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A New Approximate Birkhoff Orthogonality Type

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Abstract. In this note, we introduce a new approximate Birkhoff orthogonality type and give a characterization for inner product spaces using the approximate orthogonality. We show some general properties of the approximate Birkhoff orthogonality type as well as applications. In particular, we study the relationship between the new approximate Birkhoff orthogonality type and other approximate orthogonality types that have been defined before. Furthermore we study the approximate preserving mapping and give some properties.

1. Introduction

One of the important ideas playing a fundamental role in geometry of normed spaces is the concept of orthogonality. Many mathematicians have introduced different types of orthogonality for the normed linear spaces, cf. [2, 20, 24]. In 1934 [23], the first orthogonality type:Roberts orthogonality was introduced by Roberts. After that in 1935 [5], Birkhoff introduced one of the most important orthogonality types: x is said to be Birhoff orthogonal to y ($x \perp_B y$) if $\|x+ty\| \geq \|x\|$ for all $t \in \mathbb{R}$. Then James in 1945 [15] introduced the Pythagorean orthogonality and isosceles orthogonality: x is said to be isosceles orthogonal to y ($x \perp_I y$) if $\|x+y\| = \|x-y\|$. There are also other orthognality types related to norm limit such as ρ-orthogonality and g-orthogonality [10, 18].

Let X be inner product spaces $(X, \langle\cdot|\cdot\rangle)$, all the orthogonality types are equivalent to $x \perp y$ or equivalently, $\langle x|y\rangle = 0$. In inner product spaces a natural way to generalize orthogonality is to define the approximate orthogonality by: $x \perp^{\epsilon} y$ if and only if $|\langle x|y\rangle| \leq \epsilon\|x\|\|y\|,\ x, y \in X$ [9, 26]. Inspired by the approximate orthogonality, Dragomir [13] gave the definition of the approximate Birkhoff orthogonality $x^{\epsilon} \perp_B y : \|x+ty\| \geq (1-\epsilon)\|x\|$ for all $t \in \mathbb{R}$. It is easy to see that this type of approximate orthogonality is equivalent to $\perp^{\epsilon}$ in inner product spaces [13]. After that Jacek Chmieliński [21] introduced the approximate Birkhoff orthogonality $x \perp_B^{\epsilon} y : \|x+ty\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|ty\|$ for all $t \in \mathbb{R}$, the approximate isosceles orthogonality [11] $x \perp_I^{\epsilon} y : |\|x+y\|^2 -$

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$\|x-y\|^2| \leq 4\epsilon\|x\|\|y\|$ for all $t \in \mathbb{R}$, and $x^\epsilon \perp_I y : |\|x+y\| - \|x-y\|| \leq \epsilon\|x+y\|\|x-y\|$ for all $t \in \mathbb{R}$. Many meaningful results have been found about approximate orthogonality through the tireless efforts of mathematicians, see [12,14].

In this paper we will introduce a new approximate Birkhoff othogonality type and investigate its properties and its relationship with other approximate orthogonality types. Moreover we give a characterization of inner product spaces by approximate orthogonality types and some properties about approximately orthogonality preserving mapping.

Throughout the paper we will only consider normed spaces with dim$X \geq 2$, we use $\langle\cdot|\cdot\rangle$ denoting the inner product and $(\cdot|\cdot)$ denoting the angle between x and y, i,e, in inner product spaces $(x,y) = \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2\|x\|\|y\|}$.

2. Approximate Birkhoff Orthogonality $\perp_{B\epsilon}$

Let $\epsilon \in [0,1)$ and x, y be elements of inner product spaces X, we have the vertical relationship: $x \perp y \iff |\langle x|y\rangle| = 0$. To generalize the orthogonality, it is natural to consider the approximate orthogonality (ϵ-orthogonality: $x \perp^\epsilon y$) defined by:

$$x \perp^\epsilon y \iff |\langle x|y\rangle| \leq \epsilon\|x\|\|y\| \iff |cos(x,y)| \leq \epsilon.$$

Now we consider normed spaces, many mathematicians have introduced different types of orthogonality to represent orthogonality such as Birkhoff orthogonality [5] and Isosceles orthogonality [15]. As an extension for the orthogonality, approximately orthogonality such as approximate Birkhoff orthogonalty [13,21]:

$$x^\epsilon \perp_B y \iff \|x+ty\| \geq (1-\epsilon)\|x\| \quad t \in \mathbb{R}.$$

$$x \perp_B^\epsilon y \iff \|x+ty\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|ty\| \quad t \in \mathbb{R},$$

and approximate isosceles orthogonality [11]:

$$x \perp_I^\epsilon y \iff |\|x+y\|^2 - \|x-y\|^2| \leq 4\epsilon\|x\|\|y\| \quad t \in \mathbb{R}.$$

$$x^\epsilon \perp_I y \iff |\|x+y\| - \|x-y\|| \leq \epsilon\|x+y\|\|x-y\| \quad t \in \mathbb{R},$$

have been defined and studied. Notice that the definition of $^\epsilon\perp_B$ is quadratic while the definition of $\perp_B^\epsilon$ is of first order, we give a new approximate Birkhoff orthogonality type:

$$x \perp_{B\epsilon} y \iff \|x+ty\| \geq \|x\| - \epsilon\|ty\|,$$

which is also of first order but different from $^\epsilon\perp_B$. It is easy to see that the inequality is always correct if $t \geq \frac{\|x\|}{\epsilon\|y\|}$.

