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A New analytical Modeling for Fractional Telegraph Equation Arising in Electromagnetic

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Abstrak. Pada artikel ini, metode iterasi variasi He (VIM) dan transformasi integral Elzaki diusulkan untuk menyelesaikan persamaan telegraf fraksional linier dan nonlinier yang muncul dalam elektromagnetik. *Caputo sense* digunakan untuk mendeskripsikan *fractional derivatives*. Salah satu keuntungan dari teknik ini adalah tidak perlu menghitung pengali Lagrange dengan menghitung integrasi dalam relasi perulangan atau dengan mengambil teorema konvolusi. Selanjutnya, untuk mengurangi istilah komputasi nonlinier, polinomial Adomian diidentifikasi dengan homotopy perturbation method (HPM). Metode yang diusulkan diterapkan pada beberapa contoh persamaan telegraf fraksional linier dan nonlinier. Solusi yang diperoleh dengan teknik komputasi baru menunjukkan bahwa metode ini efisien dan memfasilitasi proses penyelesaian *time fractional differential equations*.

Keywords:
Fractional telegraph equations, variation iteration method, Elzaki integral transform, He's polynomial, homotopy perturbation method.

Abstract. In this article, He's variation iteration method (VIM) and Elzaki integral transform are proposed to analyze the time-fractional telegraph equations arising in electromagnetics. The Caputo sense is used to describe fractional derivatives. One of the advantages of this technique is that there is neither need to compute the Lagrange multiplier by calculating the integration in recurrence relations or via taking the convolution theorem. Further, to decrease nonlinear computational terms, the Adomian polynomial is identified with the homotopy perturbation method (HPM). The proposed method is applied to some examples of linear and nonlinear fractional telegraph equations. The solutions obtained by the new computational technique indicate that this method is efficient and facilitates the process of solving time fractional differential equations.

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1. Introduction

Differential equations of fractional orders can be used to simulate many scientific disciplines, which improves our understanding of how to characterize natural occurrences in a variety of scientific disciplines like engineering, electronics, biology, business, computer science, and physics. Indeed, in the improvement of fractional calculus, many scientists such as Bernoulli, Liouville, Euler, L'Hopital, and Wallis greatly contributed to this area of research. The numerical solutions are used to investigate the solutions of differential equations of fractional and integer orders, because the exact solutions of differential equations are quite difficult to be found.

Telegraph equations have applied to many problems in different fields of science which developed by Heaviside in 1880. The difference and time are described on electric transmissions with current and voltage by telegraph equation, also the proposed equation is applied for investigating the wave propagation in the cable transmission and electric signals, and it is also applied in the field of telephone lines, wireless signals, and radio frequency [1]. Telegraph equations of fractional orders have been solved, using various numerical and analytical methods, the Adomian technique [2], homotopy perturbation technique[3], Laplace decomposition combined with HPM [4], modified Adomian decomposition method (MADM) [5], and reduced differential transform technique [6]. The VIM employed to study the solution of the proposed model and obtained the same result as obtained by (ADM) with fewer computations [7], and the hyperbolic telegraph equation is analyzed by Chebyshev tau technique [8].
The researcher Inokuti was the first who study the VIM [9][10], while the Lagrange multiplier was difficult to be identified. Then, variation iteration method developed by Chinese mathematician He [11], and was applied by many researchers, see [12][13][14][15]. The homotopy perturbation method (HPM) is another crucial method which is employed to solve PDEs [16][17][18]. The solution of Voltera-Fredhom is studied by HPM[19], also the hyperbolic PDEs and many other PDEs were solved by HPM, see [20][21][22]. In the last decade, different methods have been developed to analyze the solution of PDEs of fractional orders[23][24]. Recently, Elzaki homotopy transformation perturbation method is employed to solve a class of models such, see [25][26][27]. The Elzaki transform was proposed by the Jordanian mathematician Tarig Elzaki [28], and this transform has been applied on many models to acquire their solution, see [29][30][31][32][33][34][35]. In this paper, the Elzaki transform with a new method of VIM combined with the homotopy perturbation method is utilized to study the solution of time fractional telegraph equation.

The object of the present work is to extend the implementations of EVIM and show the accuracy of the suggested technique. Therefore, the fractional telegraph equation is considered.

Nanoelectromechanical systems are playing an enormous rule in the area of sensing and actuating. However, nonlinearity effects negatively on the Nanoelectromechanical systems devise. The nonlinear vibration systems have complex behaviors that are characterized by noise, instability in response, and bifurcation phenomena. Therefore, controlling the nonlinear vibrations of Nanoelectromechanical systems is essential to obtain stable vibrations.

$$\frac{\partial^{\beta} r}{\partial x^{\beta}} + G\frac{\partial^{\alpha} w}{\partial t^{\alpha}} + Hw = 0 \tag{1}$$

$$\frac{\partial^{\beta} w}{\partial x^{\beta}} + L\frac{\partial^{\alpha} r}{\partial t^{\alpha}} + Rr = 0 \tag{2}$$

