ estimate, one can obtain the time decay estimate for $2 \leq r \leq \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$,

$$\|e^{it\Delta} f_0(x)\|_{L_x^r(\mathbb{R}^N)} \lesssim |t|^{-\frac{N}{2}} \|f_0\|_{L_x^{r'}(\mathbb{R}^N)} \tag{2.4}$$

for all $t \neq 0$.

For the local well-posedness we need estimates in both, space and time. This space-time integrability is demonstrated by Strichartz estimates. In what follows, we will always consider the case $0 \leq s < \frac{N}{2}$.

2.1. Strichartz estimates.

Definition 2.1. The pair (q, r) is called L^2 -admissible pair if $N \geq 1$ and

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2}, \quad 2 \leq q, r \leq \infty \text{ provided } (q, r, N) \neq (2, \infty, 2).$$

Remark 2.1. One can also define the \dot{H}^s -admissibility for $N \geq 1$ and $s \geq -1$ by

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - s. \tag{2.5}$$

Definition 2.2 (see [17]). The pair (q, r) is said to be *acceptable* if $N \geq 1$ and

$$1 \leq q, r \leq \infty \quad \text{and} \quad \frac{1}{q} < N \left(\frac{1}{2} - \frac{1}{r} \right), \quad \text{or } (q, r) = (\infty, 2).$$

Remark 2.2. For $s \geq 0$, every \dot{H}^s -admissible pair is acceptable.

We now recall the well-known Strichartz estimates (see [59], [35], [10]).

Lemma 2.1. *If (q, r) is an \dot{H}^s -admissible pair for $s \geq 0$ then the following linear estimate holds*

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^N)} \lesssim \|f\|_{\dot{H}_x^s(\mathbb{R}^N)}. \tag{2.6}$$

We next consider the inhomogeneous estimate (see [17]).

Lemma 2.2. *Let $1 \leq q, \tilde{q}, r, \tilde{r} \leq \infty$. If the pairs (q, r) and (\tilde{q}, \tilde{r}) are acceptable, satisfy the condition*

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{N}{2} \left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right)$$

and verify the following conditions:

- $N = 2$, we require that $r, \tilde{r} < \infty$,
- $N > 2$, we classify two cases;
 - non sharp case:

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \tag{2.7}$$

$$\frac{N-2}{N} \leq \frac{r}{\tilde{r}} \leq \frac{N}{N-2}; \tag{2.8}$$

– sharp case:

$$\frac{1}{q} + \frac{1}{\tilde{q}} = 1, \tag{2.9}$$

$$\frac{N-2}{N} < \frac{r}{\tilde{r}} < \frac{N}{N-2}, \tag{2.10}$$

$$\frac{1}{r} \leq \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{q}}. \tag{2.11}$$

Then the following estimate holds

$$\left\| \int_0^t e^{i(t-t')\Delta} F(t') dt' \right\|_{L_t^q L_x^r} + \left\| \int_t^\infty e^{i(t-t')\Delta} F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}}. \tag{2.12}$$

We are now ready to review the results on the local well-posedness of the gHartree equation.

3. LOCAL WELL-POSEDNESS

We discuss the local well-posedness results in two settings, one with the finite energy and finite mass initial data, thus, considering the H^1 space (the equation (1.1) would be energy-subcritical or critical at most). The second variant of the local well-posedness is established at the critical regularity \dot{H}^s , which allows to have local-wellposedness in the energy-supercritical cases. Both of these cases consider $p \geq 2$. Recently, some results on wellposedness for $p < 2$ were obtained in [3].

3.1. Local well-posednes in H^1 . We start with considering the initial data in H^1 space, $u_0 \in H^1(\mathbb{R}^N)$, so that we have a finite Hamiltonian (or finite energy) solutions. We consider the integral equation (2.1) with the power p as follows

$$\begin{cases} 2 \leq p \leq 1 + \frac{\gamma+2}{N-2}, & \text{if } N \geq 3 \\ 2 \leq p < \infty, & \text{if } N = 1, 2. \end{cases} \tag{3.1}$$

We mention that the local existence of H^1 solutions in the standard Hartree equation (1.2) ($p = 2$) is available from the work of Ginibre and Velo [24], see also Cazenave [10]. In the general setting ($p \geq 2$) the Cauchy problem for the equation (1.1) with the initial data $u_0 \in H^1$ was investigated by the first and second authors in [6], showing the local well-posedness in H^1 , provided $s_c < 1$. This is guaranteed by (3.1) (note that the nonlinearity in this case is always H^1 -subcritical).

Proposition 3.1. *For p as in (3.1) and $u_0 \in H^1(\mathbb{R}^N)$, there exists $T > 0$ and a unique solution $u(x, t)$ of the integral equation (2.1) on the time interval $[0, T]$ with*

$$u \in C([0, T]; H^1(\mathbb{R}^N)) \cap L^q([0, T]; W^{1,r}(\mathbb{R}^N)), \tag{3.2}$$

where (q, r) is an L^2 -admissible pair given by

$$(q, r) = \left(\frac{2p}{1 + s_c(p-1)}, \frac{2Np}{N + \gamma} \right).$$

In the energy-subcritical case $p < 1 + \frac{\gamma+2}{N-2}$, the time $T = T(\|u_0\|_{H^1}, N, p, \gamma) > 0$. In the energy-critical case $p = 1 + \frac{\gamma+2}{N-2}$ (or $s_c = 1$) an additional assumption of smallness of $\|u_0\|_{H^1_x}$ is required.

The proof relies on a fixed point argument, which can be achieved by showing that the operator

$$\Phi_{u_0}(u) = e^{it\Delta}u_0 + i \int_0^t e^{i(t-t')\Delta} \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2}u(t') dt'$$

defines a contraction on

$$\mathcal{X} = \{u \in L_t^\infty([0, T]; H_x^1(\mathbb{R}^N)) \cap L_t^q([0, T]; W^{1,r}(\mathbb{R}^N)) : \nu(u) \leq M\},$$

for some $M > 0$, where $\nu(u) = \max \left\{ \sup_{t \in [0, T]} \|u\|_{H_x^1}, \|u\|_{L_t^{q_1} W_x^{1,r_1}} \right\}$.

3.2. Local well-posedness in \dot{H}^{s_c} . One can also ask for the local well-posedness at the critical regularity \dot{H}^{s_c} for $s_c \geq 0$, which we state below. The proof can be found in [5].