Example 2.1. *Let $X = (\mathcal{R}^2, \|\cdot\|_1)$, assume that $x = (1,0),\ y = (z, 1-z),\ z \in [0,1)$. If we want $x^\epsilon \perp_B y$, then the inequality $\|x+ty\| \geq (1-\epsilon)\|x\|$ should hold for all $t \in \mathbb{R}$, thus we have:*

$$\|(1+tz, t(1-z))\| \geq 1-\epsilon \quad t \in \mathbb{R}.$$

If $t \geq 0$, the inequality is always correct. For $t < 0$, if $1 + tz \geq 0$, we have:

$$1 + tz - t + tz \geq 1 - \epsilon.$$

thus $z \leq \frac{1}{2} - \frac{\epsilon}{2t}$. By $1 + tz \geq 0$, we get $z \leq \frac{1}{2-\epsilon}$. If $1 + tz < 0$, similarly we need $t \leq \epsilon - 2$. Since $z \leq \frac{1}{2-\epsilon}$, from $1 + tz < 0$, we get $t \leq \epsilon - 2$. Thus $x^{\epsilon} \perp_B y$ iff $z \leq \frac{1}{2-\epsilon}$.

On the other hand, if we want $x \perp_{B\epsilon} y$, the inequality $\|x + ty\| \geq \|x\| - \epsilon\|ty\|$ should hold for all $t \in \mathbb{R}$, thus we have:

$$\|(1 + tz, t(1 - z))\| \geq 1 - \epsilon|t| \quad t \in \mathbb{R}.$$

If $t \geq 0$, the inequality is also always correct. For $t < 0$, if $1 + tz \geq 0$, we have:

$$1 + tz - t + tz \geq 1 + \epsilon t.$$

Thus $z \leq \frac{1+\epsilon}{2}$. Similarly we can get $\|(1 + tz, t(1 - z))\| \geq 1 - \epsilon|t|$ for $1 + tz < 0$ if $z \leq \frac{1+\epsilon}{2}$.

Thus $x \perp_{B\epsilon} y$ iff $z \leq \frac{1+\epsilon}{2}$. We have the result that $\perp_{B\epsilon}$ is not always equivalent to ${}^{\epsilon}\perp_B$ in X.

Since the definition of approximate Birkhoff orthogonality comes from the notion of approximate orthogonality $\perp^{\epsilon}$ in inner product spaces, it is natural to require the equivalence: $x \perp_{B\epsilon} y \iff x \perp^{\epsilon} y$ in inner product spaces. Now we give some basic properties about $\perp_{B\epsilon}$ before prove the equivalence.

Proposition 2.2. *Let X be normed spaces, then $\perp_{B\epsilon}$ is homogeneous., this is*

$$x \perp_{B\epsilon} y \text{ implies } \alpha x \perp_{B\epsilon} \beta y \quad (x, y \in X, \alpha, \beta \in \mathbb{R}).$$

Proof. *Since $x \perp_{B\epsilon} y$, we have $\|x + ty\| \geq \|x\| - \epsilon\|ty\|$ for any $t \in \mathbb{R}$. If $\alpha = 0$,$\alpha x \perp_{B\epsilon} \beta y$ is always correct; if $\alpha \neq 0$, we have*

$$\|\alpha x + t\beta y\| = |\alpha|\|x + \frac{\beta}{\alpha} ty\| \geq |\alpha|\{\|x\| - \epsilon\|\frac{\beta}{\alpha} ty\|\} = \|ax\| - \epsilon\|t\beta y\|.$$

Thus $\alpha x \perp_{B\epsilon} \beta y$. □

Recall that the limits [16] :

$$N_{\pm}(x; y) = \lim_{n \to \pm\infty} \|nx + y\| - |nx| = \lim_{h \to 0^{\pm}} \frac{\|x + hy\| - \|x\|}{h},$$

exist and satisfy the weakened linearity condition [4]. x, y are said to be Gateaux differentiable [1] at 0 $if N_{-}(x, y) = N_{+}(x, y)$. Moreover we have [16]:

$$N_{\pm}(x; rx + sy) = r\|x\| + s \cdot N_{\pm}(x; y), \quad for\ s \geq 0 \text{ and all } r.$$

We then give a characterization of $x \perp_{B\epsilon} y$ using the definition of $N_{\pm}(x, y)$.

