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Quasi-likelihood Estimation in Fractional Levy SPDEs from Poisson Sampling

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Abstract. We study the quasi-likelihood estimator of the drift parameter in the stochastic partial differential equations driven by a cylindrical fractional Levy process when the process is observed at the arrival times of a Poisson process. We use a two stage estimation procedure. We first estimate the intensity of the Poisson process. Then we plug-in this estimate in the quasi-likelihood to estimate the drift parameter. We obtain the strong consistency and the asymptotic normality of the estimators.

1. Introduction

Parameter estimation in infinite dimensional stochastic differential equations was first studied by Loges [20]. When the length of the observation time becomes large, he obtained consistency and asymptotic normality of the maximum likelihood estimator (MLE) of a real valued drift parameter in a Hilbert space valued SDE. Koski and Loges [18] extended the work of Loges [20] to minimum contrast estimators. Koski and Loges [17] applied the work to a stochastic heat flow problem. See the monograph Bishwal [5] for asymptotic results on likelihood inference and Bayesian inference for drift estimation of finite and infinite dimensional stochastic differential equations.

Huebner, Khasminskii and Rozovskii [12] started statistical investigation in SPDEs. They gave two contrast examples of parabolic SPDEs in one of which they obtained consistency, asymptotic normality and asymptotic efficiency of the MLE as noise intensity decreases to zero under the condition of absolute continuity of measures generated by the process for different parameters (the situation is similar to the classical finite dimensional case) and in the other they obtained these properties as the finite dimensional projection becomes large under the condition of singularity of the measures generated by the process for different parameters. The second example was extended by Huebner and Rozovskii [13] and the first example was extended by Huebner [11] to MLE for general parabolic SPDEs where the partial differential operators commute and satisfy different order conditions in the two cases.

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Huebner [10] extended the problem to the ML estimation of multidimensional parameter. Lototsky and Rozovskii [21] studied the same problem without the commutativity condition. Small noise asymptotics of the nonparametric estimation of the drift coefficient was studies by Ibragimov and Khasminskii [14].

Bishwal [3] proved the Bernstein-von Mises theorem (BVT) and obtained asymptotic properties of regular Bayes estimator of the drift parameter in a Hilbert space valued SDE when the corresponding ergodic diffusion process is observed continuously over a time interval $[0, T]$. The asymptotics are studied as $T \to \infty$ under the condition of absolute continuity of measures generated by the process. Results are illustrated for the example of an SPDE.

Bishwal [4] obtained BVT and spectral asymptotics of Bayes estimators for parabolic SPDEs when the number of Fourier coefficients becomes large. In that case, the measures generated by the process for different parameters are singular. Here we treat the case when the measures generated by the process for different parameters are absolutely continuous under some conditions on the order of the partial differential operators. Bishwal [9] studied the asymptotic properties of the posterior distributions and Bayes estimators when one has either fully observed process or finite-dimensional projections. The asymptotic parameter is only the intensity of noise. In this paper we treat the more general model with non-Gaussian noise with long memory.

On the other hand, recently long memory processes, i.e. processes with slowly decaying autocorrelation and processes with jumps have received attention in finance, engineering and physics. The simplest continuous time long memory process is the fractional Brownian motion discovered by Kolmogorov [15] and later on studied by Levy [19] and Mandelbrot and van Ness [27]. Continuous time long memory jump process is fractional Levy process. Hence fractional Levy process can also be called the *Kolmogorov-Levy process.*

We generalize fractional SPDE process to include non-normal innovations. We consider Hurst parameter greater than half. This model is interesting as it preserves both jumps and long memory.

A normalized fractional Brownian motion $\{W_t^H, t \geq 0\}$ with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process with continuous sample paths whose covariance kernel is given by

$$E(W_t^H W_s^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad s, t \geq 0.$$

The process is self similar (scale invariant) and it can be represented as a stochastic integral with respect to standard Brownian motion. For $H = \frac{1}{2}$, the process is a standard Brownian motion. For $H \neq \frac{1}{2}$, the fBm is not a semimartingale and not a Markov process, but a Dirichlet process. The increments of the fBm are negatively correlated for $H < \frac{1}{2}$ and positively correlated for for $H < \frac{1}{2}$ and in this case they display long-range dependence. The parameter H which is also called the self similarity parameter, measures the intensity of the long range dependence. The ARIMA(p, d, q) with autoregressive part of order p, moving average part of order q and fractional difference parameter $d \in (0,\ 0.5)$ process converge in Donsker sense to fBm. See Mishura [22].

The fractional Levy Ornstein-Uhlenbeck (fOU) process, is an extension of fractional Ornstein-Uhlenbeck process with fractional Levy motion (fLM) driving term. In finance, it could be useful as a generalization of fractional Vasicek model, as one-factor short-term interest rate model which could take into account the long memory effect and jump of the interest rate. The model parameter is usually unknown and must be estimated from data.