Proposition 3.2. *Let $0 < \gamma < N$ and $p \geq 2$ so that $s_c \geq 0$. Assume in addition that if p is not an even integer, then $s_c < p - 1$. Let $u_0 \in \dot{H}^{s_c}(\mathbb{R}^N)$. Then there exists a unique solution $u(x, t)$ of the equation (1.1) with data u_0 defined on $[0, T]$ for some $T > 0$, and such that*

- (1) for $s_c = 0$ and $N \geq 1$, $u \in C([0, T]; L_x^2) \cap L^q([0, T]; L_x^r)$, where $(q, r) = \left(2p, \frac{2Np}{N+\gamma}\right)$ is the L^2 -admissible pair and $x \in \mathbb{R}^N$,
- (2) for $0 < s_c < 1$ and $N \geq 1$,

$$u \in C([0, T]; \dot{H}_x^{s_c}) \cap L^{q_1}([0, T]; \dot{W}_x^{s_c, r_1}) \cap L^{q_2}([0, T]; \dot{W}_x^{s_c, r_2}),$$

where

$$(q_1, r_1) = \left(\frac{2p}{1 + s_c(p-1)}, \frac{2Np}{N + \gamma} \right), \quad (q_2, r_2) = \left(\frac{2p}{1 - s_c}, \frac{2Np}{N + \gamma + 2s_cp} \right)$$

are the L^2 -admissible pairs and $x \in \mathbb{R}^N$,

- (3) for $s_c = 1$ and $N \geq 3$, $u \in C([0, T]; \dot{H}_x^1) \cap L^q([0, T]; \dot{W}_x^{1,r})$, where $(q, r) = \left(2, \frac{2N}{N-2}\right)$ is the L^2 -admissible and $x \in \mathbb{R}^N$,

- (4) for $s_c > 1$, $u \in C([0, T]; \dot{H}_x^{s_c}) \cap L^q([0, T]; \dot{W}_x^{s_c, r})$, where $x \in \mathbb{R}^N$ and
 - (a) for $p = 2$ (thus, $N \geq 5$), $(q, r) = \left(3, \frac{6N}{3N-4}\right)$ is the L^2 -admissible pair,
 - (b) for $p > 2$ (thus, $N \geq 3$) and $0 < \gamma < \min(N, \frac{2p}{p-2})$, the L^2 -admissible pair is $(q, r) = \left(2, \frac{2N}{N-2}\right)$.

Moreover, for all $0 < \tilde{T} < T$, the continuous dependence upon the initial data holds.

The proof of this proposition is also done via the fixed point argument in the spaces given in each of the cases (1)-(4) above via the corresponding Strichartz estimates.

4. GLOBAL EXISTENCE AND SCATTERING

After establishing the local well-posedness either in H^1 and \dot{H}^{s_c} , a natural question to ask is whether it is possible to extend local in-time existence to larger time intervals. It turns out that the local existence can be extended to obtain global solutions for small data, which is the next statement. Its proof is in [6].

Proposition 4.1 (Small data theory in H^1). *Let $p \geq 2$ satisfy (3.1) with $0 < \gamma < N$ and $u_0 \in H^1(\mathbb{R}^N)$. Suppose $\|u_0\|_{H^1} \leq A$. There exists $\delta = \delta(A) > 0$ such that if $\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c})} \leq \delta$, then there exists a unique global solution u of (1.1) in $H^1(\mathbb{R}^N)$ such that*

$$\|u\|_{L_t^{\frac{2p}{1-s_c}} L_x^{\frac{2Np}{N+\gamma}}} \leq 2\|e^{it\Delta}u_0\|_{L_t^{\frac{2p}{1-s_c}} L_x^{\frac{2Np}{N+\gamma}}}$$

and

$$\|\nabla|^{s_c}u\|_{L_t^{\frac{2p}{1+s_c(p-1)}} L_x^{\frac{2Np}{N+\gamma}}} \leq 2c\|u_0\|_{H^1},$$

where c depends on constants from the Gagliardo-Nirenberg interpolation estimate and the Strichartz inequality.

A similar result is available in \dot{H}^{s_c} , which makes it possible to extend the local existence to the larger time intervals. This is proved in [5].

Proposition 4.2 (Small data theory in \dot{H}^{s_c}). *Let γ, N, p be as in Proposition 3.2 so that $s_c \geq 0$. Assume in addition that if p is not an even integer, then $s_c < p - 1$. Let $u_0 \in \dot{H}^{s_c}(\mathbb{R}^N)$ with $\|u_0\|_{\dot{H}^{s_c}} \leq A$. There exists $\delta = \delta(A) > 0$ such that if $\|e^{it\Delta}u_0\|_{W^{s_c}} \leq \delta$, then one can find a unique global solution u of (1.1) in $\dot{H}^{s_c}(\mathbb{R}^N)$ such that*

$$\|u\|_{W^{s_c}} \leq 2\|e^{it\Delta}u_0\|_{W^{s_c}},$$

and

$$\|\nabla|^{s_c}u\|_{S^0} \leq 2c_1\|u_0\|_{\dot{H}^{s_c}}.$$

Here,

$$\|u\|_{W^{s_c}} = \begin{cases} \|u\|_{L_t^{\frac{2p}{1-s_c}} L_x^{\frac{2Np}{N+\gamma}}}, & \text{for } 0 < s_c < 1, \\ \|u\|_{L_t^\infty L_x^{\frac{2N}{N-2}}}, & \text{for } s_c = 1, \\ \max\left(\|u\|_{L_t^2 L_x^{\frac{6N}{3\gamma+2}}}, \|u\|_{L_t^\infty L_x^{\frac{2N}{\gamma+2}}}\right), & \text{for } s_c > 1 \text{ and } p = 2, \\ \|u\|_{L_t^\infty L_x^{\frac{2N(p-1)}{\gamma+2}}}, & \text{for } s_c > 1 \text{ and } p > 2, \end{cases}$$

and

$$\|u\|_{S^0} = \begin{cases} \|u\|_{L_t^{\frac{2p}{1+s_c(p-1)}} L_x^{\frac{2Np}{N+7}}}, & \text{for } 0 < s_c < 1, \\ \|u\|_{L_t^2 L_x^{\frac{2N}{N-2}}}, & \text{for } s_c = 1, \\ \max\left(\|u\|_{L_t^3 L_x^{\frac{6N}{3N-4}}}, \|u\|_{L_t^\infty L_x^2}\right), & \text{for } s_c > 1 \text{ and } p = 2, \\ \|u\|_{L_t^2 L_x^{\frac{2N}{N-2}}}, & \text{for } s_c > 1 \text{ and } p > 2 \end{cases}$$

Now that we have some global solutions, one can ask about their asymptotic behavior as $t \rightarrow \pm\infty$. Specifically, if solutions eventually behave as a linear evolution (or approach a linear evolution), which is called scattering, or exhibit a nonlinear behavior. A global solution $u(t)$ to (1.1) is said to scatter in $H^s(\mathbb{R}^N)$ as $t \rightarrow +\infty$, if there exists $u^+ \in H^s(\mathbb{R}^N)$ such that

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{it\Delta} u^+\|_{H^s(\mathbb{R}^N)} = 0.$$

Global existence, asymptotic behavior of solutions and scattering theory for the standard Hartree equation (1.2) goes back to work of Ginibre and Velo [24], where the local wellposedness is established and the authors also prove asymptotic completeness for a repulsive potential (that is, the sign in front of the convolution term in (1.2) is negative, or often called the defocusing case). Hayashi and Tsutsumi [29] obtained the asymptotic completeness of wave operators in $H^m \cap L^p(|x|^\beta dx)$. Related results were established in various settings, for example, see Ginibre and Ozawa [23], Ginibre and Velo [25]-[26], and Hayashi, Naumkin and Ozawa [28].

The following scattering result in $H^1(\mathbb{R}^N)$ is proved in [6].