Proposition 2.3. *Let X be normed spaces, then*

$$x \perp_{B\epsilon} y \text{ if and only if } N_+(x,y) + \epsilon\|y\| \geq 0 \geq N_-(x,y) - \epsilon\|y\|.$$

Proof. *Let $x \perp_{B\epsilon} y$ and $t \in \mathbb{R}\backslash\{0\}$ then*

$$\frac{\|x+ty\| - \|x\|}{|t|} \geq \epsilon\|y\|.$$

Let $t \to 0^+$, we have $N_+(x,y) \geq -\epsilon\|y\|$. Similarly, let $t \to 0^-$, we have $N_-(x,y) \leq \epsilon\|y\|$. To sum up, $N_+(x,y) + \epsilon\|y\| \geq 0 \geq N_-(x,y) - \epsilon\|y\|$.

Conversely, if $N_+(x,y) \geq -\epsilon\|y\|$, for $\forall \eta > 0$, there $\exists\delta$ such that if $0 < t \leq \delta$, we have:

$$\frac{\|x+ty\| - \|x\|}{t} \geq -(\epsilon+\eta)\|y\|,$$

or equivalently

$$\|x+ty\| - \|x\| \geq -t(\epsilon+\eta)\|y\| \ for\ t \in (0,\delta].$$

Because of the convexity of $\|x+ty\|$, we have

$$\|x+ty\| - \|x\| \geq -t(\epsilon+\eta)\|y\| \ for\ t > 0.$$

Let $\delta \to 0$, we have

$$\|x+ty\| - \|x\| \geq -\epsilon\|ty\| \ for\ t > 0.$$

Similarly, using $N_-(x,y) \leq \epsilon\|y\|$, we have:$\|x+ty\| - \|x\| \geq t(\epsilon)\|y\| \ for\ t < 0$.
If $t = 0$, $\|x+ty\| - \|x\| \geq \epsilon\|ty\|$ is obvious. To conclude, we have:

$$\|x+ty\| - \|x\| \geq \epsilon\|ty\| \quad for\ t \in \mathbb{R}.$$

Thus $x \perp_{B\epsilon} y$. □

To verify the validity of the new approximate Birkhoff orthogonality, we have the following proposition:

Proposition 2.4. *Let X be normed spaces, we have:*

$$x \perp_{B\epsilon} y \text{ if and only if } x \perp^{\epsilon} y.$$

Proof. *Since $x \perp_{B\epsilon} y$, for $0 < t \leq \frac{\|x\|}{\epsilon\|y\|}$ we have*

$$\|x+ty\| \geq \|x\| - \epsilon\|ty\|.$$

Square both sides we get

$$\|x\|^2 + t^2\|y\|^2 + 2t(x,y) \geq \|x\|^2 + \epsilon^2 t^2\|y\|^2 - 2\epsilon\|x\|\|ty\|.$$

Thus $(1-\epsilon^2)t\|y\| \geq -2\|x\|(\epsilon + cos(x,y))$ When t tends to 0, $(1-\epsilon^2)t\|y\|$ tends to 0, so we have

$$\epsilon + cos(x,y) \geq 0 \iff cos(x,y) \geq -\epsilon.$$

Similarly for $0 > t \geq -\frac{\|x\|}{\epsilon\|y\|}$, we have $cos(x,y) \leq \epsilon$, thus $x \perp^{\epsilon} y$.

Conversely, if $|cos(x, y)| \leq \epsilon$*, we have*

$$\|x+ty\| - \|x\| \geq \epsilon\|ty\| \quad for\ |t| \in [0, \frac{\|x\|}{\epsilon\|y\|}].$$

On the other hand, $\|x+ty\| - \|x\| \geq \epsilon\|ty\|$ *is always correct for* $|t| \geq \frac{\|x\|}{\epsilon\|y\|}$*. To conclude,*

$$\|x+ty\| - \|x\| \geq \epsilon\|ty\| \quad for\ t \in \mathbb{R}.$$

Thus $x \perp_{B\epsilon} y$. □

From $N_{\pm}(x; rx+sy) = r\|x\| + s \cdot N_{\pm}(x; y), \quad for\ s \geq 0\ and\ all\ r$, we have the following:

Proposition 2.5. *In the normed space X, if* $x \perp_{B\epsilon} y$*, then we have* $x \perp_{B\epsilon} rx+sy$ *for* $s \geq 0,\ r$ *satisfying*

$$\epsilon(\|rx+sy\| - s\|y\|) \geq r\|x\| \geq \epsilon(s\|y\| - \|rx+sy\|).$$

Proof. *Since* $x \perp_{B\epsilon} y$*, we have* $N_+(x, y) \geq -\epsilon\|y\|$ *and* $N_-(x, y) \leq \epsilon\|y\|$*, so*

$$N_+(x, rx+sy) \geq r\|x\| + (-s\epsilon\|y\|) \geq -\epsilon\|rx+sy\|.$$

Similarly we have $N_-(x, rx+sy) \leq \epsilon\|rx+sy\|$*, thus* $x \perp_{B\epsilon} rx+sy$. □

Let X be normed spaces, it is known that [16] for any $x, y \in X$ there exists a real number a such that $x \perp_B ax+y$, moreover, such a number satisfies $|a| \leq \frac{\|y\|}{\|x\|}$. On this basis, Chmieliński [9] discovered that $x \perp_B^{\epsilon} y$ if and only if there exists a real number $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_B ax+y$. In fact, in inner product spaces, it is easy to see that $x \perp^{\epsilon} y$ if and only if there exists $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_B ax+y$ by taking $a = -\frac{\langle x|y\rangle}{\|x\|^2}x+y$ for $x \neq 0$.