Fractional Levy Process (FLP) is defined as

$$M_{H,t} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{\mathbb{R}}[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}]dM_s, \quad t \in \mathbb{R}$$

where $\{M_t, t \in \mathbb{R}\}$ is a Levy process on $\mathbb{R}$ with $E(M_1) = 0$, $E(M_1^2) < \infty$ and without Brownian component.

Here are some properties of the fractional Levy process:

1) the covariance of the process is given by

$$\text{cov}(M_{H,t}, M_{H,s}) = \frac{E(M_1^2)}{2\Gamma(2H+1)\sin(\pi H)}[|t|^{2H} + |s|^{2H} - |t-s|^{2H}].$$

2) M_H is not a martingale. For a large class of Levy processes, M_H is neither a semimartingale.

3)M_H is Hölder continuous of any order β less than $H - \frac{1}{2}$.

4) M_H has stationary increments.

5) M_H is symmetric.

6) M is self-similar, but M_H is not self-similar.

7) M_H has infinite total variation on compacts.

Thus FLP is a generalization and a natural counterpart of FBM. Fractional stable motion is a special case of FLP. First we discuss estimation in partially observed models and then we discuss estimation in directly observed model in finite dimensional set up. In finance, the log-volatility process can be modeled as a fractionally integrated moving average (FIMA) process which is defined as

$$Y_H(t) = \int_{-\infty}^{t} g_H(t-u)dM_u, \ t \in \mathbb{R}$$

where

$$g_H(t) = \frac{1}{\Gamma(H-\frac{1}{2})}\int_0^t g(t-s)s^{H-\frac{3}{2}}ds, \ t \in \mathbb{R}$$

which is the Riemann-Liouville fractional integral of order H and the kernel g is the kernel of a short memory moving average process. The log-volatility process will have slow (hyperbolic rate) decay of the auto-correlation function (acf).

The process $Y_H(t)$ can be written as

$$Y_H(t) = \int_{-\infty}^{t} g(t-u)dM_{H,u}, \ t \in \mathbb{R}.$$

We assume the following conditions on the kernel $g : \mathbb{R} \to \mathbb{R}$, namely 1) $g(t) = 0$ for all $t < 0$ (causality), 2) $|g(t)| \leq Ce^{-ct}$ for some constants $C > 0$ and $c > 0$ (short memory).

The FIMA process is stationary and is infinite divisible. It has long memory and jumps which agree empirically with stochastic volatility models. The asset return can be modeled as a COGARCH process

$$dX(t) = \sqrt{e^{Y_H(t)}}dL_t$$

where $(L_t, t \in \mathbb{R}$ is another Levy process and the initial value $Y_H(0)$ is independent of L.

Consider the kernel

$$g(t-s) = \sigma e^{-\theta(t-s)} I_{(0,\infty)}(t-s), \theta > 0$$

then

$$g_H(t) = \frac{\sigma}{\Gamma(H-\frac{1}{2})} \int_0^\infty e^{\theta(t-s)} I_{(0,\infty)}(t-s) s^{H-\frac{3}{2}} ds, \ \ t \in \mathbb{R}.$$

Note that

$$U_t^{H,\theta,\sigma} = \int_{\mathbb{R}} g_H(t-u) dM_u, \ \ t \in \mathbb{R}$$

is the fractional Levy Ornstein-Uhlenbeck (FLOU) process satisfying the fractional Langevin equation

$$dU_t = -\theta U_t dt + \sigma dM_{H,t}, \ \ t \in \mathbb{R}.$$

The process has long memory. Levy driven processes of Ornstein-Uhlenbeck type have been extensively studied over the last few years and widely used in finance, see Barndorff-Neilsen and Shephard [1]. FLOU process generalizes FOU process to include jumps. Maximum quasi-likelihood estimation in fractional Levy stochastic volatility model was studied in Bishwal [6]. Berry-Esseen inequalities for the discretely observed Ornstein-Uhlenbeck-Gamma process was studied in Bishwal [7]. Minimum contrast estimation in fractional Ornstein-Uhlenbeck process based on both continuous and discrete observations was studied in Bishwal [8].