Proposition 4.3 (H^1 scattering). *Let $u(t)$ be a global solution to (1.1) with initial data $u_0 \in H^1(\mathbb{R}^N)$. If $\|u\|_{L_t^{\frac{2p}{1-s_c}} L_x^{\frac{2Np}{N+7}}} < +\infty$ (globally finite \dot{H}^{s_c} Strichartz norm) and $\sup_{t \in \mathbb{R}^+} \|u(t)\|_{H^1} \leq B$ (uniformly bounded $H^1(\mathbb{R}^N)$ norm). Then $u(t)$ scatters in $H^1(\mathbb{R}^N)$ as $t \rightarrow +\infty$, i.e., there exists $u^+ \in H^1(\mathbb{R}^N)$ such that*

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{it\Delta} u^+\|_{H^1} = 0.$$

The next question is if the small data global existence can be extended to the global existence for large solutions, or if there is a threshold for global existence. In [6] we showed a dichotomy for global existence and scattering vs. finite time blow-up solutions, provided the initial data is in H^1 . The threshold was given by a combination of the mass-energy and the gradient comparison to that of the ground state. For the \dot{H}^s data, it is a more difficult question as the conserved quantities at the \dot{H}^s level are not available (unless $s = 0$ or $s = 1$).

In order to characterize the sharp threshold for the dichotomy, one needs a notion of a ground state, which we review next.

4.1. Ground state solutions. The equation (1.1) in the case when $s_c < 1$ admits standing wave solutions of the form $u(x, t) = e^{it}Q(x)$, where Q the nonlinear nonlocal elliptic equation

$$-Q + \Delta Q + \left(\frac{1}{|x|^{N-\gamma}} * |Q|^p\right) |Q|^{p-2}Q = 0. \tag{4.1}$$

(In the energy-critical case the above equation reduces to the one without the linear term.) The equation (4.1) is known as the nonlinear Choquard or Choquard-Pekar equation. A special case of (4.1) when $N = 3$, $p = 2$, and $\gamma = 2$,

$$\Delta Q - Q + \left(\frac{1}{|x|} * |Q|^2\right) Q = 0, x \in \mathbb{R}^N, \tag{4.2}$$

appeared back in 1954 in the work of S. I. Pekar [54] describing the quantum mechanics of a polaron at rest. Lieb in [39] mentions it in the context of the Hartree-Fock theory of plasma, pointing out that P. Choquard proposed investigating minimization of the corresponding functional in 1976. In 1996 R. Penrose proposed equation (4.2) as a model of self-gravitating matter, in which quantum state reduction is understood as a gravitational phenomenon, see [50].

The existence and uniqueness of the positive solutions to (4.2) was first proved by Lieb [39]. The general existence result of positive solutions along with the regularity and radial symmetry of solutions to (4.1) was shown by Moroz and van Schaftingen [51] (see also a review by Moroz and van Schaftingen [52] and references therein).

The uniqueness proof of Lieb in \mathbb{R}^3 for $p = 2$ with $\gamma = 2$ was extended to the dimension $N = 4$ by Krieger, Lenzmann and Raphaël in [37]; the uniqueness in the pseudo-relativistic 3d setting (1.3) was established by Lenzmann [38]. In [6, Appendix] the proof of uniqueness is written for $2 < N < 6$ (and $p = 2, \gamma = 2$). (The uniqueness and nondegeneracy of the ground state for $\gamma = 2$ and $p = 2 + \epsilon$, i.e., when p is sufficiently close to 2 in \mathbb{R}^3 was shown in [63].) In general, the uniqueness is still open.

In the case of gHartree equation when the uniqueness is known, we denote this unique positive solution, or the ground state, by Q . When it is not available, it is sufficient to use the minimizer of the Gagliardo-Nirenberg inequality of convolution type

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^{N-\gamma}} * |u|^p\right) |u|^p dx \leq C_{GN} \|\nabla u\|_{L^2}^{Np-(N+\gamma)} \|u\|_{L^2}^{N+\gamma-(N-2)p}. \tag{4.3}$$

and the unique value of the sharp constant, expressed via $\|Q\|_{L^2}$, see [6].

We next show how large the initial data can be taken to continue enjoying the property of global existence and scattering.

4.2. Dichotomy: global vs blow-up solutions. We state a dichotomy result for global vs. finite time solutions under the so-called mass-energy threshold, which also shows the H^1 scattering for the global solutions. This result was proved in [6], following the concentration-compactness and rigidity road map of Kenig and Merle [36]. This is in the spirit of [33], [14], [27], [34] for the focusing NLS equation, given as

$$iu_t + \Delta u + |u|^{p-1}u = 0, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{4.4}$$

As in [32] and [33] for the NLS equation, we observe that the quantities

$$\|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_c} \|\nabla u_0\|_{L^2(\mathbb{R}^N)}^{s_c} \quad \text{and} \quad M[u_0]^{1-s_c} E[u_0]^{s_c}$$

are scale-invariant in the gHartree equation (1.1), and for $s_c > 0$ with $\theta = \frac{1-s_c}{s_c}$ we define

- renormalized mass-energy:

$$\mathcal{ME}[u] = \frac{M[u]^\theta E[u]}{M[Q]^\theta E[Q]}$$

- renormalized gradient (which depends on t):

$$\mathcal{G}[u(t)] = \frac{\|u\|_{L^2(\mathbb{R}^N)}^\theta \|\nabla u(t)\|_{L^2(\mathbb{R}^N)}}{\|Q\|_{L^2(\mathbb{R}^N)}^\theta \|\nabla Q\|_{L^2(\mathbb{R}^N)}}.$$

For simplicity, we state the version with the zero momentum; the full version can be found in [6].

Theorem 4.4. *Let $u_0 \in H^1(\mathbb{R}^N)$ with $P[u_0] = 0$ and let $u(t)$ be the corresponding solution to (1.1) with the maximal time interval of existence (T_*, T^*) . Suppose that $\mathcal{ME}[u_0] < 1$.*

- (1) *If $\mathcal{G}[u_0] < 1$, then*
 - (a) *the solution exists globally in time with $\mathcal{G}[u(t)] < 1$ for all $t \in \mathbb{R}$, and*
 - (b) *$u(t)$ scatters in H^1 , in other words, there exists $u_{\pm} \in H^1$ such that*

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\Delta} u_{\pm}\|_{H^1(\mathbb{R}^N)} = 0.$$

- (2) *If $\mathcal{G}[u_0] > 1$, then $\mathcal{G}[u(t)] > 1$ for all $t \in (T_*, T^*)$. Moreover, if*
 - (a) *$|x|u_0 \in L^2(\mathbb{R}^N)$ (finite variance) or u_0 is radial, then the solution blows up in finite time,*
 - (b) *u_0 is of infinite variance and nonradial, then either the solution blows up in finite time or there exists a sequence of times $t_n \rightarrow +\infty$ (or $t_n \rightarrow -\infty$) such that $\|\nabla u(t_n)\|_{L^2(\mathbb{R}^N)} \rightarrow \infty$.*

In a recent work [12], Dodson and Murphy presented a simplified proof of Theorem 4.4 part (1) for (4.4) with $p = 3$ in \mathbb{R}^3 that avoids concentration-compactness route. They used a scattering criterion introduced by Tao in [61], which together with the radial Sobolev embedding and virial/Morawetz estimate was sufficient to prove scattering (in the radial setting). In [13], they extended the above approach to the non-radial case, avoiding the concentration-compactness.

The first author of this paper generalized the method of Dodson and Murphy in the radial case to the inter-critical range of the nonlinear Schrödinger equation (4.4) and also showed that it can be applied in the case of the nonlocal potential such as the gHartree equation (1.1), see [1].