In the following we will prove that it is also true for $\perp_{B\epsilon}$, that is, in normed spaces,

$$x \perp_{B\epsilon} y \text{ if and only if } there\ exists\ |a| \leq \frac{\|y\|}{\|x\|}\epsilon\ such\ that\ x \perp_B ax+y.$$

Before the proof, we need some lemma.

Lemma 2.6. *[16] Let X be normed spaces,*

$$N_-(x, y) \leq N_+(x, y).$$

Lemma 2.7. *[16] Let X be normed spaces,* $a \leq b,\ a,\ b \in X$*, if* $x \perp_B ax+y, x \perp_B bx+y$*, then*

$$x \perp_B cx+y \quad for\ c \in [a, b].$$

Lemma 2.8. *[16] Let X be normed spaces,*

$$x \perp_B ax+y \iff N_-(x, y) \leq -a\|x\| \leq N_+(x, y).$$

Theorem 2.9. *Let X be normed spaces, $x \perp_{B\epsilon} y$ if and only if there exists $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_B ax + y$.*

Proof. *If $x \perp_{B\epsilon} y$, first we have*

$$N_-(x,y) \leq \epsilon\|y\| \ , \ N_+(x,y) \geq -\epsilon\|y\|.$$

On the other hand, from Lemma 2.8, we have if

$$-\frac{N_+(x,y)}{\|x\|} \leq a \leq -\frac{N_-(x,y)}{\|x\|},$$

then $x \perp_B ax + y$. If there exists no $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_B ax + y$, then

$$-\frac{N_+(x,y)}{\|x\|} > \frac{\|y\|}{\|x\|}\epsilon \ or \ -\frac{N_-(x,y)}{\|x\|} < -\frac{\|y\|}{\|x\|}\epsilon.$$

Thus

$$N_+(x,y) < \|y\| \ or \ N_(x,y) > \|y\|\epsilon.$$

Contradict to $x \perp_{B\epsilon} y$, so there must exists $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$, such that $x \perp_B ax + y$.

Conversely, if there exists $|a| \leq \frac{\epsilon\|y\|}{\|x\|}$ such that $x \perp_B ax + y$, we have:

$$\begin{aligned} x \perp_B ax + y &\implies -N_+(x,y) \leq a\|x\| \leq -N_-(x,y) \\ &\implies N_+(x,y) + \epsilon\|y\| \geq 0 \geq N_-(x,y) - \epsilon\|y\| \\ &\implies x \perp_{B\epsilon} y. \end{aligned}$$

To conclude, in normed spaces, $x \perp_{B\epsilon} y$ if and only if there exists $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_B ax+y$.
□

Since both $x \perp_{B\epsilon} y$ and $x \perp_{B\epsilon} y$ are equivalent to there exists $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_B ax+y$. we have $x \perp_{B\epsilon} y$ if and only if $x \perp_B^\epsilon y$ in normed spaces. Now we give a direct proof for this.

Theorem 2.10. *Let X be normed spaces,*

$$x \perp_{B\epsilon} y \text{ if and only if } x \perp_B^\epsilon y.$$

Proof. *We can assume that $|t| \leq \frac{\|x\|}{\epsilon\|y\|}$ and $x \neq 0$. If $x \perp_{B\epsilon} y$, we have:*

$$\|x + ty\| \geq \|x\| - \epsilon\|ty\| \geq 0.$$

Take square on both sides, we get

$$\|x + ty\|^2 \geq \|x\|^2 - \epsilon^2\|ty\|^2 - 2\epsilon\|x\|\|ty\|.$$

Thus $\|x + ty\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|ty\|$, which means that $x \perp_B^\epsilon y$.

Conversely, if $x \perp_B^\epsilon y$, we have

$$\|x + ty\|^2 \geq \|x\|^2 - 2\epsilon\|x\|\|ty\| \geq 0.$$

Let both sides be divided by $\|x+ty\|+\|x\|$, *we get:*

$$\|x+ty\|-\|x\| \geq \frac{-2\epsilon\|x\|\|ty\|}{\|x+ty\|+\|x\|}.$$

Since $\|x+ty\|$ *tends to* $\|x\|$ *when* $t \to 0$, *for every* $2\|x\| > \delta > 0$, *we can find* $\eta > 0$, *such that*

$$\|x+ty\|+\|x\| \geq 2\|x\|-\delta \quad if \ |t| \leq \eta.$$

We then have

$$\|x+ty\|-\|x\| \geq \frac{-2\|x\|\|ty\|}{2\|x\|-\delta}.$$

Let $\delta \to 0$ *we have*

$$\|x+ty\|-\|x\| \geq \frac{-2\|x\|\|ty\|}{2\|x\|} \quad when \ t \to 0.$$

Thus

$$N_+(x,y)+\epsilon\|y\| \geq 0 \geq N_-(x,y)-\epsilon\|y\| \implies x \perp_{B\epsilon} y.$$

