Jurnal Matematika MANTIK
Vol. 7, No. 2, October 2021, pp. 140-146

ISSN: 2527-3159 (*print*) 2527-3167 (*online*)

A Comparative Study Between ADM and MDM for a System of Volterra Integral Equation

Sizar Mohammed[1], Hela Shawkat Ahmed[2]

[1,2]College Basic Education, University of Duhok, Iraq

Article history:
Received, Jul 25, 2021
Revised, Oct 19, 2021
Accepted, Oct 31, 2021

Kata Kunci:
Sistem Persamaan Integral Volterra, Adomain decomposition method, Modified decomposition method

Abstrak. Dalam penelitian ini, studi perbandingan antara *Adomain decomposition method* (ADM) dan *Modified decomposition method* (MDM) untuk sistem persamaan integral volterra. Dari contoh ilustrasi terlihat bahwa solusi eksak lebih kecil di kedua metode, metode dekomposisi yang dimodifikasi lebih bagus daripada cara tradisional, metode ini tidak terlalu rumit, membutuhkan lebih sedikit waktu untuk mendapatkan solusi dan yang terpenting solusi eksak dicapai dalam dua iterasi.

Keywords:
System of Volterra Integral Equations, Adomain decomposition method, Modified decomposition method

Abstract. In this paper, a comparative study between Adomain decomposition method (ADM) and Modified decomposition method (MDM) for a system of volterra integral equation. From the illustrate examples it is observed that the exact solution is smaller in both method, the modified decomposition method is more proficient than its traditional ones it is less complicated, needs less time to get to the solution and most importantly the exact solution is achieved in two iterations.

How to cite:
S. Mohammed and H. S. Ahmed, "A Comparative Study Between ADM and MDM for a System of Volterra Integral Equation, *J. Mat. Mantik,* vol. 7*, no. 2, pp. 140-146, October 2021.*

CONTACT:
Sizar Mohammed, *sizar@uod.go* College Basic Education, University of Duhok, Iraq

The article can be accessed here. *https://doi.org/10.15642/mantik.2021.7.2.140-146*

1. Introduction

The subject of integral equations is one of the most useful mathematical tools in both pure and Applied Mathematics, and also they have enormous applications in many physical problems, in engineering, chemistry and biological problems. Many initial and boundary value problems associated with the ordinary and partial differential equations can be transformed in to the integral equations. An integral equation is the equation in which the unknown function $u(x)$ appears inside an integral sign. In this paper we make a comparative study between two methods.

The most standard type of integral equation in $u(x)$ is of the form:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dt,$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $k(x,t)$ is a known function of two variables x and t, called the kernel. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign [3]. It is to be noted that the limits of integrations $g(x)$ and $h(x)$ may be both variables, constants or mixed.

In the past decades, many researchers have studied some numerical and analytical methods for solving different types of integral equations, for example, for example, the author in [1] investigated a numerical solution for Volterra-Fredholm integral equations via least squares method. In [2] the authors obtained the approximate solutions for the linear part of Volterra Integral Equations of Second kind by using two accurate quadrature rules and also In general Wazwaz in [3] explained different types of integral equations.

The systems of linear and nonlinear integral equations have been solved using the Adomian decomposition method in [4]. The authors in [5] are developed a Runge-Kutta method theory for integrated Volterra equations of the second kind. In [6] the author concerning the singular Cauchy kernel were used to find integral equation formulations for the Laplace equation. In [7] the authors extended smooth methods through the use of partitioned quadrature based on the collocation methods, to allow the efficient numerical solution of linear, scalar Volterra integral equations of the second kind with.

