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Weak and Strong Convergence Theorems of Modified Projection-Type Ishikawa Iteration Scheme for Lipschitz α-Hemicontractive Mappings

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Abstract. In this paper, we establish weak and strong convergence theorems of a two-step modified projection-type Ishikawa iterative scheme to the fixed point of α-hemicontractive mappings without any compactness assumption on the operator or the space. Our results extend, improve and generalize several previously known results of the existing literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $\|.,.\|$, K a nonempty convex and closed subset of H and $T : K \longrightarrow K$ a selfmap on K. We use $F(T)$ to denote the set of fixed point of T, $\mathbb{N}$ to denote the set of natural numbers and $x_n \to x$ (respectively $x_n \rightharpoonup x$) to denote the strong (weak) convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to the point x.

Definition 1.1. *Let $T : K \longrightarrow K$ be a maaping. Then*

I. T is said to be L-Lipschitizian if there exists $L > 0$ such that

$$\|Ts - Tz\| \leq \|s - z\|, \forall s, z \in K. \tag{1.1}$$

From the definition, it easy to observe that every nonexpansive mapping is Lipschitizian with $L = 1$.

II. T is called k-strictly pseudocontraction (see, for example, [9]) if there exists $k \in (0, 1]$ such that for all $s, z \in K$, the inequality

$$\|Ts - Tz\|^2 \leq \|s - z\|^2 + k\|(I - T)s - (I - T)z\|^2 \tag{1.2}$$

hods. Note that if $k = 1$ in (1.2), then T is a pseudocontraction. It well-known that in real Hilbert spaces, the class of nonexpansive mapping is a proper subclass of the class of

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k-strictly pseudocontive mapping. Also, the class of k-strictly pseudocontive mapping is a proper subclass of the class of pseudocontive mapping.

III. *T is called demicontractive mapping (see, for example, [?]) if $F(T) = \{x \in K : x = Tx\} \neq \emptyset$ and $\forall(s \times q) \in (K \times F(T))$, there exists $k \in [0, 1)$ such that the inequality*

$$\|Ts - Tq\|^2 \leq \|s - q\|^2 + k\|s - Ts\|^2 \tag{1.3}$$

hods.

IV. *T is said to satisfy condition A (see, for example [?]) $F(T) = \{x \in K : x = Tx\} \neq \emptyset$ and there exists $\lambda > 0$ such that*

$$\langle s - TS, s - q\rangle \geq \lambda\|s - Ts\|^2, \forall(s \times q) \in (K \times F(T)). \tag{1.4}$$

It is worthy to mention that the class of k-strictly pseudocontions with a nonempty fixed point set is a proper subclass of the class demicontractions. T is called hemicontraction (see, for example, [17]) if $k = 1$ in (1.3). The class of pseudocontractive maps is a proper subclass of the class of hemicontractive maps. Again, the class of demicontractive maps is a proper subclass of the class of hemicontractive maps (see, for example, [?]). These two classes of mappings have been studied extensively by many researchers (see, for example, [?], [13], [17] and the references therein).

V. *T is called α-demicontraction (see, for examole, [13]) if $F(T) = \{x \in K : x = Tx\} \neq \emptyset$ and $\forall(s \times q) \in (K \times F(T))$, there exist $\lambda > 0$ and $\alpha \geq 1$ such that the inequality*

$$\langle s - TS, s - \alpha q\rangle \geq \lambda\|s - Ts\|^2, \forall(s \times q) \in (K \times F(T)). \tag{1.5}$$

holds. Clearly, (1.5) is equivalent to

$$\|Ts - \alpha q\|^2 \leq \|s - \alpha q\|^2 + k\|s - Ts\|^2, \tag{1.6}$$

where $k = 1 - 2\lambda \in [0, 1)$.

V. *T is called α-hemicontraction (see, for examole, [17]) if $F(T) = \{x \in K : x = Tx\} \neq \emptyset$ and $\forall(s \times q) \in (K \times F(T))$, there exists $\alpha \geq 1$ such that the inequality*

$$\|Ts - \alpha q\|^2 \leq \|s - \alpha q\|^2 + \|s - Ts\|^2 \tag{1.7}$$

holds. Observe that (1.7) is equivalent to

$$\langle s - Ts, s - \alpha q\rangle \geq 0, \forall(s \times q) \in (K \times F(T)). \tag{1.8}$$