□

Recall that Dragomir gave the following definition about approximate Birkhoff orthogonality:

$$x^{\epsilon} \perp_B y \iff \|x+ty\| \geq (1-\epsilon)\|x\|.$$

It is known that [19] in normed spaces, $x \perp_B^{\epsilon} y$ implies $x^{\delta} \perp_B$, *where* $\delta = 1-\sqrt{1-4\epsilon}$. Now we give a more accurate estimate of δ as an application of the above Proposition.

Proposition 2.11. *Let* X *be normed spaces, let* $x, y \in X$, *then:*

$$x \perp_B^{\epsilon} y \implies x^{\delta} \perp_B y \ where \ \delta = 2\epsilon.$$

Proof. *Let* $f(t) = \|x+ty\|$ *and assume that* $f(t)$ *attains its minimum at* t_0, *hence*

$$\|x+t_0y+ty\| \geq \|x+t_0y\| \quad for \ all \ t \in \mathcal{R}.$$

Choose $t=-t_0$ *we have*

$$\|x\| \geq \|x+t_0y\| \geq |\|x\|-|t_0|\|y\||,$$

thus we get $|t_0| \leq \frac{2\|x\|}{\|y\|}$, *then*

$$\|x+ty\| \geq \|x+t_0y\| \geq \|x\|-\epsilon|t_0|\|y\| \geq (1-2\epsilon)\|x\| \quad for \ all \ t \in \mathcal{R}.$$

Thus $x^{\delta} \perp_B y$, *where* $\delta = 2\epsilon$. *By the equivalence between* $\perp_{B\epsilon}$ *and* $\perp_B^{\epsilon}$, *we have the result that* $x \perp_B^{\epsilon} y$ *implies* $x^{\delta} \perp_B y$. *Since* $2\epsilon \leq 1-\sqrt{1-4\epsilon}$, 2ϵ *can be seen as a more accurate estimate.* □

3. Approximate Isosceles Orthogonality And Approximate Birkhoff Orthogonality

In the following we will use the notion of approximate isosceles orthogonality [11], recall that the approximate isosceles orthogonality is defined by:

$$x \perp_I^{\epsilon} y : |\|x+y\|^2 - \|x-y\|^2| \leq 4\epsilon\|x\|\|y\|.$$

$$x^{\epsilon} \perp_I y : |\|x+y\| - \|x-y\|| \leq \epsilon(\|x+y\| + \|x-y\|).$$

It is easy to see that in inner product spaces we have:

$$x \perp_I^{\epsilon} y \iff |cos(x,y)| \leq \epsilon \iff x \perp_{B\epsilon} y,$$

and [11]

$$x^{\epsilon} \perp_I y \iff |cos(x,y)| \leq \frac{\epsilon}{1+\epsilon^2}(\|x\|^2 + \|y\|^2).$$

In the following we give some simple properties about approxiamte isosceles orthogonality.

Proposition 3.1. *Let X be normed spaces, if there exists $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ such that $x \perp_I ax + y$, then $x^{\epsilon} \perp_I y$.*

Proof. *Since $x \perp_I ax + y$ we have $\|x + ax + y\| = \|x - ax - y\|$, then*

$$|\|x+y\| - \|x-y\|| = |\|x+ax+y-ax\| - \|x-ax-y+ax\||.$$

On the other hand, by trigonometric inequality we have:

$$\|x+ax+y\| - \|ax\| - (\|x-ax-y\| + \|ax\|)$$

$$\leq \|x+ax+y-ax\| - \|x-ax-y+ax\|,$$

and

$$\|x+ax+y-ax\| - \|x-ax-y+ax\|$$

$$\leq \|x+ax+y\| + \|ax\| - (\|x-ax-y\| - \|ax\|).$$

Thus

$$|\|x+ax+y-ax\| - \|x-ax-y+ax\|| \leq 2\|ax\|,$$

then

$$|\|x+y\| - \|x-y\|| \leq 2\|ax\| \leq \epsilon(\|x+y\| + \|x-y\|),$$

thus $x^{\epsilon} \perp_I y$.