The numerical solution of the Volterra integral equations with delay was obtained using Block Methods in [8]. In [9] the author introduced a new approach which is the Galerkin method with Hermite polynomials for estimating the numerical solutions of Volterra's integral equations. In [10] the authors applied the Galerkin Residual Weighted Method to solve Volterra integral equations of the first and second type with normal and single kernel. The author in [11] obtained the numerical solution by aggregation method that was formulated and justified for Fredholm equations of the second type. In reference [12], the authors apply both the aggregation method and Chebyshev polynomials to obtain numerical solutions to the Volterra integral equations. The authors In [13] used a modified trapezoid quadrature method to solve linear integral equations of the second kind. Tahmasbi in [14] is introduced a new approach, namely the power series method for solving Volterra integral equation of the second kind.

The Galerkin weighted residual approximation method was applied to obtain a numerical approach to Volterra's integral equations in [15] and in [16] Wazwaz focused on recent developments in approximate methods for solving linear and nonlinear integral equations with applications. The aim of this study is to solve a system of Volterra integral equation of the second kind by two accurate methods, which are the Adomian decomposition method and the modified decomposition method.

2. Adomain Decomposition Method

The Adomain decomposition method (ADM) was introduced and developed by George Adomain [17-18]. It consists of decomposing the unknown function $u(x)$ of any equation in to a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x)\,, \tag{1}$$

Or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots, \tag{2}$$

The decomposition method is concerned with finding the components $u_0, u_1, u_2, \dots.$ individually. To establish the recurrence relation, we substitute (1) into equation [3] to get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x k(x,t)(\sum_{n=0}^{\infty} u_n(x))dt \tag{3}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= f(x) + \lambda \int_0^x k(x,t)\,[u_0(t) + u_1(t) + u_2(t) + \cdots]dt. \tag{4}$$

The components $u_j(x),\ j \geq 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$u_0(x) = f(x),\ \ u_{n+1}(x) = \lambda \int_0^x k(x,t)u_n(t)dt,\ \ n \geq 0, \tag{5}$$

or equivalently

$$\begin{aligned} u_0(x) &= f(x), \\ u_1(x) &= \lambda \int_0^x k(x,t)u_0\,(t)dt, \\ u_2(x) &= \lambda \int_0^x k(x,t)u_1(t)dt, \\ u_3(x) &= \lambda \int_0^x k(x,t)u_2(t)dt, \end{aligned} \tag{6}$$

and so on for other components. As a result the components $u_1(x),\ u_2(x),\ u_3(x), \dots$ are completely determined, and then the solution $u(x)$ of the Volterra integral equation (6) is readily obtained in a series from by using the series assumption in (1).

The decomposition method converts the integral equation into an elegant determination of components. If an exact solution exists for the problem, then the obtained series converges very rapidly to that exact solution. However, for concrete problems, where a closed from solution is not obtainable. The more components we use the higher accuracy we obtain [16].

Example 1. Consider the Volterra integral equation of the second kind

$$u(x) = 6x - 3x^2 + \int_0^x u(t)dt, \tag{7}$$

Using the Adomain decomposition method, we notice that $f(x) = 6x - 3x^2, \lambda = 1, k(x,t) = 1.$ Recall that the solution $u(x)$ is assumed to have series from given in (1). Substituting the decomposition series (1) into both sides of (7) gives

$$\sum_{n=0}^{\infty} u_n(x) = 6x - 3x^2 + \int_0^x \sum_{n=0}^{\infty} u_n(t)dt,$$

or equivalently

$$u_0(x)+u_1(x)+u_2(x)+\cdots=6x-3x^2+\int_0^x[u_0(t)+u_1(t)+u_2(t)+\cdots]dt.$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x)=6x-3x^2,$$

$$u_{n+1}(x)=\int_0^x u_n(t)dt,\ n\geq 0.$$

So that

$$u_0(x)=6x-3x^2,$$

$$u_1(x)=\int_0^x u_0(t)dt=\int_0^x 6t-3t^2dt=3x^2-x^3,$$

$$u_2(x)=\int_0^x u_1(t)dt=\int_0^x(3t^2-t^3)dt=x^3-\frac{x^4}{4},$$

$$u_3(x)=\int_0^x u_2(t)dt=\int_0^x(t^3-\frac{t^4}{4})dt=\frac{x^4}{4}-\frac{x^5}{20},$$