Differentiate the equation (1) with respect to t and (2) with respect to x, then solving the system, the following equation is obtained

$$\frac{\partial^{2\beta}w}{\partial x^{2\beta}} + R\left[-G\frac{\partial^{\alpha}w}{\partial t^{\alpha}} - Hw\right] + L\left[-G\frac{\partial^{2\alpha}w}{\partial t^{2\alpha}} - H\frac{\partial^{\alpha}w}{\partial t^{\alpha}}\right] = 0 \tag{3}$$

Assume that $\varepsilon = \frac{R}{L}, \epsilon = \frac{H}{G}, \delta^2 = \frac{1}{LG}$, substituting these values in the equation (3), we obtain equation (4) is telegraph equation which arises in electromagnetic waves.

$$\frac{\partial^{2\alpha}w}{\partial t^{2\alpha}} + (\varepsilon + \epsilon)\frac{\partial^{\alpha}w}{\partial t^{\alpha}} + \varepsilon\epsilon w = \delta^2\frac{\partial^{2\beta}w}{\partial x^{2\beta}} \tag{4}$$

2. Preliminaries

Here, several basic definitions and characteristics of fractional calculus and the proposed transform are given.

Definition2.1.[36] A function $g(y), y > 0$ is considered to be a real valued function and belong to the space $C_\sigma, \sigma \in R$. assume that the real number $d > \sigma$, such that $g(y) = y^d g_1(y)$ where $g_1(y) \in C(0,\infty)$, and it is said to be in the space C_σ^n if $g^n \in R_\sigma, n \in N$.

Definition2.2.[37] The function f(u) is called Riemann-Liouvill fractional integral of order $\alpha > 0$ if it defines as:

$$J^\alpha f(u) = \frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1} f(t)dt \qquad , t > 0.$$

In particular $J^0 f(u) = f(u)$.
For $\theta \geq 0$ and $\vartheta \geq -1$, we have the following properties:

1. $J^\alpha J^\theta f(u) = J^{\alpha+\theta} f(u)$,
2. $J^\alpha J^\theta f(u) = J^\theta J^\alpha f(u)$,
3. $J^\alpha x^\vartheta = \frac{\Gamma(\vartheta+1)}{\Gamma(\alpha+\vartheta+1)} x^{\alpha+\vartheta}$.

Definition2.3.[37] Assume that function $f \in C_{-1}^n$, $n \in N$. The function f is called Caputo fractional derivative and defined by

$$D^\alpha f(u) = \frac{1}{\Gamma(n-\alpha)}\int_0^u (u-t)^{n-\alpha-1} f^n(t)dt, \qquad n-1 < \alpha \leq n.$$

Definition2.4.[28] The function f(u) is called Elzaki-transform if defined as follows:

$$E[f(u)] = T(v) = v\int_0^\infty f(u)e^{\frac{-u}{v}}du \qquad u > 0.$$

Assume that f is piecewise continuous, then Elzaki transform of the Caputo derivative

$$E\left[\frac{\partial^n f(x,t)}{\partial u^n}\right] = \frac{T(x,v)}{v^n} - \sum_{i=0}^{n-1} v^{2-n+i}\frac{\partial^i f(x,0)}{\partial u^i}. \tag{5}$$

The Caputo fractional derivative of Laplace transform is defined as follows

$$L\left(D_x^\alpha g(x,u)\right) = s^\alpha G(s) - \sum_{i=0}^{n-1} s^{\alpha-1-i} g^{(i)}(x,0) \qquad n-1 < \alpha \leq n. \tag{6}$$

Where $G(s)$ represents the Laplace transform of $g(x)$.

Theorem 2.1 Let $B = \{f(x,u)|\ there\ exist\ M, m_1, m_2 > 0\ s.t|f(x,u)| < Me^{\frac{|u|}{m_j}}$, $u \in (-1)^j \times [0,\infty)\}$ and let $f(x,u) \in B$. The Elzak transform $T(v)$ of $f(u)$is

$$T(v) = vG\left(\frac{1}{v}\right),$$

where $G(s)$ is the Laplace transform of $g(x)$.

Theorem 2.2 Assume $T(v)$ is the Elzaki transform of the function $f(x,u)$. Thus

$$E\left(D_u^{\alpha} f(x,u)\right) = \frac{T(v)}{v^{\alpha}} - \sum_{i=0}^{n-1} v^{i-\alpha+2} f^{(i)}(x,0) \qquad n-1 < \alpha \leq n$$

Proof: by Theorem 1 $E\{D^{\alpha} f(x,u), v\} = vL\left\{D^{\alpha} f(x,u), \frac{1}{v}\right\}$.
Using equation (6), we obtain

$$E\{D^{\alpha} f(x,u), v\} = \frac{v}{v^{\alpha}} G\left(\frac{1}{v}\right) - v \sum_{i=0}^{n-1} \left(\frac{1}{v}\right)^{\alpha-i-1} f^{(i)}(x,0)$$

$$\frac{vG\left(\frac{1}{v}\right)}{v^{\alpha}} - \sum_{i=0}^{n-1} v^{i-\alpha+2} f^{(i)}(x,0) \qquad n-1 < \alpha \leq n$$

$$= \frac{T(v)}{v^{\alpha}} - \sum_{i=0}^{n-1} v^{i-\alpha+2} f^{(i)}(x,0).$$

3. Applications of HETM

Recently, the Lagrange multiplier is introduced in new manners [38][39][40]. In this work, the Elzaki transform is used and multiply it by Lagrange multiplier in order to obtain the recurrence relation that is restricted in order to define the Lagrange multiplier. To avoid the convolution terms and integral evaluations, we use this technique. Due to the limitations of Elzaki transform on nonlinear parts, the HPM is used to decrease the computations. The innovative and modified scheme is constructed as follows:

Taking the Elzaki transform of the proposed model and multiplying it via the Lagrange multiplier to obtain the recurrence relation that identifies the Lagrange multiplier, via variation approach. The Adomin polynomial is used to evaluate the nonlinear terms, and then the well-known HPM is used to find the series solution of the proposed problem.

$$Ru - Nu - k = 0.$$

Taking the Elzaki transform, the following relation is obtained

$$E[Ru - Nu - k] = 0.$$

Now, we take the Lagrange multiplier $\mu(v)$,

$$\mu(v)\{E[Ru - Nu - k]\} = 0.$$

Here, we can have the following recurrence relation

$$U_{j+1}(v) = U_j(v) + \mu(v)\{E[Ru - Nu - k]\}. \tag{7}$$

The recurrence relation represents the modified Elzaki variation; we apply the optimal condition using the following relation to introduce the Lagrange multiplier $\mu(v)$

$$\frac{\rho U_{j+1}(v)}{\rho U_j(v)} = 0.$$

Here, the inverse Elzaki transform is applied on (7) to achieve the solution of equation (4)

$$u_{j+1}(v) = u_j(v) + E^{-1}\left[\mu(v)\left\{E\left[Ru_j\right] - E[A_j + k]\right\}\right]. \quad j = 0,1,2,3,\dots$$

where A_j represents the Adomian polynomial as follows:

$$A_j = \frac{1}{j!}\frac{d^j}{d\tau^j}\left(N\left(\sum_{j=0}^{\infty} u_j \tau^j\right)\right). \tag{8}$$

Finally, to investigate the series approximate solution, the HPM is considered by equating the powers of the embedded parameter p.

4. Homotopy perturbation

In this portion, we study the concept of HPM for the solution of our problem.
Consider the following differential equation

$$Ru - Nu = k, \tag{9}$$

where k be a source term, R is linear term and N is the non-linear term, and u the solution function.
By the Homtopy theory $H(w,p)$, $H(w,p): R \times [0,1] \to R$ which satisfies the following:

$$H(w,p) = (1-p)[R(w) - R(w_0)] + p[R(w) - N(w) - k] = 0,$$

Simple calculations, we obtain

$$R(w) - pN(w) = k, \tag{10}$$

The parameter $p \in [0,1]$, u_0 is the initial term of (4), and w is the homotopy function with $R(u_0) = k$.
Since, w can be written as:

$$w = \lim_{p\to 1}(w_0 + pw_1 + p^2 w_2 + \cdots). \tag{11}$$

Using (10) and (11), we have

$$w_0 + pw_1 + p^2 w_2 + p^3 w_3 \ldots = k + pN(w).$$

Equating the powers of p can be written as follows:

p^0: $w_0 = k,$
p^1: $w_1 = N(w_0),$
p^2: $w_2 = w_1 N'(w_0),$
p^3: $w_3 = N'(w_0) + \frac{{w_1}^2 N''(w_0)}{2},$
$\vdots$

Finally, as p approach to 1, the following series solution is the solution of (4):

$$u = w_0 + w_1 + w_2 + w_3 \ldots . \tag{12}$$

Indeed, the convergence of the solution (12) is studied in[41] [42].

5. Applications

The approximate patterns are employed to show the importance of the new method for solving the time fractional telegraph equation. Here, the following models of time fractional differential equations is given:

Example 4.1 Consider the following linear telegraph equation of fractional order

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^{2\alpha} z}{\partial t^{2\alpha}} + 2\frac{\partial^{\alpha} z}{\partial t^{\alpha}} + z, \qquad t \geq 0, 0 < \alpha \leq 1. \tag{13}$$

with the initial conditions $z(x,0) = e^x, z_t(x,0) = -2e^x$. The exact solution of equation (13) is:

$$z(x,t) = e^{x-2t}.$$

Taking the Elzaki transform of equation (13)

$$E\left[\frac{\partial^{2\alpha} z}{\partial t^{2\alpha}} + 2\frac{\partial^{\alpha} z}{\partial t^{\alpha}} + z - \frac{\partial^2 z}{\partial x^2}\right] = 0.$$

Now, we multiply both sides of above equation by $\mu(v)$

$$\mu(v)E\left[\frac{\partial^{2\alpha}z}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}z}{\partial t^{\alpha}}+z-\frac{\partial^{2}z}{\partial x^{2}}\right]=0.$$

The recurrence relation has the following form

$$Z_{j+1}(x,v)=Z_j(x,v)+\mu(v)E\left[\frac{\partial^{2\alpha}z}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}z}{\partial t^{\alpha}}+z-\frac{\partial^{2}z}{\partial x^{2}}\right]. \quad (14)$$