Proposition 3.2. *Let X be normed spaces, if for every $\|x\| = \|y\| = 1$, there is no $0 \leq \epsilon < 1$ such that $x \perp_I^{\epsilon} y$, then X is a strictly convex space.*

Proof. *For any $\|x\| = \|y\| = \frac{\|x+y\|}{2} = 1$, if $x \neq y$,we have*

$$|\|x+y\|^2 - \|x-y\|^2| = |4 - \|x-y\|| < 4,$$

thus there must exist a $0 \leq \epsilon < 1$ *such that* $|4 - \|x - y\|^2| \leq 4\epsilon$ *which means that* $x \perp_I^\epsilon y$, *contradict to the condition. so there must be* $x = y$. *From the equivalent characterization of strictly convex space [23]. we get the result that* X *must be a strictly convex space.*

It is known that in inner product spaces, different orthogonality types such as isosceles, pythagorean, and Birkhoff orthogonality is equivalent [3]. Using the notions of orthogonality in normed linear spaces it is possible to give different characterizations for inner product spaces. For instance [17], if $x \perp_I y \implies x \perp_B y$ in a normed space X, then X must be inner product spaces. Inspired by this, now we give a characterization for inner product spaces using approximate orthogonality.

Theorem 3.3. *Let* X *be normed spaces, then* X *is inner product spaces iff the following two conditions are satisfied.*
(1) *If there exists* $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ *such that* $x \perp_I ax + y$, *then* $x \perp_I^\epsilon y$.
(2) $x \perp_I^\epsilon y$ *implies*$x \perp_{B\epsilon} y$.

Proof. *If* X *is an inner product space, we have*

$$x \perp_I y \iff x \perp_B y \ \ and \ \ x \perp_I^\epsilon y \iff x \perp_{B\epsilon} y.$$

Thus (2) *is satisfied. If there exists* $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ *such that* $x \perp_I ax + y$, *we have:*

$$x \perp_B ax + y, \ |a| \leq \frac{\|y\|}{\|x\|}\epsilon.$$

Thus $x \perp_{B\epsilon} y$ *which implies* $x \perp_I^\epsilon y$. *Thus both* (1) *and* (2) *are satisfied.*
Conversely, assume that both (1) *and* (2) *are satisfied, let* $x \perp_I y, \ x \neq 0$. *If* $|a| \leq \epsilon\frac{\|ax+y\|}{\|x\|}$, *let* $b = -a$, *then* $|b| \leq \epsilon\frac{\|ax+y\|}{\|x\|}$ *and* $x \perp_I bx + ax + y$, *thus form* (1) *we have* $x \perp_I^\epsilon ax + y$. *To conclude we have:*

$$x \perp_I^\epsilon ax + y \ if \ |a| \leq \epsilon\frac{\|ax + y\|}{\|x\|}.$$

Now define: $f(t) = \frac{\|tx+y\|}{\|x\|}\epsilon$, *we have*

$$f(t) = \frac{\|tx + y\|}{\|x\|}\epsilon \leq \frac{\|tx\| + \|y\|}{\|x\|}\epsilon = \epsilon|t| + \frac{\|y\|}{\|x\|}\epsilon.$$

Since $0 \leq \epsilon < 1$, *when* $|t|$ *tends to infinite,* $f(t) < |t|$.
When $t = 0, \ f(0) = \frac{\|y\|}{\|x\|}\epsilon > 0$. *By the convexity of* $f(t)$ *we have*

$$x \perp_I^\epsilon t_1x + y, \quad t_1 < 0, \frac{\|t_1x + y\|}{\|x\|}\epsilon = -t_1.$$

$$x \perp_I^\epsilon t_2x + y, \quad t_2 > 0, \frac{\|t_2x + y\|}{\|x\|}\epsilon = t_2.$$

From (2) *we have* $x \perp_{B\epsilon} t_1x + y$ *and* $x \perp_{B\epsilon} t_2x + y$. *from proposition 2.9,there must exist*

$$|a_1| \leq \frac{\|t_1x + y\|}{\|x\|}\epsilon = -t_1 \ and \ |a_2| \leq \frac{\|t_2x + y\|}{\|x\|}\epsilon = t_2$$

such that $x \perp_B a_1 + t_1 + y,\ x \perp_B a_2 + t_2 + y$.
By $a_1 + t_1 \leq 0$, $a_2 + t_2 \geq 0$ *and lemma 2.7, we have* $x \perp_B y$. *Thus we have*

$$x \perp_I y \implies x \perp_B y,$$

which means that X *is inner product spaces.*

Example 3.4. *Let* $X = (\mathcal{R}^2, \|\cdot\|_\infty)$, *that is,* $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$, *assume that* $x = (1, 0), y = (z, 1), |z| < 1$. *In order to satisfy* $x \perp_{B\epsilon} y$ *or equivalently* $\|x + ty\| \geq \|x\| - \epsilon\|ty\|$, *the following inequality shuold hold for all* $t \in \mathbb{R}$:

$$\|(1 + tc, t)\| \geq 1 - \epsilon|t|.$$

Since $\|(1 + tc, t)\| \geq \|t\|$, *we know the above inequality is always correct if* $|t| \geq \frac{1}{1+\epsilon}$, *then we may assume that* $|t| < \frac{1}{1+\epsilon} \leq 1$.
when $1 > t \geq 0$, *If* $1 + tc \geq t$ *which means that* $t \leq \frac{1}{1-c}$, *we have*