In [[17], Example 2.2], Osilike and Onah gave an example of α-hemicontractive mapping with $\alpha > 1$ which is not hemicontractive mapping, and also showed that there are hemicontractive (1-hemicontractive) mappings which are not α-hemicontraction for $\alpha > 1$(see [[17], Example 2.1] for details). Again, Osilike and Onah [17] presented an example of a mapping which is hemicontractive (1-hemicontractive) and $alpha$-hemicontractive mapping for $\alpha > 1$ but *neither demicontractive (1-demicontractive) nor α-demicontractive mapping for $\alpha > 1$(see [17], Example 2.3 for details). For further characterisation of α-hemicontractive mapping, interested reader should consult [17].*

A mapping $T : H \longrightarrow H$ is called ν-strongly monotone if there exists $\nu > 0$ such that

$$\langle s - Ts, s - z\rangle \geq \nu\|s - z\|^2, \forall s, z \in H.. \tag{1.9}$$

Iterative method for approximating fixed point of L-Lipschitz pseudocontractive mapping has been an active area of investigation in recent times (see, for example, [?], [?], [20], [14], [26], [27] and the references contained in them). In [24], Voluhan introduced the modified projection-type Ishikawa iterative method in the following way: Let H be a Hilbert space, K nonempty, closed and convex subset of H and $T : K \longrightarrow K$ be an L-Lipshitz pseudocontractive mapping. For an arbitrary $x0 \in K$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively as follows.

$$\begin{cases} x_{n+1} = P_K[(1 - \alpha_n - \gamma_n)x_n + \gamma_n T y_n] \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, n \geq 1, \end{cases} \tag{1.10}$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \in (0, 1)$ and P_K is a projection map from H *onto* K. Using (1.10), she proved the following theorem.

Theorem 1.1. *Let H be a Hilbert space, D a nonempty closed convex subset of H and $T : D \longrightarrow D$ an L-Lipschitz pseudocontractive mapping such that $F(T) \neq \emptyset$. For any given $x_0 \in H$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by* (1.10). *Assume the sequences $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty} \in (0, 1)$ satisfy*

(1) $\beta_n(1 - \alpha_n) > \gamma_n, \forall n \geq 1;$
(2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty;$
(3) $0 < \alpha \leq \gamma_n \leq \beta_n \leq \beta < \dfrac{1}{\sqrt{1 + L^2} + 1}, \forall n \geq 1.$

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ strongly converges to the fixed point of T.

Remark 1.1. *If $\alpha_n = 0, \forall n \geq 1$, and P_K is an identity,* (1.10) *reduces to the well-known Ishikawa iteration method*

$$\begin{cases} x_{n+1} = (1 - \gamma_n)x_n + \gamma_n T y_n \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, n \geq 1, \end{cases} \tag{1.11}$$

which has been used by several researchers to approximate the fixed points of different operators or operator equations in different spaces.

Motivated and inspired by the works in [17], [24] and some ongoing research in this direction, it is our purpose in this paper to extend the results in [24] and other related results from Lipschitz pseudocontractive mapping to the more general α-hemicontractive mapping. Our results is more general and also more applicable because fewer and simpler conditions are required to attain convergence.

2. Preliminary

The following definitionS and lemmas will be needed to prove our main results.

Definition 2.1. *(see [27]) Let H and K be as defined above. For each $x \in H$, there exists a unique nearest point of K, denoted by $P_K x$, such that*

$$\|x - P_K x\| \leq \|x - y\|, \forall y \in K.$$

Such a P_K is called metric projection from H onto K. It is well-known that P_K is firmly nonexpansive mapping from H onto K; that is,

$$\|P_K x - P_K y\|^2 \leq \langle P_K x - P_K y, x - y\rangle, \forall x, y \in H.$$

Also, for any $x \in H$ and $z \in K, z = P_K x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \forall y \in K.$$

Definition 2.2. *The Banach space Z is said to have Opial property, if for each weakly convergent sequence $\{z_n\}_{n=0}^{\infty}$with weak limit $z \in Z$, the following inequality holds:*

$$\limsup_{n\to\infty} \|z_n - z\| < \|z_n - y\|, \forall y \in Z with z \neq y.$$

Note that all finite dimensional Banach spaces, all Hilbert spaces and $\ell^p(0 \leq p < \infty)$ satisfy the Opial property. But $L_p(1 < p < \infty . p \neq 2)$ do not satisfies the Opial property.