Taking the variation of the above equation and using Elzaki property (5), we obtain

$$\rho Z_{j+1}(x,v)=\rho Z_j(x,v)+\mu(v)\rho\left\{\frac{Z_j(x,v)}{v^{2\alpha}}-v^{2-2\alpha}\hat{Z}_j(x,0)-v^{3-2\alpha}\frac{\partial^{\alpha}\hat{Z}_j(x,0)}{\partial t}+E\left[2\frac{\partial^{\alpha}\hat{z}_j}{\partial t^{\alpha}}+\hat{z}_j-\frac{\partial^{2}\hat{z}_j}{\partial x^{2}}\right]\right\}. \quad (15)$$

Here, $\hat{z}_j=\hat{z}_j(x,0)=\hat{Z}_j(x,0)$ are restricted variables, it means that $\rho\hat{z}_j(x,0)=\rho\hat{Z}_j(x,0)=0$ and since $\frac{Z_{j+1}(x,0)}{Z_j(x,0)}=0$.
Substituting restricted variables in equation (15), gives

$$\rho Z_{j+1}(x,v)=\rho Z_j(x,v)+\frac{1}{v^{2\alpha}}\mu(v)\rho Z_j(x,v).$$

Therefore, the Lagrange multiplier $\mu(v)=-v^{2\alpha}$.
Substituting the Lagrange multiplier in equation (14), we obtain

$$Z_{j+1}(x,v)=Z_j(x,v)-v^{2\alpha}E\left[\frac{\partial^{2\alpha}z_j}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}z_j}{\partial t^{\alpha}}+z-\frac{\partial^{2}z_j}{\partial x^{2}}\right].$$

Applying Elzaki inverse, we have

$$z_{j+1}(x,v)=z_j(x,v)-E^{-1}\left[v^{2\alpha}E\left[\frac{\partial^{2\alpha}z_j(x,t)}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}z_j(x,t)}{\partial t^{\alpha}}+z_j-\frac{\partial^{2}z_j(x,t)}{\partial x^{2}}\right]\right]$$

Since $\frac{\partial^{2\alpha}z_j}{\partial t^{2\alpha}}=0, j=0,1,2,...$, to obtain He's polynomial the homotopy perturbation method is utilized

$$z_0+pz_1+p^2z_2+p^3z_3\dots=z_j(x,t)-pE^{-1}\left[v^{2\alpha}E\left[2\frac{\partial^{\alpha}z_j}{\partial t^{\alpha}}+z_j-\frac{\partial^{2}z_j}{\partial x^{2}}\right]\right]$$

$$=z_j(x,t)-pE^{-1}\left[v^{2\alpha}E\left[\left(2\frac{\partial^{\alpha}z_0}{\partial t^{\alpha}}+z_0-\frac{\partial^{2}z_0}{\partial x^{2}}\right)+p\left(2\frac{\partial^{\alpha}z_1}{\partial t^{\alpha}}+z_1-\frac{\partial^{2}z_1}{\partial x^{2}}\right)+p^2\left(2\frac{\partial^{\alpha}z_2}{\partial t^{\alpha}}+z_2-\frac{\partial^{2}z_2}{\partial x^{2}}\right)+p^3\left(2\frac{\partial^{\alpha}z_3}{\partial t^{\alpha}}+z_3-\frac{\partial^{2}z_3}{\partial x^{2}}\right)\right]\right]$$

Equating the highest powers of p

$$p^0\ :\ z_0=z_0(x,t)+tz_{0_t}(x,t)$$

$$p^1\ :\ z_1=-E^{-1}\left[v^{2\alpha}E\left[2\frac{\partial^{\alpha}z_0}{\partial t^{\alpha}}+z_0-\frac{\partial^{2}z_0}{\partial x^{2}}\right]\right]$$

$$p^2\ :\ z_2=-E^{-1}\left[v^{2\alpha}E\left[2\frac{\partial^{\alpha}z_1}{\partial t^{\alpha}}+z_1-\frac{\partial^{2}z_1}{\partial x^{2}}\right]\right]$$

$$p^3\ :\ z_3=-E^{-1}\left[v^{2\alpha}E\left[2\frac{\partial^{\alpha}z_2}{\partial t^{\alpha}}+z_2-\frac{\partial^{2}z_2}{\partial x^{2}}\right]\right]$$

Therefore, we obtain

$$z_0=e^x-2te^x.$$

Since by Theorem 2, we have

$$z_1 = -E^{-1}[-v^{\alpha+3}4e^x],$$

$$z_1 = \frac{4e^x t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

Similarly,

$$z_2 = \frac{-8e^x t^{2\alpha+1}}{\Gamma(2\alpha+2)}, z_3 = \frac{16e^x t^{3\alpha+1}}{\Gamma(3\alpha+2)}, \cdots.$$

One can expressed these results in a series such as:

$$z = z_0 + z_1 + z_2 + z_3 + \cdots.$$

$$z = e^x - 2te^x + \frac{4e^x t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{8e^x t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{16e^x t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \cdots,$$

when $\alpha = 1$, the HETM solution for equation (13) is

$$z = e^x - 2te^x + \frac{4e^x t^2}{2!} - \frac{8e^x t^3}{3!} + \frac{16e^x t^4}{4!} - \cdots.$$

$$z = e^{x-2t}.$$

Thus, the obtained result using HETM can compute to the exact solution $z = e^{x-2t}$, when $\alpha = 1$.