$$u_4(x)=\int_0^x u_3(t)dt=\int_0^x(\frac{t^4}{4}-\frac{t^5}{20})dt=\frac{t^5}{20}-\frac{t^6}{120},$$

the solution in a series from is given by

$$u(x)=6x-3x^2+3x^2-x^3+x^3-\frac{x^4}{4}+\frac{x^4}{4}-\frac{x^5}{20}+\frac{x^5}{20}-\frac{x^6}{120}+\cdots\ .$$

We can easily notice the appearance of identical of terms with opposite signs this phenomenon of such terms is called noise term phenomenon canceling the identical terms with opposite terms gives the exact solution:

$$u(x)=6x.$$

3. Modified Decomposition Method

As shown before, the Adomain decomposition method provides the solution in an infinite series of components. The components $u_j, j\geq 0$ are easily computed if the inhomogeneous term $f(x)$ in the Volterra integral equation:

$$u(x)=f(x)+\lambda\int_0^x k(x,t)u(t)dt, \tag{8}$$

consists of a polynomial. However, if the function $f(x)$ consists of a combination of two or more of polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components $u_j, j\geq 0$ require more work. A reliable modification of the Adomain decomposition method was developed by Wazwaz [4]. The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution. This will be applied whenever it is appropriate to all integral equations and differential equations of any order. It is important to note that the modified decomposition method relies mainly on splitting the function $f(x)$ into two parts; therefore it can not be used if the function $f(x)$ consists of only one term. To explain this technique, we recall that the standard Adomain decomposition method admits the use of the recurrence relation:

$$u_0(x)=f(x),$$

$$u_{n+1}(x) = \lambda \int_0^x k(x,t) u_n(t) dt, \quad n \geq 0, \tag{9}$$

where the solution $u(x)$ is expressed by an infinite sum of components defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{10}$$

In virtue of (9), the components $u_n, n \geq 0$ can easily be evaluated. The modified decomposition method introduces slight variation to the recurrence relation (9).
That will lead to be determination of the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x) \tag{11}$$

In virtue of (11), we introduce a qualitative change in the formation of the recurrence relation (9). To reduce the calculations, we will introduce of the modified decomposition method into recurrence relation:

$$u_0(x) = f_1(x)$$

$$u_1(x) = f_2(x) + \lambda \int_0^x k(x,t)\, u_0(t) dt, \tag{12}$$

$$u_{n+1}(x) = \lambda \int_0^x k(x,t) u_n(t) dt, \quad n \geq 1.$$

This shows that the formation of the first two components $u_0(x) and\, u_1(x)$ is only the difference between the standard recurrence relation (9) and the modified recurrence relation (12). The others components $u_j, j \geq 2$ remain
The same in the two recurrence relations. This variation in the formation of $u_0(x) and\, u_1(x)$ is important to accelerate the convergence of the solution and in minimizing the size of computational work [3].
Example 2. Consider the Volterra integral equations of the second kind

$$u(x) = 6x - 3x^2 - \int_0^x u(t) dt.$$

Using the modified decomposition method, we first split $f(x)$

$$f(x) = 6x - 3x^2,$$

into two parts, namely

$$f_1(x) = 6x,$$
$$f_2(x) = -3x^2.$$

Next, use the modified recurrence formula (2.12) to obtain

$$u_0(x) = f_1(x) = 6x,$$

$$u_1(x) = 6x - 3x^2 - \int_0^x u_0(t) dt = 0,$$

$$u_{n+1}(x) = -\int_0^x k(x,t) u_n(t) dt = 0, \quad n \geq 1.$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by

$$u(x) = 6x.$$

4. Conclusion

It is clearly seen that the decomposition method converted the integral equation into an elegant determination of computable components. It was formally shown that if an exact solution exists for such problems, then the obtained series converges very rapidly to that exact solution. We here emphasize on the two important remarks, first, by proper selection of the functions $f_1(x)$ and $f_2(x)$, the exact solution solution $u(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the proper choice of $f_1(x)$ and $f_2(x)$, and this can be made through trials only. Second, if $f(x)$ consist of one term only, the modified decomposition method cannot be used in this case.