5. BLOW-UP CRITERION

A similar question about the global existence for large data at the critical regularity \dot{H}^{s_c} can be considered, or if there is a threshold for global existence. For the \dot{H}^{s_c} data, this is a more difficult question to answer, since the conserved quantities at the \dot{H}^{s_c} level are not available (unless $s_c = 0$ or $s_c = 1$). What is possible to answer is to show that large data may blow-up in finite time. We give a sufficient condition, blow-up criterion, for the blow-up in finite-time in the generalized Hartree equation (1.1), which follows the ideas in [31, 15, 43, 44] except that now we have to find a bound for the convolution term. To state the result we define the variance, $V(t) \stackrel{\text{def}}{=} \|xu(t)\|_{L^2(\mathbb{R}^N)}^2$, and note that similar to the NLS case, finite variance solutions with negative energy blow up in finite time in gHartree equation by a similar virial (or convexity-type) argument modified to the gHartree case.

Theorem 5.1. *Let $u_0 \in H^1$ if $s_c \leq 1$ and $u_0 \in H^{s_c}$ if $s_c > 1$. Assume also $V(0) < \infty$ and $E[u] > 0$. The following is a sufficient condition for the blow-up in finite time for the solutions to the gHartree equation (1.1) with initial data u_0 in the mass-supercritical case ($s_c > 0$):*

$$\frac{\partial_t V(0)}{\omega M[u_0]} < 4\sqrt{2} f\left(\frac{E[u_0]V(0)}{(\omega M[u_0])^2}\right), \tag{5.1}$$

where

$$\omega^2 = \frac{N^2(N(p-2) + b - 2)}{8(N(p-2) + b)}$$

and the function f is defined as (here, $k = s_c(p-1)$)

$$f(x) = \begin{cases} \sqrt{\frac{1}{kx^k} + x - \frac{1+k}{k}} & \text{if } 0 < x < 1 \\ -\sqrt{\frac{1}{kx^k} + x - \frac{1+k}{k}} & \text{if } x \geq 1. \end{cases} \tag{5.2}$$

The proof of Theorem 5.1 can be found in [5], where the authors show examples of Gaussian data with thresholds in various cases (such as the energy-subcritical, critical and supercritical cases). Those examples play an essential role in studying the actual dynamics of stable finite time blow-up. In [66] the dynamics of stable blow-up is investigated (including rates and profiles) for the gHartree equation in the L^2 -critical and supercritical cases, and it was compared with the known stable blow-up dynamics of the (local) nonlinear Schrödinger equation. We discuss that next.

6. NUMERICAL INVESTIGATION

We have shown that there are solutions which blow-up in finite time in the case $s \geq 0$. The next question is to understand the dynamics of such solutions, in particular, how the blow-up happens. For that we need to separate the L^2 -critical and supercritical case (similar to the NLS equation). In this section we show the numerical investigations of the stable L^2 -critical blow-up and that it happens in the self-similar regime. We point out that the minimal mass blow-up in the L^2 -critical setting occurs at the threshold $M[u_0] = M[Q]$ and is similar to the NLS (see, for example, [37]), however, the minimal mass blow-up is not stable and is not possible to observe numerically. In what follows we consider the initial mass larger than $M[Q]$ (again, note that this quantity is uniquely defined), afterwards we discuss the analytical methods available and challenges in them for blow-up studies.

6.1. Direct numerical simulation for blow-up solutions. To investigate the blow-up behavior numerically, we note that a blow-up solution can behave as a “delta” function, and hence, standard numerical methods cannot be applied. Thanks to the scaling invariance (2.2), one can apply the dynamic rescaling method to investigate the blow-up dynamics. We refer to [60] as well as [65], [66] for details on this method.

Recalling the scaling (2.2), we write

$$u(x, t) = \frac{1}{L^{2/(p-1)}} v(\xi, \tau) \quad \text{with } \xi = \frac{x}{L}, \quad \tau = \int_0^t \frac{1}{L^2(s)} ds.$$

Substituting the above into the gHartree equation (1.1) yields

$$iv_\tau + ia(\tau)\left(\frac{2}{p-1}v + \xi v_\xi\right) + \Delta v + (I_\gamma * |v|^p) |v|^{p-2}v = 0, \tag{6.1}$$

where $a(\tau) = -L L_t = -\frac{d(\ln L)}{d\tau}$ and we choose

$$L(t) = \left(\frac{1}{\|u(\cdot, t)\|_\infty}\right)^{(p-1)/2}$$

(for the choice on $L(t)$ see [65]). Understanding the behavior of the parameter $L(t)$ as t approaches the blow-up time T , or equivalently, $L(\tau)$ as $\tau \rightarrow \infty$, will reveal the rates of the blow-up as well as the convergence to a blow-up profile.

To study the self-similar profile in the blow-up, we separate variables $v(\xi, \tau) = e^{i\tau} Q(\xi)$ and obtain

$$\Delta_\xi Q - Q + ia(\tau)\left(\frac{2}{p-1}Q + \xi Q_\xi\right) + (I_\gamma * |Q|^p) |Q|^{p-2}Q = 0, \tag{6.2}$$

here, $\Delta_\xi := \partial_{\xi\xi} + \frac{d-1}{\xi}\partial_\xi$ denotes the Laplacian with radial symmetry. It was shown that $a(\tau)$ converges to a constant a (and in the L^2 -critical case $a = 0$, however, the convergence is very slow), thus, instead of (6.2) we study

$$\begin{cases} \Delta_\xi Q - Q + ia\left(\frac{2}{p-1}Q + \xi Q_\xi\right) + (I_\gamma * |Q|^p) |Q|^{p-2}Q = 0, \\ Q_\xi(0) = 0, \quad Q(0) \in \mathbb{R}, \quad Q(\infty) = 0. \end{cases} \tag{6.3}$$

The first condition for Q indicates that the local maximum is at zero. The second condition on Q shows that we fix the phase of the solutions, since the equation is phase invariant; the last condition means that $Q(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$. Actually, in the L^2 -critical case one advantage is that the profile solution will be a ground state solution from (4.1), since $a = 0$. Thus, (6.3) is simply reduced to (4.1). (The parameter a is non-zero in the L^2 -supercritical case, see [66] or [65], and we need to study the non-zero a case in the L^2 -critical case, since that allows us to track the blow-up rate with the logarithmic corrections.)

We mention that numerically we study only $\gamma = 2$ case (the convolution is then inverse Laplacian up to a dimensional constant, or in other words, a fundamental solution of the Poisson equation), and solving (6.3) with $a = 0$ numerically produces a unique ground state solution Q to (4.2) (iterations always converge to the same Q , see Remark 6.1 in [66]).

We return to the equation (6.1) and note that it is well-defined for $\tau > 0$, and thus, can be solved with a standard numerical method with respect to ξ and τ , for details refer to [66]. We investigate the blow-up dynamics in the L^2 -critical gHartree equation (with $\gamma = 2$) in dimensions $3 \leq N \leq 7$, the snapshots while tracking the blow-up solution in the 4d case ($N = 4, p = 2$) is shown in Figure 1. In other dimensions, the snapshots look similar. One can note that the solution converges to the rescaled ground state Q slowly (recall that in 4d the ground state Q from (4.2) is proved to be unique).