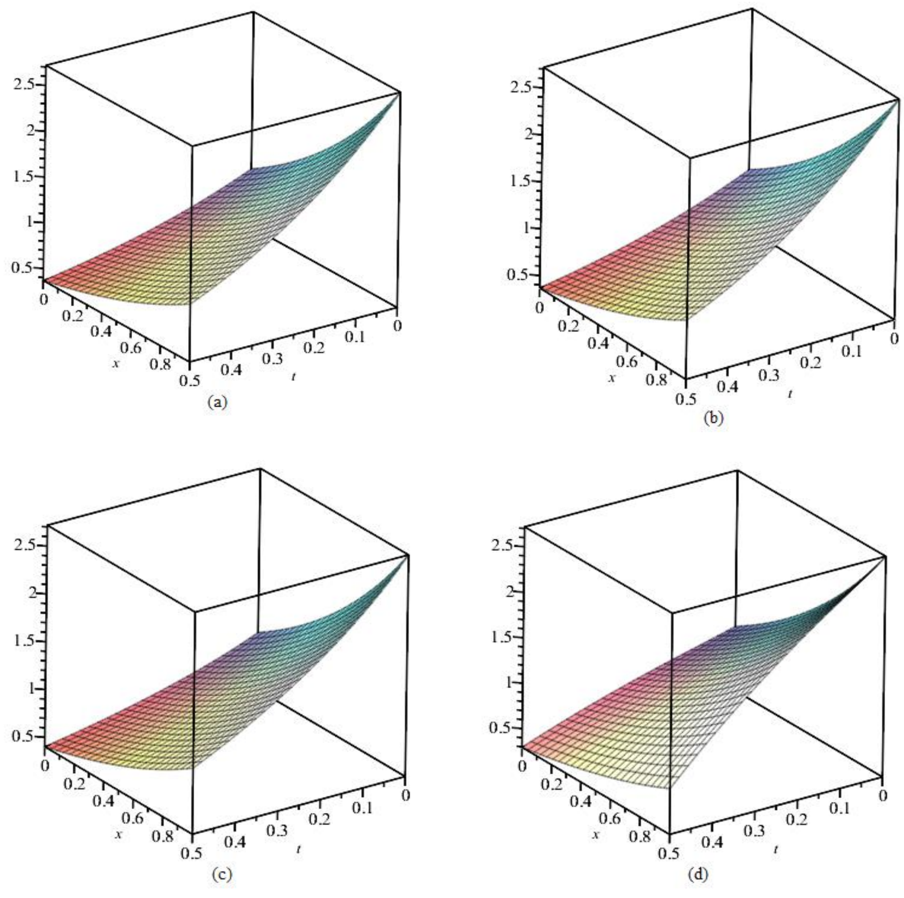

Figure 1. (a) Exact solution and (b) HETM solution of $z(x,t)$ of equation (13) at $\alpha = 1$. The HETM solution of $z(x,t)$ of equation (13) at (c) $\alpha = 0.8$ and (d) $\alpha = 0.2$.

Example 4.2 Consider the following time fractional telegraph equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^{2\alpha} z}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha} z}{\partial t^{\alpha}}+2z, t\geq 0,0<\alpha\leq 1, \tag{16}$$

with the initial conditions $z(x,y,0)=e^{x+y}, z_t(x,y,0)=-3e^{x+y}$. The exact solution of the equation (16) is

$$z(x,y,t)=e^{x+y-3t}.$$

Taking the Elzaki transform of equation (16)

$$E\left[\frac{\partial^{2\alpha} z}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha} z}{\partial t^{\alpha}}+2z-\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right]=0.$$

Now, we multiply both sides of above equation by $\mu(v)$

$$\mu(v)E\left[\frac{\partial^{2\alpha} z}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha} z}{\partial t^{\alpha}}+2z-\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}\right]=0.$$

The recurrence relation has the following form

$$Z_{j+1}(x,v)=Z_j(x,v)+\mu(v)E\left[\frac{\partial^{2\alpha} z_j}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha} z_j}{\partial t^{\alpha}}+2z_j-\frac{\partial^2 z_j}{\partial x^2}-\frac{\partial^2 z_j}{\partial y^2}\right]. \tag{17}$$

Taking the variation of the above equation and using Elzaki property (5), we obtain

$$\rho Z_{j+1}(x,y,v)=\rho Z_j(x,y,v)+\mu(v)\rho\left\{\frac{Z_j(x,y,v)}{v^{2\alpha}}-v^{2-2\alpha}\hat{Z}_j(x,y,0)-v^{3-2\alpha}\frac{\partial^{\alpha}\hat{Z}_j(x,y,0)}{\partial t}+E\left[\frac{\partial^{2\alpha}\hat{z}_j(x,y,0)}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha}\hat{z}_j(x,y,0)}{\partial t^{\alpha}}+2\hat{z}_j(x,y,0)-\frac{\partial^2\hat{z}_j(x,y,0)}{\partial x^2}-\frac{\partial^2\hat{z}_j(x,y,0)}{\partial y^2}\right]\right\}. \tag{18}$$

The variables $\hat{z}_j=\hat{z}_j(x,y,0)=\hat{Z}_j(x,y,0)$ are restricted variable, since $\rho\hat{z}_j(x,y,0)=\rho\hat{Z}_j(x,y,,0)=0$ and $\frac{\hat{Z}_{j+1}(x,y,0)}{\hat{Z}_j(x,y,0)}=0$.