Consider the asset return driven by fractional Levy process

$$dS_{H,t} = \sigma_{t-} dL_{H,t}, \ \ t > 0, \ \ S_0 = 0,$$

with log-volatility

$$\log \sigma_t^2 = \mu + X_t, \ \ t \geq 0$$

where the Levy driven OU process X satisfies

$$dX_t = -\theta X_t dt + dM_t, \ \ t > 0$$

with $\theta \in \mathbb{R}^+$ and the driving compound Poisson process M is a Levy process with Levy symbol

$$\psi_M(u) = -\frac{u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1) \Phi_{0,1/\lambda}(dx),$$

where $\Phi_{0,1/\lambda}$ being a normal distribution with mean 0 and variance $1/\lambda$. This means that M is the sum of a standard Brownian motion W and a compound Poisson process $J_t = \sum_{k=1}^{N_t} Z_k, \ \ J_{-t} = \sum_{k=1}^{-N_{-t}} Z_{-k}, \ \ t \geq 0$ where $(N_t, t \in \mathbb{R})$ is an independent Poisson process with intensity $\lambda > 0$ and jump times $(t_k)_{k\in\mathbb{Z}}$, i.e., $M_t = W_t + J_t$. The Poisson process N is also independent from the i.i.d. sequence of jump sizes $(Z_k)_{k\in\mathbb{Z}}$ with $Z_1 \sim N(0, 1/\lambda)$. The Levy process M in this case is given by

$$M_t = \sum_{k=1}^{N_t}(\alpha Z_k + \gamma|Z_k|) - Ct, \ \ t > 0 \ \ \text{and} \quad C := \gamma \int_{\mathbb{R}} |x|\lambda\Phi_{0,1/\lambda}(dx) = \sqrt{\frac{2\lambda}{\pi}}\gamma.$$

$\{M_{-t}, t \geq 0\}$ is defined analogously. The stationary log-volatility is given by

$$\log\sigma_t^2 = \mu + \int_{-\infty}^{t} e^{-\theta(t-s)}dM_s.$$

We observe S at n consecutive jump times $0 = t_0 < t_1 < \ldots < t_n < T < t_{n+1}$, $n \in \mathbb{Z}$ over the time interval $[0, T]$. The state process X has then the following autoregressive representation

$$\begin{aligned} X_{t_i} &= e^{-\theta\Delta t_i}X_{t_{i-1}} + \sum_{k=N_{t_{i-1}}+1}^{N_{t_i}} e^{-\theta(t_i-t_k)}[\alpha Z_k + \gamma|Z_k|] - \int_{t_{i-1}}^{t_i} e^{-\theta(t_i-s)}Cds \\ &= e^{-\theta\Delta t_i}X_{t_{i-1}} + \alpha Z_i + \left(|Z_i| - \frac{C}{\theta}(1-e^{-\theta\Delta t_i})\right) \end{aligned}$$

where $\Delta t_i := t_i - t_{i-1}$, $i = 1, 2, \ldots, n$ and $N_{t_{i-1}} + 1 = N_{t_i} = i$.

We do the parameter estimation in two steps. The rate λ of the Poisson process N can be estimated given the jump times t_i, therefore it is done at a first step. Since we observe total number of jumps n of the Poisson process N over the T intervals of length one, the MLE of λ is given by $\hat{\lambda}_n := \frac{n}{T}$.

To estimate the remaining parameters (α, θ, μ), we use the quasi maximum likelihood estimation procedure in conditionally heteroscedastic time series models developed by Straumann [25].

Assuming that $S^{\Delta t_i}_{H,t_i}$ given $S^{\Delta t_{i-1}}_{H,t_{i-1}}, \ldots, S^{\Delta t_1}_{H,t_1}, X_0$ is conditionally normally distributed with mean zero and variance $\sigma^2_{t_i-}/\lambda$, the conditional log-likelihood given the initial value X_0 has the representation

$$\mathcal{L}(\vartheta|S^{\Delta}_H, \lambda) := -\frac{n}{2}\log(2\pi) - \frac{1}{2}\left(\sum_{i=1}^{n}\log(\sigma^2_{t_i-}/\lambda) - \sum_{i=1}^{n}\frac{(S^{\Delta t_i}_{H,t_i})^2}{\sigma^2_{t_i-}/\lambda}\right).$$

where $S^{\Delta t_i}_{H,t_i} = S_{H,t_i} - S_{H,t_{i-1}}$ is the return at time t_i. Since the volatility is unobservable, this log-likelihood can not be evaluated numerically. The quasi log-likelihood function for $\vartheta = (\theta, \alpha, \gamma, \mu)$ given the data $S^{\Delta}_H := (S^{\Delta t_1}_{H,t_1}, S^{\Delta t_2}_{H,t_2}, \ldots, S^{\Delta t_n}_{H,t_n})$ and the MLE $\hat{\lambda}_n$ is defined as

$$\mathcal{L}(\vartheta|S^{\Delta}_H, \hat{\lambda}_n) := -\frac{1}{2}\sum_{i=1}^{n}\log(\hat{\sigma}^2_{H,t_i}(\vartheta, \hat{\lambda}_n)) - \frac{1}{2}\sum_{i=1}^{n}\frac{(S^{\Delta t_i}_{H,t_i})^2}{\hat{\sigma}^2_{H,T_i}(\vartheta, \hat{\lambda}_n)/\hat{\lambda}_n}$$