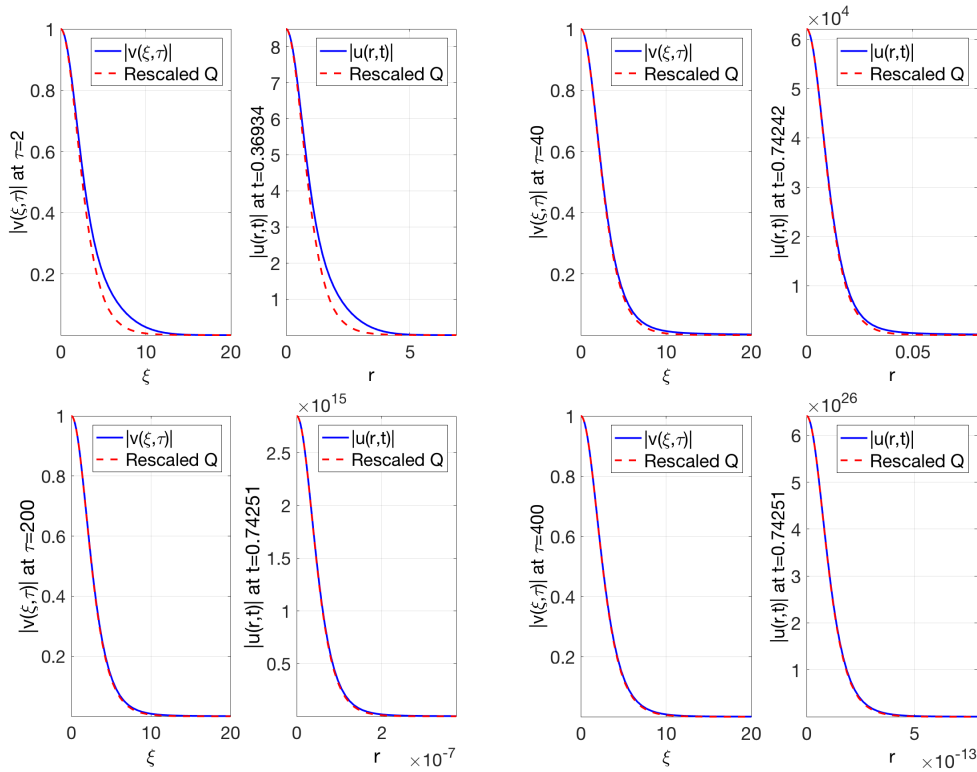


FIGURE 1. The 4d Hartree ($p = 2, \gamma = 2$): snapshots of the blow-up dynamics, converging to the ground state Q at different time t . The snapshots are given in pairs: the left figure is a rescaled solution v from (6.1) and the right is the actual solution compared to the rescaled Q , note the height on the vertical axis and concentration on the horizontal axis.

Next, we study the blow-up rate. For that we track the quantities $L(t)$ and $a(\tau)$ in Figure 2. The left subplot in Figure 2 shows that $\ln L(t)$ depends on $\ln(T - t)$ linearly with the slope 0.50, which means

$L(t) \sim \sqrt{T-t}$. We check the convergence of the parameter $a(\tau)$ to see if there are any corrections to the rate. The middle subplot in Figure 2 shows that $a(\tau)$ decays (very) slowly to zero. This affects the rate of the blow-up, or convergence to the blow-up profile Q , hence, we investigate further the dependence of $a(\tau)$ on possible logarithmic corrections. The right subplot in Figure 2 shows $a(\tau)$ decays at least at a rate of $1/\ln(\tau)$, possibly with further corrections. It is quite challenging to track an extra logarithmic correction, however, we do functional fitting (see [66]) as well as the asymptotic analysis. This leads to the conclusion that the square root rate has a “log-log” correction term, similar to the NLS equation, and in the L^2 cases that we have tracked, the dynamics of blow-up is very similar to the NLS. We refer the interested reader to [66] for further details on asymptotic analysis.

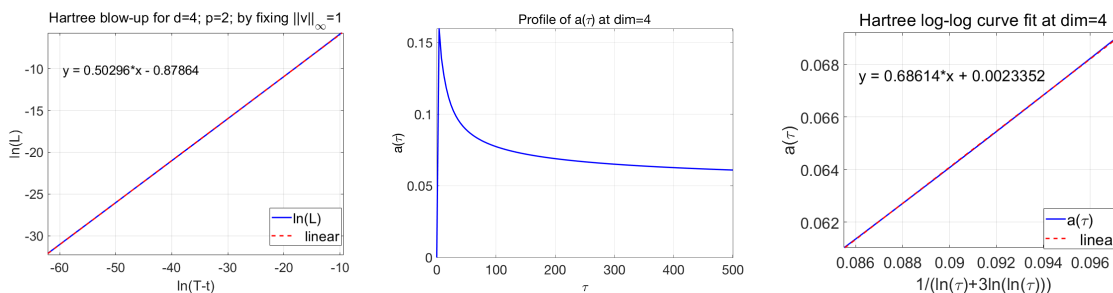


FIGURE 2. The 4d Hartree ($p = 2, \gamma = 2$). Left: the slope of $L(t)$ vs. $T - t$ on a \log scale. Middle: the behavior of $a(\tau)$, indicating a very slow decay to zero. Right: the fitting $a(\tau)$ vs. $1/\ln(\tau) - a(\tau)$ decays as $1/\ln(\tau)$.

7. STABLE BLOW-UP DYNAMICS

Numerical simulations and asymptotic analysis in Section 6 show that stable blow-up dynamics in the L^2 -critical gHartree equation (in the considered cases of $\gamma = 2$ and dimensions $3 \leq N \leq 7$) follows the *log-log* regime, similar to the known results in the L^2 -critical NLS equation, which had an interesting history. We mention some of it.

In the L^2 -critical NLS, the numerical and heuristical investigations of stable blow-up solutions go back to 1970’s and attracted an enormous amount of attention (see [64] for a review). The search of the correct blow-up rate was especially involved, as it has a correction and it is a challenging task to understand what the correction should be (numerically it is not possible to track double logarithm correction). The first rigorous analytical proof of the stable *log-log* blow-up regime was done at the turn of this century by Galina Perelman [55] for the 1d quintic NLS equation, which was followed by a systematic study in a series of papers by Merle and Raphaël [45, 46, 47, 48, 49], obtaining a detailed description of the stable blow-up dynamics for solutions with mass slightly higher than the mass of the ground state solution. The proof requires certain coercivity properties on some bilinear forms, often referred to as the Spectral Property (see Section 7.1, also [47, Section 4.4(D)] or [64]). In the 1d case, the spectral property is proved analytically, since the ground state in the NLS equation is explicit (a rescaled version of $\text{sech}^{1/2} x$), for example see [47, Appendix A]. In higher dimensions the available proofs are numerically-assisted due to the fact that Q is not explicit as well as certain signs of the inner products are also computed numerically (and since the signs are robust to perturbations, it is sufficient for the validity of the Spectral Property). For dimensions $2 \leq N \leq 5$, see [16]; for dimensions $2 \leq N \leq 12$, see [64].

There is very little known about the blow-up dynamics (how it happens, what rates, profiles and other characterizations) for the other forms of nonlinearities, in particular, nonlocal, convolution-type

nonlinearity. We mention that understanding the blow-up dynamics for the convolution nonlinearity, as it is in the Hartree, or gHartree equation, is relevant for the development of theories for a gravitational collapse of, for example, boson stars (as mentioned in the introduction) modeled by the equation (1.3). Fröhlich and Lenzmann [19] proved the existence of finite time blow-up solutions in the pseudo-relativistic Hartree equation (1.3) in regards to the theory of gravitational collapse.