Substituting restricted variables in equation (18), gives

$$\rho Z_{j+1}(x,y,v)=\rho Z_j(x,y,v)+\frac{1}{v^{2\alpha}}\mu(v)\rho Z_j(x,y,v).$$

Therefore, the Lagrange multiplier $\mu(v)=-v^{2\alpha}$.
Substituting the Lagrange multiplier in equation (17), we obtain

$$Z_{j+1}(x,y,v)=Z_j(x,y,v)-v^{2\alpha}E\left[\frac{\partial^{2\alpha} z_j}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha} z_j}{\partial t^{\alpha}}+2z_j-\frac{\partial^2 z_j}{\partial x^2}-\frac{\partial^2 z_j}{\partial y^2}\right].$$

Applying Elzaki inverse, we get

$$z_{j+1}(x,y,v)=z_j(x,y,v)-E^{-1}\left[v^{2\alpha}E\left[\frac{\partial^{2\alpha} z_j}{\partial t^{2\alpha}}+3\frac{\partial^{\alpha} z_j}{\partial t^{\alpha}}+2z_j-\frac{\partial^2 z_j}{\partial x^2}-\frac{\partial^2 z_j}{\partial y^2}\right]\right]$$

Since $\frac{\partial^{2\alpha} z_j}{\partial t^{2\alpha}}=0, j=0,1,2,...$ to obtain He's polynomial, we apply HPM

$$z_0+pz_1+p^2z_2+p^3z_3\,...=z_j(x,y,t)-pE^{-1}\left[v^{2\alpha}E\left[3\frac{\partial^{\alpha} z_j}{\partial t^{\alpha}}+2z_j-\frac{\partial^2 z_j}{\partial x^2}-\frac{\partial^2 z_j}{\partial y^2}\right]\right]$$

$$= z_j(x,y,t) - pE^{-1}\left[v^{2\alpha}E\left[\left(3\frac{\partial^\alpha z_0}{\partial t^\alpha} + 2z_0 - \frac{\partial^2 z_0}{\partial x^2} - \frac{\partial^2 z_0}{\partial y^2}\right)\right.\right.$$
$$+ p\left(3\frac{\partial^\alpha z_1}{\partial t^\alpha} + 2z_1 - \frac{\partial^2 z_1}{\partial x^2} - \frac{\partial^2 z_1}{\partial y^2}\right)$$
$$+ p^2\left(3\frac{\partial^\alpha z_2}{\partial t^\alpha} + 2z_2 - \frac{\partial^2 z_2}{\partial x^2} - \frac{\partial^2 z_2}{\partial y^2}\right)$$
$$\left.\left.+ p^3\left(3\frac{\partial^\alpha z_3}{\partial t^\alpha} + 2z_3 - \frac{\partial^2 z_3}{\partial x^2} - \frac{\partial^2 z_3}{\partial y^2}\right)\right]\right].$$

Equating the highest powers of p

$$p^0 \ : \ z_0 = z_0(x,y,t) + t{z_0}_t(x,y,t)$$
$$p^1 \ : \ z_1 = -E^{-1}\left[v^{2\alpha}E\left[3\frac{\partial^\alpha z_0}{\partial t^\alpha} + 2z_0 - \frac{\partial^2 z_0}{\partial x^2} - \frac{\partial^2 z_0}{\partial y^2}\right]\right]$$
$$p^2 \ : \ z_2 = -E^{-1}\left[v^{2\alpha}E\left[3\frac{\partial^\alpha z_1}{\partial t^\alpha} + 2z_1 - \frac{\partial^2 z_1}{\partial x^2} - \frac{\partial^2 z_1}{\partial y^2}\right]\right]$$
$$p^3 \ : \ z_3 = -E^{-1}\left[v^{2\alpha}E\left[3\frac{\partial^\alpha z_2}{\partial t^\alpha} + 2z_2 - \frac{\partial^2 z_2}{\partial x^2} - \frac{\partial^2 z_2}{\partial y^2}\right]\right]$$
$$p^4 \ : \ z_4 = -E^{-1}\left[v^{2\alpha}E\left[3\frac{\partial^\alpha z_3}{\partial t^\alpha} + 2z_3 - \frac{\partial^2 z_3}{\partial x^2} - \frac{\partial^2 z_3}{\partial y^2}\right]\right].$$

Therefore, we obtain

$$z_0 = e^{x+y} - 3te^{x+y}.$$

Since by Theorem 2.2, we have

$$z_1 = -E^{-1}[-v^{\alpha+3}9e^{x+y}],$$
$$z_1 = \frac{9e^{x+y}t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

Similarly,

$$z_2 = \frac{-27e^{x+y}t^{2\alpha+1}}{\Gamma(2\alpha+2)}, z_3 = \frac{81e^{x+y}t^{3\alpha+1}}{\Gamma(3\alpha+2)}, \ldots$$

Here, the HETM for equation (16) is

$$z(x,y,t) = z_0 + z_1 + z_2 + z_3 + \cdots.$$
$$z(x,y,t) = e^{x+y}\left(1 - 3t + \frac{9t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{27t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{81t^{3\alpha+1}}{\Gamma(3\alpha+2)} - \cdots\right),$$

when $\alpha = 1$, the HETM for equation (16) is

$$z(x,y,t) = e^{x+y} - 3te^{x+y} + \frac{9e^{x+y}t^2}{2!} - \frac{27e^{x+y}t^3}{3!} + \frac{81e^{x+y}t^4}{4!} - \cdots.$$
$$z(x,y,t) = e^{x+y-3t}.$$

Thus, exact solution of model (15) is obtained when $\alpha = 1$.