where the estimates of the volatility σ^2_{H,t_i}, $i = 1, 2, \ldots, n$ are given by

$$\hat{\sigma}^2_{H,t_i}(\vartheta, \lambda_n) := \exp(\mu + e^{-\alpha\Delta T_i}X_{H,t_{i-1}}(\vartheta, \lambda) - \hat{C}\Delta t_i), ß = 1, 2, \ldots, n$$

and given the parameters ϑ and λ the estimates of the state process X are given by the recursion

$$\hat{X}_{H,t_i} = e^{-\theta\Delta t_i}\hat{X}_{H,t_{i-1}} + \alpha\frac{S_{H,t_i}}{\hat{\sigma}_{t_i}(\vartheta, \lambda)} + \left(\frac{S_{H,t_i}}{\hat{\sigma}_{t_i}(\vartheta, \lambda)} - \hat{C}\Delta t_i\right), i = 1, 2, \ldots, n$$

Note that $E(|W|)=\sqrt{\frac{2}{\pi\lambda}},\ W\sim N(0,1/\lambda)$.

Here the approximation $(1-e^{-z})\approx z$ for small z is used and $\frac{S_{H,t_i}}{\hat{\sigma}_{t_i}(\vartheta,\lambda)}$ approximates the innovation Z_i. The recursion needs a starting value $\hat{X}_{H,0}$ which will be set equal to the mean value of the stationary distribution of x which is zero. the mean value zero of the stationary distribution of X.

QMLE of ϑ is defined as

$$\hat{\vartheta}_n := \arg\max_{\vartheta\in\Theta}\mathcal{L}(\vartheta|S_H^\Delta,\hat{\lambda}_n).$$

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$ be the stochastic basis on which is defined the Ornstein-Uhlenbeck process X_t satisfying the Itô stochastic differential equation

$$dX_t=-\theta X_t dt+dM_t^H,\quad t\geq 0,$$

where $\{M_t^H\}$ is a fractional Levy motion with $H>1/2$ with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\theta\in\mathbb{R}^+$ is the unknown parameter to be estimated on the basis of completely directly observed continuous observation of the process $\{X_t\}$ on the time interval $[0,T]$. Observe that

$$X_t=\int_{-\infty}^t e^{-\theta(t-s)}dM_s^H.$$

This process is stationary and is a process with long memory. It can be shown that X_{t_i} is a stationary discrete time AR(1) process with autoregression coefficient $\phi\in(0,1)$ with the following representation

$$X_{t_i}=\phi X_{t_{i-1}}+\epsilon_{t_{i-1}}$$

where

$$\phi=e^{-\theta\Delta}\quad\text{and}\quad \epsilon_{t_{i-1}}=\int_{t_{i-1}}^{t_i}e^{-\theta(t_i-u)}dM_u^H.$$

Then the problem is a AR(1) estimation with non-Gaussian non-martingale error. For equidistant sampling, one can study the least squares estimator which boils down to the study of error distribution for non-semimartingales. One can specialize to the case when M is a either a gamma process or an inverse Gaussian process in order to have infinite number of jumps in a finite time interval unlike the compound Poissoan case which have finite number of jumps in a finite time interval. These fractional Gamma and fractional inverse Gaussian Ornstein-Uhlenbeck (FLOU) processes are LOU processes which include long memory. In the next section we deal with completely observed process.

The rest of the paper is organized as follows : Section 2 contains model, assumptions and preliminaries. Section 3 contains the asymptotic properties of quasi likelihood estimator.

2. FLSPDE Model and Preliminaries

In order to introduce fractional Levy stochastic partial differential equation (FLSODE) we proceed as follows. Let us fix θ_0, the unknown true value of the parameter θ. Let $(\Omega,\mathcal{F},P)$ be a complete probability space and $W(t,x)$ be a process on this space with values in the Schwarz space of distributions $D'(G)$ such that for $\phi,\psi \in C_0^\infty(G), \|\phi\|^{-1}_{L^2(G)}$ $\langle W(t,\cdot),\phi(\cdot)\rangle$ is a one dimensional Wiener process and

$$E(\langle W(s,\cdot),\phi(\cdot)\rangle\langle W(t,\cdot),\psi(\cdot)\rangle) = (s\wedge t)(\phi,\psi)_{L^2(G)}.$$

This process is usually referred to as the cylindrical Brownian motion (C.B.M.).