Definition 2.3. *(see [27]) Let E be a real Banach space. A mapping T, with domain $D(T) \in E$, is said to be demiclosed at 0 if for any sequence $z_n \subset E, z_n \upharpoonright q \in D(T)$ and $\|z_n - Tz_n\| \to 0$, then $Tq = q$.*

Lemma 2.1. *(see [27]) Let H be a real Hilbert space. Then, the following inequality holds:*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|, \forall \lambda \in [0, 1], \forall x, y \in H.$$

Lemma 2.2. *(see [27]) Let $\{s_n\}_{n\in N}$ be a sequence of nonnegative real numbers satisfying the inequality:*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \delta_n, \forall n \geq 1,$$

where $\{\gamma_n\}_{n\in N}$ and $\{\delta_n\}_{n\in N}$ satisfy the following conditions:

(i) $\{\gamma_n\}_{n\in N} \subset (0, 1)$;
(ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Suppose $\sum_{n=1}^{\infty} \delta_n < \infty$, then,$\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *(see [4]) Let E be a real Hilbert space. Then, for all $x, y \in H$, the following inequalities hold:*

$$I. \quad \|x-y\|^2 \leq \|x\|^2 - 2\langle y,(x+y)\rangle + \|y\|^2;$$
$$II. \quad \|x-y\|^2 \leq \|x\|^2 - 2\langle y,(x+y)\rangle.$$

Lemma 2.4. *(see [?]) Let D be a sunset of a real Hilbert space, $T : D \longrightarrow H$ be a nonexpansive mapping and z a weak cluster point of the sequence $\{y_n\}_{n=0}^{\infty}$. If $\|Ty_n - y_n\| \to 0$, then $z \in F(T)$*

Proposition 2.5. *(see [27]) Let D be a nonempty subset of a real Hilbert space amd $\Gamma : D \longrightarrow D$ an α-demicontractive mapping. Assume that $x \in D$ and $\alpha \geq 1$. Then, Γ is Lipschitizian.*

Theorem 2.6. *(see [4]) A Banach space E is reflexive if and only if every (normed) bounded sequence in E has a subsequence which converges weakly to an element of E.*

3. Convergence Results

Now, we prove our main results.

Theorem 3.1. *Let H be a real Hilbert space, K a nonempty closed convex subset of H and $T : K \longrightarrow K$ an L-Lipschitz α-hemicontractive mapping. For any arbitrary $x_0 \in H$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively as follows:*

$$\begin{cases} x_{n+1} = P_K[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n] \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, n \geq 1, \end{cases} \tag{3.1}$$

where the sequences $\{\delta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in (0,1)$ satisfy the following conditions:

(i) $0 < \delta \leq \delta_n \leq \beta_n \leq \gamma_n \leq \gamma \leq \dfrac{1-\delta}{1+L^2}$;

(ii) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (3.1) weakly and strongly converges to the fixed point of T.

Proof. Since $F(T)$ is nonempty, let $\alpha q \in F(T)$ *and* $x \in K$. Using (3.1), Lemma 2.1 and the fact that T *is* L-Lipschitizian, we estimate as follows:

$$\begin{aligned} \|x_{n+1}-\alpha q\|^2 &= \|P_K[(1-\delta_n-\gamma_n)x_n + \gamma_n T y_n] - \alpha q\| \\ &\leq \|(1-\delta_n-\gamma_n)x_n + \gamma_n T y_n - \alpha q\| \\ &= \|(1-\delta_n-\gamma_n)(x_n-\alpha q) + \gamma_n(Ty_n-\alpha q) - \delta_n \alpha q\| \\ &\leq \|(1-\delta_n-\gamma_n)(x_n-\alpha q) + \gamma_n(Ty_n-\alpha q)\| + \delta_n\|\alpha q\|. \end{aligned} \tag{3.2}$$

Set $Q_n = \|(1-\delta_n-\gamma_n)(x_n-\alpha q) + \gamma_n(Ty_n-\alpha q)\|^2$ and observe that

$$Q_n = \|(1-\delta_n)(x_n-\alpha q) - (1-\gamma_n)(x_n-\alpha q) + \gamma_n(Ty_n-\alpha q)\|^2. \tag{3.3}$$

Since

$$(1-\delta_n)(x_n-\alpha q)=(1-\delta_n)(1-\gamma_n)(x_n-\alpha q)+\gamma_n(1-\delta_n))(x_n-\alpha q) \tag{3.4}$$

and

$$\gamma_n(Ty_n-\alpha q)=\gamma_n(1-\delta_n)(Ty_n-\alpha q)+\gamma_n\delta_n(Ty_n-\alpha q), \tag{3.5}$$

it follows from (3.3) that

$$\begin{aligned} Q_n &= \|(1-\delta_n)(1-\gamma_n)(x_n-\alpha q)+\gamma_n(1-\delta_n))(x_n-\alpha q)-(1-\gamma_n)(x_n-\alpha q) \\ &\quad +\gamma_n(1-\delta_n)(Ty_n-\alpha q)+\gamma_n\delta_n(Ty_n-\alpha q)\|^2 \\ &= \|(1-\delta_n)[(1-\gamma_n)(x_n-\alpha q)+\gamma_n(Ty_n-\alpha q)]+\delta_n\gamma_n(Ty_n-x_n)\|^2. \end{aligned} \tag{3.6}$$