Example 4.3 Consider the following time fractional telegraph equation

$$\frac{\partial^\alpha z(x,t)}{\partial t^\alpha} - \frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} - z^2(x,t) + xz(x,t)z_x(x,t), t, x \geq 0, 1 < \alpha \leq 2, \tag{19}$$

with the initial terms $z(x,0) = x, z_t(x,0) = x$.

Appling the Elzaki transform of model (19), we obtain

$$E\left[\frac{\partial^\alpha z(x,t)}{\partial t^\alpha}-\frac{\partial^2 z(x,t)}{\partial x^2}-\frac{\partial z(x,t)}{\partial t}+z^2(x,t)-xz(x,t)z_x(x,t)\right]=0.$$

Now, we multiply both sides of above equation by $\mu(v)$

$$\mu(v)E\left[\frac{\partial^\alpha z(x,t)}{\partial t^\alpha}-\frac{\partial^2 z(x,t)}{\partial x^2}-\frac{\partial z(x,t)}{\partial t}+z^2(x,t)-xz(x,t)z_x(x,t)\right]=0.$$

The recurrence relation has the following form

$$Z_{j+1}(x,v)=Z_j(x,v)+\mu(v)E\left[\frac{\partial^\alpha z_j(x,t)}{\partial t^\alpha}-\frac{\partial^2 z_j(x,t)}{\partial x^2}-\frac{\partial z_j(x,t)}{\partial t}+{z_j}^2(x,t)-xz_j(x,t){z_j}_x(x,t)\right]. \quad (20)$$

Taking the variation of the above equation and using Elzaki property (5), we obtain

$$\rho Z_{j+1}(x,v)=\rho Z_j(x,v)+\mu(v)\rho\left\{\frac{Z_j(x,v)}{v^{2\alpha}}-v^{2-2\alpha}\hat{Z}_j(x,0)-v^{3-2\alpha}\frac{\partial^\alpha \hat{Z}_j(x,0)}{\partial t}-E\left[\frac{\partial^2 \hat{z}_j(x,t)}{\partial x^2}+\frac{\partial \hat{z}_j(x,t)}{\partial t}-{\hat{z}_j}^2(x,t)+x\hat{z}_j(x,t){\hat{z}_j}_x(x,t)\right]\right\}. \quad (21)$$

$$\rho Z_{j+1}(x,v)=\rho Z_j(x,v)+\frac{1}{v^{2\alpha}}\mu(v)\rho Z_j(x,v).$$

The variables $\hat{z}_j=\hat{z}_j(x,0)=\hat{Z}_j(x,0)$ are restricted variables, since $\rho\hat{z}_j(x,0)=\rho\hat{Z}_j(x,0)=0$ and $\frac{\rho Z_{j+1}(x,v)}{\rho Z_j(x,v)}=0$.
Therefore, the Lagrange multiplier $\mu(v)=-v^{2\alpha}$.

Substituting the Lagrange multiplier in (20), the following relation is acquired:

$$Z_{j+1}(x,v)=Z_j(x,v)-v^{2\alpha}E\left[\frac{\partial^\alpha z_j(x,t)}{\partial t^\alpha}+\frac{\partial^2 z_j(x,t)}{\partial x^2}+\frac{\partial z_j(x,t)}{\partial t}-{z_j}^2(x,t)+xz_j(x,t){z_j}_x(x,t)\right].$$

Applying Elzaki inverse, we get

$$z_{j+1}(x,v)=z_j(x,v)-E^{-1}\left[v^{2\alpha}E\left[\frac{\partial^\alpha z_j(x,t)}{\partial t^\alpha}+\frac{\partial^2 z_j(x,t)}{\partial x^2}+\frac{\partial z_j(x,t)}{\partial t}-{z_j}^2(x,t)+xz_j(x,t){z_j}_x(x,t)\right]\right].$$

Since $\frac{\partial^\alpha z_j}{\partial t^\alpha}=0, j=0,1,2,3\ldots$ to get He's polynomial, we apply HPM

$$z_0+pz_1+p^2z_2+p^3z_3\ldots=z_j(x,t)-pE^{-1}\left[v^{2\alpha}E\left[\frac{\partial^2 z_j(x,t)}{\partial x^2}+\frac{\partial z_j(x,t)}{\partial t}-A_j+xB_j\right]\right], \quad (22)$$

where A_j and $B_j$are the Adomian polynomials of $(z_0,\ z_1,\ z_2, z_3\ldots)$, we use (8) to calculate the Adomian polynomials:

$$A_0=z_0^2,\qquad B_0=z_0{z_0}_x,$$
$$A_1=2z_0z_1,\qquad B_1=z_0{z_1}_x+{z_0}_xz_1,$$
$$A_2=2z_0z_2+z_1^2,\qquad B_2=z_0{z_2}_x+z_1{z_1}_x+z_2{z_0}_x,$$
$$A_3=2z_0z_3+2z_1z_2,\qquad B_3=z_3{z_0}_x+z_2{z_1}_x+z_1{z_2}_x+z_0{z_3}_x,$$

Table 1: The numerical and exact solutions at various values of α and t for equation (19), where $x=0.5$.