We assume that there exists a complete orthonormal system $\{h_i\}_{i=1}^\infty$ in $L_2(G))$ such that for every $i=1,2,\dots, h_i \in W_0^{m,2}(G)\cap C^\infty(\overline{G})$ and

$$\Lambda_\theta h_i = \beta_i(\theta)h_i, \text{ and } \mathcal{L}_\theta h_i = \mu_i(\theta)h_i \text{ for all } \theta\in\Theta$$

where $\mathcal{L}_\theta$ is a closed self adjoint extension of A^θ, $\Lambda_\theta := (k(\theta)I-\mathcal{L}_\theta)^{1/2m}$, $k(\theta)$ is a constant and and the spectrum of the operator Λ_θ consists of eigen values $\{\beta_i(\theta)\}_{i=1}^\infty$ of finite multiplicities and $\mu_i = -\beta_i^{2m}+k(\theta)$.

CFLP $M_H(t)$ can be expanded in the series

$$M_H(t,x) = \sum_{i=1}^\infty M_{H,i}(t)h_i(x)$$

where $\{M_{H,i}(t)\}_{i=1}^\infty$ are independent one dimensional FLPs, see Peszat and Zabczyk [24]. The latter series converges P-a.s. in $H^{-\nu}$ for $\nu > d/2$. Indeed

$$\|M_H(t)\|^2_{-\nu} = \sum_{i=1}^\infty M^2_{H,i}(t)\|h_i\|^2_{-\nu} = \sum_{i=1}^\infty M^2_{H,i}(t)\beta_i^{-2\nu}$$

and the later series converges P-a.s.

Consider the parabolic SPDE

$$du^\theta(t,x) = \theta u^\theta(t,x) + \frac{\partial^2}{\partial x^2}u^\theta(t,x)dt + dM_H(t,x),\ \ t\ge 0,\ \ x\in[0,1] \tag{2.1}$$

$$u(0,x) = u_0(x)\in L_2([0,1]) \tag{2.2}$$

$$u^\theta(t,0) = u^\theta(t,1),\ \ t\in\ [0,T], \tag{2.3}$$

Here $\theta\in\Theta\subseteq\mathbb{R}$ is the unknown parameter to be estimated on the basis of the observations of the field $u^\theta(t,x), t\ge 0,\ \ x\in[0,1]$. For $x\in[0,1]$, we observe the process $\{u_t, t\ge 0\}$ at times $\{t_0,t_1,t_2,....\}$. We assume that the sampling instants $\{t_i, i=0,1,2...\}$ are generated by a Poisson process on $[0,\infty)$, i.e., $t_0=0, t_i = t_{i-1}+\xi_i,\ \ i=1,2,...$ where ξ_i are i.i.d. positive random variables with a common exponential distribution $F(x) = 1-\exp(-\lambda x)$. Note that intensity parameter $\lambda>0$ is the average sampling rate which is assumed to be known. It is also assumed that the sampling process $t_i, i=0,1,2,...$ is independent of the observation process $\{X_t, t\ge 0\}$. We note that the probability density function of $t_{k+i}-t_k$ is independent of k and is given by the gamma density

$$f_i(t) = \lambda(\lambda t)^{i-1}\exp(-\lambda t)I_t/(i-1)!,\ \ i=0,1,2,.... \tag{2.4}$$

where $I_t = 1$ if $t\ge 0$ and $I_t = 0$ if $t<0$.

Consider the Fourier expansion of the process

$$u(t,x) = \sum_{t=1}^{\infty} u_i(t)\phi_i(x) \tag{2.5}$$

corresponding to some orthogonal basis $\{\phi_i(x)\}_{i=1}^{\infty}$. Note that the Fourier coefficients $\{u_i^{\theta}(t), i \geq 1\}$ are independent one dimensional Ornstein-Uhlenbeck processes

$$du_i^{\theta}(t) = \mu_i^{\theta} u_i^{\theta}(t)dt + \beta_i^{-\nu} dM_{H,i}(t) \tag{2.6}$$

$$u_i^{\theta}(0) = u_{0i}^{\theta},$$

Recall that $\mu_i(\theta) = k(\theta) - \beta_i^{2m}$. Thus

$$du_i^{\theta}(t) = (k(\theta) - \beta_i^{2m})u_i^{\theta}(t)dt + \beta_i^{-\nu} dM_{H,i}(t) \tag{2.7}$$

The random field $u(t,x)$ is observed at discrete times t and discrete positions x. Equivalently, the Fourier coefficients $u_i^{\theta}(t)$ are observed at discrete time points.