$$1 + tc \geq 1 - \epsilon t$$

which implies that $c \geq -\epsilon$. *If* $1 + tc < t$ *or equivalently* $t > \frac{1}{1-c}$, *we have* $t \geq 1 - \epsilon t$ *that is* $t \geq \frac{1}{1+\epsilon}$. *so there must be*

$$\frac{1}{1-c} \geq \frac{1}{1+\epsilon}$$

which implies that $c \geq -\epsilon$. *Similarly when* $-1 < t \leq 0$, *we can get* $c \leq \epsilon$. *To conclude we have:* $(1, 0) \perp_{B\epsilon} (c, 1)$ *if* $|c| \leq \epsilon$.
In order to satisfy $x \perp_{I\epsilon} y$ *or equivalently* $|\|x+y\|^2 - \|x-y\|^2| \leq 4\epsilon\|x\|\|y\|$, *the following inequality shuold hold:*

$$|\|(1 + c, 1)\|^2 - \|(1 - c, -1)\|^2| \leq 4\epsilon.$$

If $c \geq 0$, *we have* $(1 + c)^2 - 1 \leq 4\epsilon \implies 0 \leq c < -1 + \sqrt{1 + 4\epsilon}$.
If $c < 0$, *we have* $(1 - c)^2 - 1 \leq 4\epsilon \implies 1 - \sqrt{1 + 4\epsilon} \leq c < 0$.
To conclude we have $(1, 0) \perp_I^\epsilon (c, 1)$ *if* $|c| \leq -1 + \sqrt{1 + 4\epsilon}$. *Since* $\epsilon \leq -1 + \sqrt{1 + 4\epsilon}$, *it can be seen as an example that* $x \perp_I^\epsilon y$ *does not imply* $x \perp_{B\epsilon} y$.

Example 3.5. *Let* $X = (\mathcal{R}^2, \|\cdot\|_\infty)$, *we assume that* $x = (1, 1), y = (-1 - \frac{\sqrt{2}}{2}\epsilon, 1 - \frac{\sqrt{2}}{2}\epsilon)$, $z = (-1, 1)$. *We have* $x \perp_I z$, *and*

$$z = -\frac{\epsilon}{\sqrt{2}}x + y,\ \epsilon \leq \frac{\|y\|}{\|x\|}\epsilon.$$

Thus $|-\frac{\epsilon}{\sqrt{2}}| \leq \frac{\|y\|}{\|x\|}\epsilon$, *which implies that there exists* $|a| \leq \frac{\|y\|}{\|x\|}\epsilon$ *such that* $x \perp_I ax + y$.
On the other hand, since

$$\|x + y\| = \|(-\frac{\sqrt{2}}{2}\epsilon, 2 - \frac{\sqrt{2}}{2}\epsilon)\| = 2 - \frac{\sqrt{2}}{2}\epsilon.$$

$$\|x - y\| = \|(2 + \frac{\sqrt{2}}{2}\epsilon, \frac{\sqrt{2}}{2}\epsilon)\| = 2 + \frac{\sqrt{2}}{2}\epsilon.$$

we have $|\|x+y\|^2 - \|x-y\|^2| = 4\sqrt{2}\epsilon \geq 4\sqrt{2}$. *so* $x \not\perp_I^\epsilon y$, *thus it can be seen as an example that condition* (1) *is not satisfied.*

A mapping $T : \mathbb{H} \to \mathbb{K}$ which satisfies the condtion

$$x \perp y \implies T(x) \perp T(y).$$

is called orthogonality preserving(o.p.) [7,8], and T is said to be an isometry maping [22] if $\|Tx\| = \|x\|$. To promote the concept, Jacek Chmielinski [6] introduced the notion of approximately orthogonality preserving $(a.o.p.)$ mapping and have studied the properties of mapping that is approximarely isosceles orthogonality preserving(T: $x \perp_I y \implies T(x) \perp_I^\epsilon T(y)$). After that many mathematicians have show great interest in the $a.o.p$ mapping [25], and Aleksej Turnšek [19] studied the mapping that is approximately Birkhoff orthogonality preserving($T : \ x \perp_B y \implies T(x) \perp_B^\epsilon T(x)$)in normed spaces. Now we try to study the approximarely orthogonality preserving mapping types:

$$T : \ x \perp_I y \implies Tx \perp_B^\epsilon Ty,$$

and

$$T : \ x \perp_I y \implies Tx \;{}^\epsilon x \perp_B Ty.$$

Proposition 3.6. *Let* $T : \ X \to Y$ *be a nontrivial linear mapping satisfying*

$$x \perp_I y \implies Tx \perp_B^\epsilon Ty, \quad x, y \in X.$$

Then T *is a bounded and bounded from below,* $\|Tx\| \geq \frac{(1-\epsilon)^2}{3-\epsilon^2+2\sqrt{2-\epsilon^2}}$.