(3.6) and Lemma 2.1 imply that

$$\begin{aligned} Q_n &= (1-\delta_n)\|(1-\gamma_n)(x_n-\alpha q)+\gamma_n(Ty_n-\alpha q)\|^2+\delta_n\|\gamma_n(Ty_n-x_n)\|^2 \\ &\quad -\delta_n(1-\delta_n)\|x_n-\alpha q\|^2. \end{aligned} \tag{3.7}$$

If we denote $V_n=\|(1-\gamma_n)(x_n-\alpha q)+\gamma_n(Ty_n-\alpha q)\|^2$ and use similar technique as above, then we get

$$V_n = (1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n\|Ty_n-\alpha q\|^2-\gamma_n(1-\gamma_n)\|x_n-Ty_n\|^2. \tag{3.8}$$

(3.7) and (3.8) imply

$$\begin{aligned} Q_n &= (1-\delta_n)[(1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n\|Ty_n-\alpha q\|^2-\gamma_n(1-\gamma_n)\|x_n-Ty_n\|^2] \\ &\quad +\delta_n\gamma_n^2\|Ty_n-x_n\|^2-\delta_n(1-\delta_n)\|x_n-\alpha q\|^2 \\ &= (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2+(1-\delta_n)\gamma_n\|Ty_n-\alpha q\|^2-\gamma_n(1-\gamma_n)(1-\delta_n)\|x_n-Ty_n\|^2 \\ &\quad +\delta_n\gamma_n^2\|Ty_n-x_n\|^2-\delta_n(1-\delta_n)\|x_n-\alpha q\|^2 \\ &\leq (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2+(1-\delta_n)\gamma_nL^2\|y_n-\alpha q\|^2 \\ &\quad -(\gamma_n-\delta_n\gamma_n-\gamma_n^2+\gamma_n^2\delta_n)\|x_n-Ty_n\|^2+\delta_n\gamma_n^2\|Ty_n-x_n\|^2-\delta_n(1-\delta_n)\|x_n-\alpha q\|^2 \\ &= (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_nL^2\|y_n-\alpha q\|^2-\delta_n\gamma_nL^2\|y_n-\alpha q\|^2 \\ &\quad -(\gamma_n-\delta_n\gamma_n-\gamma_n^2)\|x_n-Ty_n\|^2-\delta_n(1-\delta_n)\|x_n-\alpha q\|^2. \end{aligned} \tag{3.9}$$

Observr that

$$\begin{aligned} |x_n-Ty_n\| &\leq (\|x_n-\alpha q\|+L\|y_n-\alpha q\|)^2 \\ &= \|x_n-\alpha q\|^2+L(2\|x_n-\alpha q\|\|y_n-\alpha q\|)+L^2\|y_n-\alpha q\|^2 \\ &\leq \|x_n-\alpha q\|^2+L\|x_n-\alpha q\|^2+L\|y_n-\alpha q\|^2+L^2\|y_n-\alpha q\|^2 \\ &= (1+L)\|x_n-\alpha q\|^2+L(1+L)\|y_n-\alpha q\|^2. \end{aligned} \tag{3.10}$$

(3.9) and (3.10) imply

$$\begin{aligned} Q_n &\le (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n L^2\|y_n-\alpha q\|^2-\delta_n\gamma_n L^2\|y_n-\alpha q\|^2 \\ &\quad -(\gamma_n-\delta_n\gamma_n-\gamma_n^2)[(1+L)\|x_n-\alpha q\|^2+L(1+L)\|y_n-\alpha q\|^2]-\delta_n(1-\delta_n)\|x_n-\alpha q\|^2 \\ &= (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2-(1+L)(\gamma_n-\delta_n\gamma_n-\gamma_n^2)\|x_n-\alpha q\| \\ &\quad -[(\gamma_n-\delta_n\gamma_n-\gamma_n^2)L-L^2\gamma_n^2]\|y_n-\alpha q\|^2-\delta_n(1-\delta_n)\|x_n-\alpha q\|^2 \qquad (3.11) \end{aligned}$$