Now we focus on the fundamental semimartingale behind the O-U model. Define

$$\begin{aligned}
\kappa_H &:= 2H\Gamma(3/2-H)\Gamma(H+1/2), \\
k_H(t,s) &:= \kappa_H^{-1}(s(t-s))^{\frac{1}{2}-H}, \\
\eta_H &:= \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(3/2-H)}, \\
v_t \equiv v_t^H &:= \eta_H^{-1} t^{2-2H}, \\
\mathcal{M}_t^H &:= \int_0^t k_H(t,s)dM_s^H.
\end{aligned}$$

For using Girsanov theorem for Brownian motion, since a Radon-Nikodym derivative process is always a martingale, a central problem is how to construct an appropriate martingale which generates the same filtration, up to sets of measure zero, as the non-semimartingale called the *fundamental martingale*.

Extending Norros *et al.* [23] it can be shown that $\mathcal{M}_t^H$ is a martingale, called the fundamental martingale whose quadratic variation $\langle \mathcal{M}^H \rangle_t$ is v_t^H. Moreover, the natural filtration of the martingale $\mathcal{M}^H$ coincides with the natural filtration of the FLP M^H since

$$M_t^H := \int_0^t K(t,s)d\mathcal{M}_s^H$$

holds for $H \in (1/2, 1)$ where

$$K_H(t,s) := H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, \quad 0 \leq s \leq t$$

and for $H = 1/2$, the convention $K_{1/2} \equiv 1$ is used.

Define

$$Q_i(t) := \frac{d}{dv_t}\int_0^t k_H(t,s)u_i(s)ds, i \geq 1.$$

It is easy to see that

$$Q_i(t) = \frac{\eta_H}{2(2-2H)} \left\{ t^{2H-1} Z_i(t) + \int_0^t r^{2H-1} dZ_i(s) \right\}.$$

Define the process $Z_i = (Z_i(t), t \in [0, T])$ by

$$Z_i(t) := \int_0^t k_H(t,s) du_i(s).$$

Extending Kleptsyna and Le Breton [16], we have:

(i) Z_i is the fundamental semimartingale associated with the process u_i.

(ii) Z_i is a $(\mathcal{F}_t)$ -semimartingale with the decomposition

$$Z_i(t) = \mu_i(\theta) \int_0^t Q_i(s) dv_s + \beta_i^{-\nu} \mathcal{M}_t^H.$$

(iii) u_i admits the representation

$$u_i(t) = \int_0^t K_H(t,s) dZ_i(s).$$

(iv) The natural filtration $(\mathcal{Z}_i(t))$ of Z_i and $(\mathcal{U}_i(t))$ of u_i coincide.

We focus on our obserbations now. Note that for equally spaced data (homoscedastic case)

$$v_{t_k} - v_{t_{k-1}} = \eta_H^{-1} \left(\frac{T}{n} \right)^{2-2H} [k^{2-2H} - (k-1)^{2-2H}], \quad k = 1, 2, \cdots, n. \tag{2.8}$$

For $H = 0.5$,

$$v_{t_k} - v_{t_{k-1}} = \eta_H^{-1} \left(\frac{T}{n} \right)^{2-2H} [k^{2-2H} - (k-1)^{2-2H}] = \frac{T}{n}, \quad k = 1, 2, \ldots, n.$$

We have

$$\begin{aligned} Q_i(t) &= \frac{d}{dv_t} \int_0^t k_H(t,s) u_i(s) ds \\ &= \kappa_H^{-1} \frac{d}{dv_t} \int_0^t s^{1/2-H} (t-s)^{1/2-H} u_i(s) ds \\ &= \kappa_H^{-1} \eta_H t^{2H-1} \frac{d}{dt} \int_0^t s^{1/2-H} (t-s)^{1/2-H} u_i(s) ds \\ &= \kappa_H^{-1} \eta_H t^{2H-1} \int_0^t \frac{d}{dt} s^{1/2-H} (t-s)^{1/2-H} u_i(s) ds \\ &= \kappa_H^{-1} \eta_H t^{2H-1} \int_0^t s^{1/2-H} (t-s)^{-1/2-H} u_i(s) ds. \end{aligned} \tag{2.9}$$

The process Q_i depends continuously on u_i and therefore, the discrete observations of u_i does not allow one to obtain the discrete observations of Q_i. The process Q_i can be approximated by

$$\widetilde{Q}_i(n) = \kappa_H^{-1} \eta_H n^{2H-1} \sum_{j=0}^{n-1} j^{1/2-H} (n-j)^{-1/2-H} u_i(j). \tag{2.10}$$

It is easy to show that $\widetilde{Q}_i(n) \to Q_i(t)$ almost surely as $n \to \infty$, see Tudor and Viens [26].

Define a new partition $0 \le r_1 < r_2 < r_3 < \cdots < r_{m_k} = t_k, \quad k = 1, 2, \cdots, n$. Define

$$\widetilde{Q}_i(t_k) = \kappa_H^{-1} \eta_H t_k^{2H-1} \sum_{j=1}^{m_k} r_j^{1/2-H} (r_{m_k} - r_j)^{-1/2-H} u_i(r_j)(r_j - r_{j-1}), \tag{2.11}$$

$k = 1, 2, \cdots, n$.