Table 1. demonstrate the comparison of exact and approximate solutions of (19) for different values of t and α using HETM. It is clear that the value $\alpha = 2$ using HETM gives almost the exact solution of model (19).

t	Exact Solution	$\alpha = 2$	$\alpha = 1.90$	$\alpha = 1.80$
0.0	0.5	0.5	0.5	0.5
0.5	0.8243606355	0.8243606355	0.8377559990	0.8531832410
1.0	1.359140914	1.359140914	1.398076434	1.439803212
1.5	2.240844535	2.240844535	2.312171661	2.385449643
2.0	3.694528050	3.694528049	3.803171996	3.911485870

Equating the highest powers of p, and substituting the Adomian polynomials in (22), leads

$$p^0 \ : \ z_0 = z_0(x,t) + t{z_0}_t(x,t)$$

$$p^1 \ : \ z_1 = -E^{-1}\left[v^{2\alpha}E\left[\frac{\partial z_0}{\partial t} + \frac{\partial^2 z_0}{\partial x^2} - z_0^2 + xz_0{z_0}_x\right]\right]$$

$$p^2 \ : \ z_2 = -E^{-1}\left[v^{2\alpha}E\left[\frac{\partial z_1}{\partial t} + \frac{\partial^2 z_1}{\partial x^2} - 2z_0z_1 + x\left(z_0{z_1}_x + {z_0}_x z_1\right)\right]\right]$$

$$p^3 \ : \ z_3 = -E^{-1}\left[v^{2\alpha}E\left[\frac{\partial z_2}{\partial t} + \frac{\partial^2 z_2}{\partial x^2} - 2z_0z_2 - \ z_1^2 + x\left(z_0{z_2}_x + z_1{z_1}_x + z_2{z_0}_x\right)\right]\right]$$

Therefore, we obtain

$z_0 = x(1+t), z_1 = \frac{xt^\alpha}{\Gamma(\alpha+1)}, z_2 = \frac{xt^{\alpha+1}}{\Gamma(\alpha+2)}, z_3 = \frac{xt^{\alpha+2}}{\Gamma(\alpha+3)}, \ldots.$

Here, the HETM solution for equation (19) is

$$z = z_0 + z_1 + z_2 + z_3 + \cdots.$$

$$z = x\left(1 + t + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \cdots\right),$$

when $\alpha = 2$, the HETM solution for equation (19) is

$$z = x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right).$$

$$z = xe^t.$$

Thus, the exact solution of equation (19) is obtained when $\alpha = 2$.

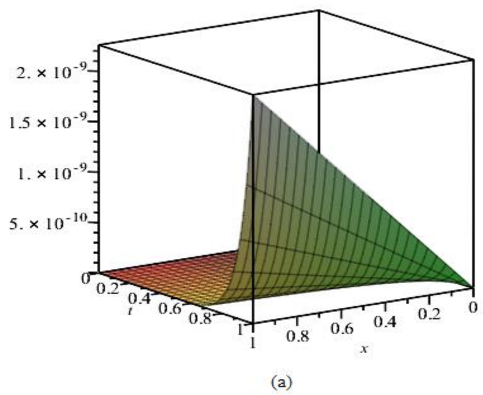

Figure 2. (a) The error plot of $z(x,t)$ of equation (19) at $\alpha = 2$.

In this paper, several models of time fractional telegraph equations are studied using a novel numerical approach. The present outcomes are compared with the analytic solutions via tables and illustrative graphs. Table 1. illustrates the comparison between the approximate solutions acquired via the proposed technique for various orders of fractional derivative α with the exact solution. In Figure 1. graph (a) the exact solution is given, graph (b) the solution of HETM for $\alpha = 2$ is given, and graphs (c) and (d) the solutions of HETM for $\alpha = 0.8$ and $\alpha = 0.2$ are given, respectively. Finally, the error plot of equation (19) is given in Figure 2. As a result, it can be observed that there is an excellent agreement between the present results and the exact solution.

6. Conclusion

In this work, a novel computational method called Elzaki integral transform combined with a new technique of He's variation iteration technique to investigate the solution of linear and nonlinear telegraph equations of fractional orders. The Caputo sense is used to describe the fractional derivatives. This method is implemented on the several models of telegraph equations; the exact and approximate solutions are obtained for each model. The advantage of the proposed technique is that for defining the Lagrange multiplier, there is no need to integration or convolution theorem in recurrence relation. Because of the limitations of Elzaki transform on nonlinear parts, the HPM is used to reduce the computations. Finally, the present results show the accuracy of the novel computational method according to the obtained results. In future, the proposed method can be used to investigate the solutions of the differential equations.

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