This confirms our belief that the Adomian decomposition method and the modified decomposition method introduce the solution of volterra integral equation in the form of a rapidly convergent power series with elegantly computable term. However, if $f(x)$ consist of more than one term, the modified decomposition method minimizes the volume of the computational work.

The obtained result showed that the modified decomposition method is more accurate and effective than Adomian decomposition method, needs less time to get to the solution and most importantly the exact solution is achieved in two iterations. The essential condition for that to succeed is that the zeroth component should include the exact solution.

References

[1] S. S. Ahmed, Numerical solution for Volterra-Fredholm integral equation of the second kind by using least squares technique. Iraqi Journal of Science 52, no. 4 (2011): 504-512.

[2] M. U. S. A. Aigo, On the numerical approximation of Volterra integral equations of the second kind using quadrature rules. International Journal of Advanced Scientific and Technical Research 1, no. 3 (2013): 558-564.

[3] A. M. Wazwaz, A First course in Integral Equations, World Scientific Publishing, London,(2015).

[4] J. Biazar and M. Pourabd, A Maple Program for Solving Systems of Linear and Nonlinear Integral Equations by Adomian Decomposition Method. Int. J. Contemp. Math. Sciences 2, no. 29 (2007): 1425-1432.

[5] H. Brunner, E. Hairer and S. P. Njersett, Runge-Kutta theory for Volterra integral equations of the second kind. MATHEMATICS of computation 39, no. 159 (1982): 147-163.

[6] W. Hackbusch, Integral Equations: Theory and Numerical Treatment, Birkhauser Verlag, Basel, (1995).

[7] A. Isaacson and M. Kirby, Numerical solution of linear Volterra integral equations of the second kind with sharp gradients. Journal of computational and applied mathematics 235, no. 14 (2011): 4283-4301.

[8] M. Mustafa, Numerical solution of volterra integral equations with Delay using Block methods. AL-Fatih journal 36 (2008).

[9] M. M. Rahman, Numerical Solution s of Volterra Integral Equations Using Calerkin method with Hernite polynomials, pure and Appl. Math.,(2013).

[10] Md A. Rahman, Md S. Islam, and M. M. Alam, Numerical solutions of Volterra integral equations using Laguerre polynomials. Journal of scientific research 4, no. 2 (2012): 357-357

[11] A. G. Ramm, A collocation method for solving integral equations. International Journal of Computing Science and Mathematics 2, no. 3 (2009): 222-228.

[12] J. Rashidinia, E. Najafi and A. Arzhang, An iterative scheme for numerical solution of Volterra integral equations using collocation method and Chebyshev polynomials, Rashidinia et al. Mathematical Sciences 2012.

[13] J. Saeri-Nadja and M. Heidari, Solving linear integral equations of the second kind with repeated modified trapezoid quadrature method. Appl . Math. Comput., 189 (2007), 980-985.
[14] A. Tahmasbi, A new approach to the numerical solution of linear Volterra integral equations of the second kind. Int. J. Contemp. Math. Sciences 3, no. 32 (2008): 1607-1610.
[15] M. S. Islam and Md. A. Rahman, Solutions of Linear and Nonlinear Volterra Integral Equations Using Hermite and Chebyshev Polynomials. International Journal of Computers & Technology 11 (2013): 2910-2920.
[16] A. M. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications (2011). Higher Education, Springer, Beijing, Berlin.
[17] G. Adomian, Nonlinear stochastic systems and Applications to Physics, Kluwer,(1989).
[18] G. Adomian, G.E. Adomian, A global method for solution of complex systems, Math. Model, 5 (1984) 521-568.