It is easy to show that $\widetilde{Q}_i(t_k) \to Q_i(t)$ almost surely as $m_k \to \infty$ for each $k = 1, 2, \cdots, n$.

We use this approximate observation in the calculation of our estimators. Thus our observations are

$$u_i(t) \approx \int_0^t K_H(t,s) d\widetilde{Z}_i(s) \quad \text{where} \quad \widetilde{Z}_i(t) = \theta \int_0^t \widetilde{Q}_i(s) dv_s + \mathcal{M}_t^H. \tag{2.12}$$

observed at Poisson arrivals $t_1, t_2, \ldots, t_n$. We observe just one such approximate Fourier coefficient $u_i(t)$ which we denote by $u(t)$ and the corresponding observations are denoted by $u_{t_1}, u_{t_2}, \ldots, u_{t_n}$ and let $n \to \infty$. Ideally we are in a large time asymptotic framework.

Now we focus on the estimation methodology. Define

$$\rho := \rho(\lambda, \theta) = \frac{\lambda}{\lambda - \kappa(\theta) + \beta_i^{2m}}. \tag{2.13}$$

The quasi likelihood estimator is the solution of the estimating equation:

$$G_n^*(\theta) = 0 \tag{2.14}$$

where

$$G_n^*(\theta) = \frac{\beta_i^{2\nu} \lambda (\rho(\lambda, \theta))^2}{\rho(\lambda, 2\theta)} \sum_{i=1}^{n} u_{t_{i-1}} \left((u_{t_{i-1}} \theta \rho(\lambda, \theta))^2 + \lambda \right)^{-1} (u_{t_i} - \rho(\lambda, \theta) u_{t_{i-1}}) \tag{2.15}$$

We call the solution of the estimating equation the quasi likelihood estimator. There is no explicit solution for this equation.

The optimal estimating function for estimation of the unknown parameter θ is

$$G_n(\theta) = \beta_i^{2\nu} \sum_{i=1}^{n} u_{t_{i-1}} [u_{t_i} - \rho(\lambda, \theta) u_{t_{i-1}}]. \tag{2.16}$$

The martingale estimation function (MEF) estimator of ρ is the solution of $G_n(\theta) = 0$ and is given by

$$\hat{\rho}_n := \frac{\sum_{i=1}^n u_{t_{i-1}} u_{t_i}}{\sum_{i=1}^n u_{t_{i-1}}^2}. \tag{2.17}$$

3. Main Results

We do the parameter estimation in two steps: The rate λ of the Poisson process can be estimated given the arrival times t_i, therefore it is done at a first step. Since we observe total number of arrivals n of the Poisson process over the T intervals of length one, the MLE of λ is given by

$$\hat{\lambda}_n := \frac{n}{T}. \tag{3.1}$$

Theorem 3.1 We have

$$\hat{\lambda}_n \to \lambda \text{ a.s.} \quad \text{as} \quad n \to \infty,$$

$$\sqrt{n}(\hat{\lambda}_n - \lambda) \to^{\mathcal{D}} \mathcal{N}(0, \ e^{\lambda}(1 - e^{-\lambda})) \text{ as } \ n \to \infty.$$

Proof. Let V_i be the number of arrivals in the interval $(i-1, i]$. Then $V_i, \ i = 1, 2, \ldots, n$ are i.i.d. Poisson distributed with parameter λ. Since Φ is continuous, we have $I_{\{0\}}(V_i) = I_{\{0\}}(u(t_i))$ a.s. $i = 1, 2, \ldots, n$. Note that

$$\frac{1}{n}\sum_{i=1}^{n} I_{\{0\}}(u_{t_i}) \to^{a.s.} E(I_{\{0\}}V_1) = P(V_1 = 0) = e^{-\lambda} \text{ as } \ n \to \infty.$$

LLN and CLT and delta method applied to the sequence $I_{\{0\}}(u_{t_i}), \ i = 1, 2, \ldots, n$ give the results. □

The CLT result above allows us to construct confidence interval for the jump rate λ.

Corollary 3.1 A $100(1-\alpha)\%$ confidence interval for λ is given by

$$\left[\frac{n}{T} - Z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}}, \quad \frac{n}{T} + Z_{1-\frac{\alpha}{2}}\sqrt{\frac{1}{n} - \frac{1}{T}}\right]$$

where $Z_{1-\frac{\alpha}{2}}$ is the $(1-\frac{\alpha}{2})$-quantile of the standard normal distribution.