Proof. *Take two arbitrary unit vectors* x *and* y *and note that* $\frac{x+y}{2} \perp_I \frac{x-y}{2}$, *it follows that*

$$T(x+y) \perp_B^\epsilon T(x-y),$$

hence for all $\lambda \in \mathbb{R}$ *we have*

$$\|T(x+y) + \lambda T(x-y)\|^2 \geq \|T(x+y)\|^2 - 2\epsilon\|T(x+y)\|\|\lambda T(x-y)\|,$$

by the triangle inequality and the linearity of T *it follows that*

$$\|T(x+y)\|^2 \leq \|(1+\lambda)Tx + (1-\lambda)Ty\|^2 + 2\epsilon|\lambda|\|Tx + Ty\|^2.$$

On the other hand we have

$$\|T(x+y)\|^2 \geq (\|Tx\| - \|Ty\|)^2,$$

thus we get:

$$\|Tx\|^2+\|Ty\|^2-2\|Tx\|\|Ty\| \leq (1+\lambda)^2\|Tx\|^2+(1-\lambda)^2\|Ty\|^2+2(1-\lambda^2)\|Tx\|\|Ty\|+2\epsilon|\lambda|\|Tx+Ty\|^2.$$

If $Tx = 0$, *then let* $y \in X$ *such that* $Ty \neq 0$, *substitute* x, y *into the above formula,we have*

$$0 \leq \lambda^2 - 2\lambda + 2\epsilon|\lambda| \quad for\ all\ \lambda \in \mathbb{R}.$$

It is impossible cause $0 \leq \epsilon < 0$*, so we can divide both sides of the inequality by* $\|Tx\|^2$ *and denote* $z = \frac{\|Ty\|}{\|Tx\|}$*. we get:*

$$(z-1)^2\lambda^2 + \big(2 - 2z^2 + 2\epsilon(1+z)^2\big)\lambda + 4z \geq 0 \quad for\ all\ \lambda \geq 0.$$

the inequality is satisfied when $-\frac{b}{2a} \leq 0$ *or* $\Delta = b^2 - 4ac \leq 0$.

$$\Delta = 4\big((1-z^2)^2 + \epsilon^2(1+z)^4 + 2\epsilon(1+z)^2(1-z^2) - (4z)(z-1)^2\big).$$

$$(1-z^2)^2 + \epsilon^2(1+z)^4 + 2\epsilon(1+z)^2(1-z^2) - (4z)(z+1)^2 \leq 0 \implies \Delta \leq 0.$$

Thus we get $z \leq \frac{3-\epsilon^2+2\sqrt{2-\epsilon^2}}{(1-\epsilon)^2}$ *which implies* $\|Ty\| \leq \frac{3-\epsilon^2+2\sqrt{2-\epsilon^2}}{(1-\epsilon)^2}\|Tx\|$*. Since* x, y *are arbitrary,* T *is bounded and*

$$\|Tx\| \geq \frac{(1-\epsilon)^2}{3-\epsilon^2+2\sqrt{2-\epsilon^2}}\|T\|\|x\|.$$

Theorem 3.7. *Let* $T : X \to Y$ *be a nontrivial linear mapping satisfying*

$$x \perp_I y \implies Tx \ {}^{\epsilon}\!\perp_B Ty, \quad x, y \in X.$$

Then T *is a scalar mutiple of a isometric mapping, i.e., for some* $\gamma > 0$, $\|Tx\| = \gamma\|x\|$.

Proof. *Take two arbitrary unit vectors* x *and* y *and note that* $\frac{x+y}{2} \perp_I \frac{x-y}{2}$*, it follows that*

$$T(x+y)^{\epsilon} \perp_B T(x-y).$$

Hence for all $\lambda \in \mathbb{R}$ *we have*

$$\|T(x+y) + \lambda T(x-y)\| \geq (1-\epsilon)\|T(x+y)\|,$$

thus

$$\big(\|(1+\lambda)Tx\| + \|(1-\lambda)Ty\|\big)^2 \geq (1-\epsilon)^2\|Tx+Ty\|^2 \geq (1-\epsilon)^2(\|Tx\| - \|Ty\|)^2.$$

If $Tx = 0$*, then let* $y \in X$ *such that* $Ty \neq 0$*, substitute* x, y *into the above formula, we have*

$$\|(1-\lambda)Ty\|^2 \geq (1-\epsilon)^2\|Ty\|^2 \implies (1-\lambda)^2 \geq (1-\epsilon)^2 \quad for\ all\ \lambda \in \mathbb{R},$$

it is impossible, then we can divide both sides by $\|Tx\|^2$ *like before and denote* $z = \frac{\|Ty\|}{\|Tx\|}$*, we get*

$$(1-z)^2\lambda^2 + 2(1-z^2)\lambda + (z+1)^2 - (1-\epsilon)^2(1-z)^2 \geq 0 \quad for\ all\ \lambda \in \mathbb{R}.$$

If $z \neq 1$*, then* $\Delta = 4 \cdot (1-z)^4(1-\epsilon)^2 > 0$*. Thus the inequality is satisfied only when* $z = 1$*, so* $z = 1$ *which means that* $\|Tx\| = \|Ty\|$*, thus* T *must be a scalar mutiple of a isometric mapping.*

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