In [4] there is a first attempt to analytical study of the stable blow-up dynamics for the L^2 -critical gHartree equation. Most of the results in that paper hold for the general L^2 -critical gHartree equation (with $\gamma = 2$), however, the Spectral Property we were able to verify only in \mathbb{R}^3 , see subsection 7.1 and [7]. It is an open question to prove analytically the *log-log* blow-up dynamics in other dimensions in the L^2 -critical setting of the gHartree equation, for example, as shown in Figures 1 and 2 (or for other examples, see [66]).

As mentioned in Section 6 we take $\gamma = 2$ (allowing us to write the convolution as the inverse Laplacian). Then the L^2 -critical exponent for (1.1) is $p = 1 + \frac{4}{N}$, and the equation (1.1) becomes

$$iu_t + \Delta u + \left(\frac{1}{|x|^{N-2}} * |u|^{1+\frac{4}{N}} \right) |u|^{\frac{4}{N}-1} u = 0. \tag{7.1}$$

The corresponding ground state equation is

$$-Q + \Delta Q + \left(\frac{1}{|x|^{N-2}} * Q^{1+\frac{4}{N}} \right) Q^{\frac{4}{N}} = 0. \tag{7.2}$$

We gave numerical confirmation in Section 6 to the following conjecture (originally stated in [66]) and in the rest of this survey we will give the sketch of the proof of this conjecture in the 3d case with the one log correction rate.

Conjecture 7.1. *A stable blow-up solution to the L^2 -critical gHartree equation has a self-similar structure and comes with the rate*

$$\lim_{t \rightarrow T} \|\nabla u(\cdot, t)\|_{L_x^2} = \left(\frac{\ln |\ln(T-t)|}{2\pi(T-t)} \right)^{\frac{1}{2}} \quad \text{as } t \rightarrow T,$$

known as the log-log rate. The solution blows up in a self-similar regime with profile converging to a rescaled profile Q , which is a ground state solution of (7.2), namely,

$$u(x, t) \sim \frac{1}{L(t)^{\frac{d}{2}}} Q \left(\frac{x - x(t)}{L(t)} \right) e^{i\gamma(t)}$$

with time depending parameters $L(t)$, $x(t)$ and $\gamma(t)$, converging when $t \rightarrow T$ as follows: $x(t) \rightarrow x_c$ (the blow-up center), $\gamma(t) \rightarrow \gamma_0$ (for some $\gamma_0 \in \mathbb{R}$) and

$$L(t) \sim \left(\frac{2\pi(T-t)}{\ln |\ln(T-t)|} \right)^{\frac{1}{2}}.$$

Thus, the stable blow-up dynamics in the L^2 -critical gHartree equation is similar to the stable blow-up dynamics in the L^2 -critical NLS equation.

In [4] the following blow-up result is proved (see also [2]).

Theorem 7.2. *Let $N = 3$ and consider the L^2 -critical gHartree equation (7.1) with $p = \frac{7}{3}$*

$$iu_t + \Delta u + \left(\frac{1}{|x|} * |u|^{\frac{7}{3}} \right) |u|^{\frac{1}{3}} u = 0. \tag{7.3}$$

Consider $u_0 \in H^1(\mathbb{R}^3)$ such that

$$M[Q] < M[u_0] < M[Q] + \alpha, \quad \text{for some } \alpha > 0, \tag{7.4}$$

and

$$W[u_0] < 0, \quad \text{Im} \left(\int_{\mathbb{R}^3} \bar{u}_0 \nabla u_0 \, dx \right) = 0.$$

Let $u(t)$ be the corresponding solution to (7.3). Then there exist $\alpha_0 > 0$ such that for all $\alpha < \alpha_0$

(1) there exist time depending parameters $(\lambda(t), x(t), \gamma(t)) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$ such that

$$u(t) = \frac{1}{\lambda(t)^{3/2}} (Q + \varepsilon) \left(\frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)},$$

and $\|\varepsilon(t)\|_{H^1(\mathbb{R}^3)} \lesssim \sqrt{\alpha_0}$,

(2) $u(t)$ blows up in finite time, i.e., there exists $0 < T < +\infty$ such that

$$\lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^3)} = +\infty; \quad \text{and}$$

(3) for t close to the blow-up time T , we have

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^3)} \leq C \left(\frac{|\ln(T-t)|}{T-t} \right)^{\frac{1}{2}}, \tag{7.5}$$

for some universal constant $C > 0$.

We outline the strategy of the proof below and refer readers to [4] for a detailed analysis. Note that we start with the general setting (in any dimension), and we point out where the proof is only possible to carry in 3d.

The variational characterization of the ground state Q along with (7.4) and conservation laws allows us to decompose the solution $u(x, t)$ to (7.1) around Q

$$\varepsilon(y, t) = e^{i\gamma(t)} \lambda(t)^{N/2} u(\lambda(t)y + x(t), t) - Q(y), \tag{7.6}$$

where $\|\varepsilon\|_{H^1(\mathbb{R}^N)} \leq \delta(\alpha_0)$, $\lambda(t) > 0$, $x(t) \in \mathbb{R}^N$ and $\gamma(t) \in \mathbb{R}$ are C^1 functions of time. We rescale the time variable by $\frac{ds}{dt} = \frac{1}{\lambda(t)^2}$, and write $\varepsilon = \varepsilon(y, s)$, observe that in this time rescaling we have $s \in [0, \infty)$. This decomposition allows us to transform the analysis to $\varepsilon = \varepsilon_1 + i\varepsilon_2$. Before we write the equations for each component ε_1 and ε_2 , we define the scaling generator

$$\Lambda f = \frac{N}{2} f + x \cdot \nabla f. \tag{7.7}$$

We obtain the following equations

$$(\varepsilon_1)_s - L_- \varepsilon_2 = \frac{\lambda_s}{\lambda} \Lambda Q + \frac{x_s}{\lambda} \cdot \nabla Q + \frac{\lambda_s}{\lambda} (\Lambda \varepsilon_1) + \frac{x_s}{\lambda} \cdot \nabla \varepsilon_1 + \tilde{\gamma}_s \varepsilon_2 - R_2(\varepsilon),$$

$$(\varepsilon_2)_s + L_+ \varepsilon_1 = -\tilde{\gamma}_s Q + \frac{\lambda_s}{\lambda} (\Lambda \varepsilon_2) + \frac{x_s}{\lambda} \cdot \nabla \varepsilon_2 - \tilde{\gamma}_s \varepsilon_1 + R_1(\varepsilon),$$

where $\tilde{\gamma}_s = -s - \gamma_s$, the operators L_{\pm} are defined by

$$\begin{aligned} L_+ \varepsilon_1 := & -\Delta \varepsilon_1 + \varepsilon_1 - \frac{4}{N} \left(|y|^{-(N-2)} * Q^{1+\frac{4}{N}} \right) Q^{\frac{4}{N}-1} \varepsilon_1 \\ & - \left(1 + \frac{4}{N} \right) \left(|y|^{-(N-2)} * \left(Q^{\frac{4}{N}} \varepsilon_1 \right) \right) Q^{\frac{4}{N}}, \end{aligned}$$

$$L_- \varepsilon_2 := -\Delta \varepsilon_2 + \varepsilon_2 - \left(|y|^{-(N-2)} * Q^{1+\frac{4}{N}} \right) Q^{\frac{4}{N}-1} \varepsilon_2,$$

and the remainders R_1, R_2 are quadratic in ε . We choose the modulation parameters $\lambda(s), x(s), \gamma(s)$ such that ε satisfies the following orthogonality conditions

$$\varepsilon_1 \perp y_j Q, \quad \varepsilon_1 \perp \Lambda Q + \Lambda^2 Q \quad \text{and} \quad \varepsilon_2 \perp \Lambda^2 Q. \tag{7.8}$$