We obtain the strong consistency and asymptotic normality of the MEF estimator.

Theorem 3.2 We have

$$\hat{\rho}_n \to \rho \text{ a.s.} \quad \text{as} \quad n \to \infty,$$

$$\sqrt{n}(\hat{\rho}_n - \rho) \to^{\mathcal{D}} \mathcal{N}(0, \ \lambda^{-i}(1 - e^{-\rho})) \text{ as } \ n \to \infty.$$

Proof: By using the fact that every stationary mixing process is ergodic, it is easy to show that if u_t is a stationary ergodic O-U process and t_i is a process with nonnegative i.i.d. increments which is independent of u_t, then $\{u_{t_i}, i \geq 1\}$ is a stationary ergodic process. Hence $\{u_{t_i}, i \geq 1\}$ is a stationary ergodic process.

Observe that $u_i^{\theta}(t) := v_i$ is stationary ergodic and $v_i \sim \mathcal{N}(0, \sigma^2)$ where σ^2 is the variance of u_0. Thus by SLLN for zero mean square integrable martingales, we have as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i} \to^{a.s.} E(u_{t_0}u_{t_1}) = \rho E(u_{t_0}^2)$$

$$\frac{1}{n}\sum_{i=1}^{n} u_{t_{i-1}}^2 \to^{a.s.} E(u_{t_0}^2)$$

Thus

$$\frac{\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}}{\sum_{i=1}^{n} u_{t_{i-1}}^2} \to^{a.s.} \rho.$$

Further,

$$\sqrt{n}(\hat{\rho}_n - \rho) = \frac{n^{-1/2}\sum_{i=1}^{n} u_{t_{i-1}}(u_{t_i} - \theta u_{t_{i-1}})}{n^{-1}\sum_{i=1}^{n} u_{t_{i-1}}^2}.$$

Since

$$E(u_{t_1}u_{t_2}|u_{t_1}) = \theta u_{t_1}^2$$

it follows by Lemma 3.1 in Bibby and Srensen [2]

$$n^{-1/2}\sum_{i=1}^{n} u_{t_{i-1}}(u_{t_i} - \theta u_{t_{i-1}})$$

converges in distribution to normal distribution with mean zero and variance equal to

$$E[(u_{t_1}u_{t_2}) - E(u_{t_1}u_{t_2}|u_{t_1})]^2 = 1 - e^{2(\theta-\beta_1\delta)}\{2(\beta_1-\theta)(\beta_i+1)\}^{-1}.$$

Applying delta method the result follows. □

In the next step, we use the estimator of λ to estimate θ.
Note that

$$\frac{1}{\hat{\rho}_n} = \frac{\sum_{i=1}^{n} u_{t_{i-1}}^2}{\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}}.$$

Hence

$$1 + \frac{\beta_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{i=1}^{n} u_{t_{i-1}}^2}{\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}}.$$

Thus

$$\frac{\beta_1^{2m} - \kappa(\theta)}{\lambda} = \frac{\sum_{i=1}^{n} u_{t_{i-1}}^2}{\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}} - 1 = -\frac{\sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - u_{t_{i-1}}]}{\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}}.$$

Now replace λ by its estimator MLE $\hat{\lambda}_n$.

$$\beta_1^{2m} - \kappa(\theta) = -\frac{\sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - u_{t_{i-1}}]}{\frac{T}{n}\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}}.$$

Thus

$$\hat{\theta}_n = \kappa^{-1}\left(\beta_1^{2m} + \frac{\sum_{i=1}^{n} u_{t_{i-1}}[u_{t_i} - u_{t_{i-1}}]}{\frac{T}{n}\sum_{i=1}^{n} u_{t_{i-1}}u_{t_i}}\right).$$

Since the function $\kappa^{-1}(\cdot)$ is a continuous function, by application of delta method, the following result is a consequence of Theorem 3.2.

Theorem 3.3

$$\hat{\theta}_n \to^{a.s.} \theta \text{ as } \; n \to \infty,$$

$$\sqrt{n}(\hat{\theta}_n - \theta) \to^{\mathcal{D}} \mathcal{N}(0,\; (\kappa'(\theta))^{-2}\lambda^2(1 - e^{-2\lambda^{-1}(\kappa(\theta)-\beta_1^{2m})})) \text{ as } \; n \to \infty.$$

In the second stage, we plug-in λ by its estimator $\hat{\lambda}_n$.

Remark Sub-fractional Brownian motion, which has main properties of the fractional Brownian motion, excluding the stationarity of increments, has the covariance function

$$C_H(s,t) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right], \quad s,t > 0.$$

One can gereneralize this to sub-fractional Levy process by plug-in method which would have nonstationary increments and corresponding SPDE models could be used for modeling in finance and biology.

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