Now to perform the blow-up analysis one needs to have the virial identity which is given by

$$\frac{d}{dt} \int |x|^2 |u(x, t)|^2 dx = 4 \operatorname{Im} \left(\int \bar{u} x \cdot \nabla u \right) = -16 |E[u_0]| t + C.$$

However, we rewrite the above virial identity for the ε , which will be given by calculating the time derivative (in s) of the quantity $\Psi(\varepsilon(s)) = \operatorname{Im} \left(\int \bar{\varepsilon} x \cdot \nabla \varepsilon \right) (s)$. Thus, evaluating $\Psi(u(t))$ via the decomposition (7.6), we get

$$\Psi(u(t)) = \operatorname{Im} \left(\int (\bar{\varepsilon} + Q) y \cdot \nabla (\varepsilon + Q) \right) = -4 |E[u_0]| t + \frac{C}{4},$$

which is equivalent to

$$\Psi(\varepsilon(s)) - 2 \int \varepsilon_2 \Lambda Q = -4 |E[u_0]| t + \frac{C}{4}.$$

Taking the derivative of above expression with respect to s and using $\frac{dt}{ds} = \lambda^2(s)$, we obtain

$$(\Psi(\varepsilon))_s(s) = 2(\varepsilon_2, \Lambda Q)_s(s) - 4\lambda^s(s) |E[u_0]|.$$

Thus, for the virial identity in ε we compute $(\varepsilon_2, \Lambda Q)_s$ and obtain the following

$$(\varepsilon_2, \Lambda Q)_s = H(\varepsilon, \varepsilon) + 2\lambda^2 |E_0| - \tilde{\gamma}_s(\varepsilon_1, \Lambda Q) - \frac{\lambda_s}{\lambda} (\varepsilon_2, \Lambda^2 Q) - \frac{x_s}{\lambda} (\varepsilon_2, \nabla(\Lambda Q)) + G(\varepsilon),$$

where $G(\varepsilon)$ is cubic in ε and $H(\varepsilon, \varepsilon)$ is given by (7.16) (or (7.17)).

The next step would be to show coercivity of the bilinear form H and then proceed with the bounds on $(\varepsilon, \Lambda Q)_s$, which will allow us to obtain the blow-up rate with the log correction.

This is a point, where the Spectral Property is needed to proceed. We pause the proof here, and discuss the Spectral Property in a separate subsection.

Before we give the sketch of the proof, we discuss the Spectral Property needed to prove Theorem 7.2, which will indicate why we only consider the 3d case.

7.1. Spectral Property. We recall the scaling generator $\Lambda f = \frac{N}{2} f + x \cdot \nabla f$. We define the two operators L_1 and L_2 as

$$L_1 f = \frac{1}{2} [L_+(\Lambda f) - \Lambda(L_+ f)]; \tag{7.9}$$

$$L_2 f = \frac{1}{2} [L_-(\Lambda f) - \Lambda(L_- f)]. \tag{7.10}$$

Definition 7.1. Let $N > 2$. Given $L_{1,2}$ and a skew-adjoint operator Λ , consider the two real Schrödinger operators

$$L_{1,2} = -\Delta + V_{1,2},$$

defined by the commutator relations

$$L_{1,2} f = \frac{1}{2} [L_{1,2}(\Lambda f) - \Lambda(L_{1,2} f)].$$

Let the real quadratic form for $\mathbf{z} = (u, v)^T \in H_r^1 \times H_r^1$ with radial symmetry be

$$\mathcal{B}(\mathbf{z}, \mathbf{z}) = \mathcal{B}_1(u, u) + \mathcal{B}_2(v, v).$$

The system is said to satisfy a spectral property with radial symmetric assumption on the subspace $\mathcal{U} \in H_r^1 \times H_r^1$ if there exists a universal constant $\delta_0 > 0$ such that $\forall \mathbf{z} \in \mathcal{U}$,

$$\mathcal{B}(\mathbf{z}, \mathbf{z}) \geq \delta_0 \int \left(|\nabla \mathbf{z}|^2 + e^{-|\mathbf{y}|} |\mathbf{z}|^2 \right) d\mathbf{y}.$$

We establish the following results (for the proofs with numerical assistance see [7]).

Theorem 7.3. *The spectral property holds for the 3d generalized Hartree equation (1.1) for $(f, g)^T \in \mathcal{U} \subset L^2 \times L^2$ specified by the following orthogonality conditions*

$$\langle f, Q \rangle = 0, \quad \langle f, \Lambda^2 Q \rangle = 0; \tag{7.11}$$

$$\langle \Lambda Q, g \rangle = 0, \quad \langle \Lambda^2 Q, g \rangle = 0, \tag{7.12}$$

where $\Lambda Q = \frac{N}{2}Q + x \cdot \nabla Q$ as in (7.7), and $\Lambda^2 Q = \Lambda(\Lambda Q)$.

Theorem 7.4. *If we treat the dimension N as a parameter, since we are under the radial symmetric assumption, we have the following results:*

1. *Let the dimensions $\alpha_1 \leq N \leq \alpha_2$ and assume the subspace $\mathcal{U} \subset L_r^2 \times L_r^2$ with the orthogonal conditions*

$$\langle f, Q \rangle = 0, \quad \langle f, \Lambda Q \rangle = 0; \quad \langle \Lambda Q, g \rangle = 0, \quad \langle \Lambda^2 Q, g \rangle = 0. \tag{7.13}$$

Then, the spectral property holds for $(f, g)^T \in \mathcal{U}$ with $\alpha_1 \approx 2.02$ and $\alpha_2 \approx 2.6$.

2. *Let the dimensions $\alpha_3 \leq N \leq \alpha_4$ and assume the subspace $\mathcal{U} \subset L_r^2 \times L_r^2$ with the orthogonal conditions*

$$\langle f, Q \rangle = 0, \quad \langle f, \lambda^2 Q \rangle = 0; \quad \langle \Lambda Q, g \rangle = 0, \quad \langle \Lambda^2 Q, g \rangle = 0. \tag{7.14}$$

Then, the spectral property holds for $(f, g)^T \in \mathcal{U}$ with $\alpha_3 \approx 2.7$ and $\alpha_4 \approx 3.1$.

Note that in the above Theorem the only acceptable integer is $N = 3$ (between α_3 and α_4). For the purpose of analytical proof later, we need a modified version of the above spectral property to incorporate the span of ΛQ , which we state next.

Theorem 7.5. *The spectral property holds for the 3d generalized Hartree equation for $(f, g)^T \in \mathcal{U} \subset L^2 \times L^2$ in the space orthogonal to the spans*

$$\langle f, Q \rangle = 0, \quad \langle f, \Lambda Q + \alpha \Lambda^2 Q \rangle = 0; \quad \langle \Lambda Q, g \rangle = 0, \quad \langle \Lambda Q, g \rangle = 0, \tag{7.15}$$

with α in the range $\alpha < \alpha_1^$ or $\alpha > \alpha_2^*$, where $\alpha_1^* \approx -0.44601$ and $\alpha_2^* \approx 0.69022$.*

Remark 7.1. Theorem 7.5 actually holds for $2.8 \leq N \leq 3.1$ with slightly different values of α_1^* and α_2^* depending on the value of N . We point that the 3d case is of the most interest (as this is the only integer dimension that fits the above spectral property).

The numerically-assisted proof of the above theorems consists of the following steps:

1. Identify the number of negative eigenvalues (indices) of L_1 and L_2 .
2. Show that the indices of L_1 and L_2 are stable under perturbations.
3. Justify that the chosen orthogonal conditions produce the negative spans.

We refer the readers to [7] for further details. Note that the reason that we cannot consider the case $N = 4$ is due to the fact that in 4d the potentials in Definition 7.1 decay as $\frac{C}{|x|^2}$ with a large constant C , which leads to infinitely many negative eigenvalues, and thus, in the step (1.) we would get infinitely many directions (or orthogonal conditions) to deal with, see [56].

Thus, reformulating the above spectral property in terms of the bilinear form H , we have two real-valued operators L_1 and L_2 , defined in (7.9) and (7.10), and the associated real-valued quadratic form $H(\varepsilon, \varepsilon)$ for $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1(\mathbb{R}^3)$ defined as

$$H(\varepsilon, \varepsilon) = H_1(\varepsilon_1, \varepsilon_1) + H_2(\varepsilon_2, \varepsilon_2) \tag{7.16}$$

$$= (L_1 \varepsilon_1, \varepsilon_1) + (L_2 \varepsilon_2, \varepsilon_2). \tag{7.17}$$

Then there exists a universal constant $\delta_0 > 0$ such that for any $\varepsilon \in H^1(\mathbb{R}^3)$, the quadratic form H is positive, or more precisely,

$$H(\varepsilon, \varepsilon) \geq \delta_0 \int \left(|\nabla \varepsilon|^2 + |\varepsilon|^2 e^{-2|y|} \right) dy,$$

provided

$$(\varepsilon_1, Q) = (\varepsilon_1, \Lambda Q + \Lambda^2 Q) = 0 \quad \text{and} \quad (\varepsilon_2, \Lambda Q) = (\varepsilon_2, \Lambda^2 Q) = 0.$$

7.2. Completing the 3d blow-up rate proof. We are now ready to finish the sketch of the proof of the stable *log-log* blow-up in the 3d case.

We show why the choice of the orthogonality condition $\Lambda Q + \Lambda^2 Q$ comes into play. We fix the dimension $N = 3$ and using the Spectral Property discussed above we proceed as follows. Let $\varepsilon \in H^1(\mathbb{R}^3)$ with $(\varepsilon_1, y_i Q) = (\varepsilon_1, \Lambda Q + \Lambda^2 Q) = (\varepsilon_2, \Lambda^2 Q) = 0$ (i.e., ε satisfies (7.8), which implies that it verifies the modulation theory). We set

$$\tilde{\varepsilon} = \varepsilon - aQ - b\Lambda Q - ic\Lambda Q.$$

Observe that $(\tilde{\varepsilon}_1, y_i Q) = (\tilde{\varepsilon}_2, \Lambda^2 Q) = 0$. Also, $(\tilde{\varepsilon}_1, Q) = 0$ and $(\tilde{\varepsilon}_1, \Lambda Q + \Lambda^2 Q) = 0$ with

$$a = \frac{(\varepsilon_1, Q)}{\|Q\|_{L^2(\mathbb{R}^3)}^2} = b.$$

Similarly, $(\tilde{\varepsilon}_2, \Lambda Q) = 0$ with $c = \frac{(\varepsilon_2, \Lambda Q)}{\|\Lambda Q\|_{L^2(\mathbb{R}^3)}^2}$. Hence, $\tilde{\varepsilon}$ now satisfies both the spectral property and the modulation theory.

We evaluate

$$\begin{aligned} H(\varepsilon, \varepsilon) &= H(\tilde{\varepsilon}, \tilde{\varepsilon}) + 2a(\tilde{\varepsilon}_1, L_1 Q) + 2b(\tilde{\varepsilon}_1, L_1(\Lambda Q)) + a^2 H_1(Q, Q) + b^2 H_1(\Lambda Q, \Lambda Q) \\ &\quad + 2ab(L_1 Q, \Lambda Q) + 2c(\tilde{\varepsilon}_2, L_2(\Lambda Q)) + c^2 H_2(\Lambda Q, \Lambda Q) \\ &\geq \tilde{\delta}_0 \int |\nabla \tilde{\varepsilon}|^2 - C(a^2 + c^2) \geq \delta_0 \int |\nabla \varepsilon|^2 - \frac{1}{\delta_0} \left((\varepsilon_1, Q)^2 + (\varepsilon_2, \Lambda Q)^2 \right) \end{aligned} \tag{7.18}$$

for some fixed universal constant $\delta_1 > 0$ small enough. Here, we have used the fact that $H_1(Q, Q) < 0$, $H_1(\Lambda Q, \Lambda Q) = 0$, $H_2(\Lambda Q, \Lambda Q) < 0$ and $(L_1 Q, \Lambda Q) < 0$.

We now give a maximum principle type property, which gives the sign structure of the quantity $(\varepsilon_2, \Lambda Q)$, which says that there exists a unique $s_0 \in \mathbb{R}$ such that for all $s < s_0$, $(\varepsilon_2, \Lambda Q)(s) < 0$, for all $s > s_0$, $(\varepsilon_2, \Lambda Q)(s) > 0$ and $(\varepsilon_2, \Lambda Q)(s_0) = 0$. This together with the relation involving scaling parameter and the quantity $(\varepsilon_2, \Lambda Q)$ of the form $\frac{\lambda^a}{\lambda} \sim -(\varepsilon_2, \Lambda Q)$ yields the monotonicity of the scaling parameter $\lambda(t)$, i.e., for all $s_2 \geq s_1 \geq s_0$, $\lambda(s_2) < 2\lambda(s_1)$.

Using the monotonicity property of scaling parameter, we establish the preliminary weaker upper bound on the blow-up rate, given by

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{\sqrt{|E[u(0)]|(T-t)}}.$$

We then use the fact that the quadratic form $H(\varepsilon, \varepsilon)$ has a non-trivial kernel to establish a refined version of the the localized virial relation, which then allows us to prove a superior control on the scaling parameter, namely,

$$\lambda^2(s) \leq \exp\left(-\frac{C}{(\varepsilon_2, \Lambda Q)(s)}\right), \quad \text{or equivalently,} \quad (\varepsilon_2, \Lambda Q)(s) \geq \frac{C}{|\ln(\lambda(s))|}. \tag{7.19}$$

We then consider a sequence of times t_n such that $\lambda(t_n) = 2^{-n}$ and use (7.19) along with the monotonicity of scaling parameter to deduce that

$$\lambda^2(t) |\ln(\lambda(t))| \geq C\lambda^2(t_n) |\ln(\lambda(t_n))| \geq C(T - t_n) \geq C(T - t).$$

This allows us to conclude the desired upper bound (7.5)

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^3)} \leq C \left(\frac{|\ln(T - t)|}{T - t} \right)^{\frac{1}{2}},$$

completing the proof.

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