Again, from (3.1), we get

$$\|y_n-\alpha q\|^2=\|(1-\beta_n)(x_n-\alpha q)+\beta_n(Tx_n-\alpha q)\|^2 \qquad (3.12)$$

Since T is α-hemicontractive mapping, it follows from (3.12) and Lemma 2.1 that

$$\begin{aligned} \|y_n-\alpha q\|^2 &\le (1-\beta_n)\|x_n-\alpha q\|^2+\beta_n[\|x_n-\alpha q\|^2\|^2+\|x_n-Tx_n\|^2]-\beta_n(1-\beta_n)\|x_n-Tx_n\|^2 \\ &= (1-\beta_n)\|x_n-\alpha q\|^2+\beta_n^2\|x_n-Tx_n\|^2. \qquad (3.13) \end{aligned}$$

Putting (3.13) into (3.11), we have

$$\begin{aligned} Q_n &\le (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2-(1+L)(\gamma_n-\delta_n\gamma_n-\gamma_n^2)\|x_n-\alpha q\| \\ &\quad -[(\gamma_n-\delta_n\gamma_n-\gamma_n^2)L-L^2\gamma_n^2]\{(1-\beta_n)\|x_n-\alpha q\|^2+\beta_n^2\|x_n-Tx_n\|^2\} \\ &\quad -\delta_n(1-\delta_n)\|x_n-\alpha q\|^2 \\ &\le (1-\delta_n)(1-\gamma_n)\|x_n-\alpha q\|^2-[(\gamma_n-\delta_n\gamma_n-\gamma_n^2)(1+L)+\delta_n(1-\delta_n)-L^2\gamma_n^2]\|x_n-\alpha q\|^2 \\ &\quad -\beta_n^2[(\gamma_n-\delta_n\gamma_n-\gamma_n^2)L-L^2\gamma_n^2]\|x_n-Tx_n\|^2. \qquad (3.14) \end{aligned}$$

Since from condition (i), $(\gamma_n-\delta_n\gamma_n-\gamma_n^2)-L^2\gamma_n^2\ge 0$, it follows from (3.14) that

$$Q_n\le(1-\delta_n)^2\|x_n-\alpha q\|^2 \qquad (3.15)$$

(3.2) and (3.15) imply

$$\begin{aligned} |x_{n+1}-\alpha q\| &\le (1-\delta_n)\|x_n-\alpha q\|^2+\delta_n\|\alpha q\| \\ &\le max\{\|x_n-\alpha q\|^2,\|\alpha q\|\},\forall n\in\mathbb{N}. \end{aligned}$$

It is easy to see, using mathematical induction, that

$$\begin{aligned} |x_{n+1}-\alpha q\| &\le max\{\|x_n-\alpha q\|^2,\|\alpha q\|\} \\ &= \|x_0-\alpha q\|^2. \qquad (3.16) \end{aligned}$$

Hence, $\{x_n\}_{n=0}^{\infty}$ is bounded.

Furthermore, since from (3.1),

$$\begin{aligned}\|x_{n+1}-\alpha q\|^2 &= \|P_K[(1-\delta_n-\gamma_n)x_n+\gamma_n Ty_n]-\alpha q\|^2\\ &\leq \|(1-\delta_n-\gamma_n)x_n+\gamma_n Ty_n-\alpha q\|^2\\ &= \|x_n-\alpha q-\gamma_n(x_n-Ty_n)-\delta_n x_n\|^2,\end{aligned}$$

it follows from Lemma 2.3(i) that

$$\|x_{n+1}-\alpha q\|^2 \leq \|x_n-\alpha q-\gamma_n(x_n-Ty_n)\|^2-2\delta_n\langle x_n,x_{n+1}-\alpha q\rangle. \tag{3.17}$$

Since

$$\begin{aligned}\|x_n-\alpha q-\gamma_n(x_n-Ty_n)\|^2 &= \|(1-\gamma_n)(x_n-\alpha q)+\gamma_n(\alpha q-Ty_n)\|^2\\ &= (1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n\|\alpha q-Ty_n\|^2-\gamma_n(1-\gamma_n)\|Ty_n-x_n\|^2\\ &\leq (1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n L^2\|y_n-\alpha q\|^2\\ &\quad -\gamma_n(1-\gamma_n)\|Ty_n-x_n\|^2,\end{aligned} \tag{3.18}$$

it follows from (3.10) that

$$\begin{aligned}\|x_n-\alpha q-\gamma_n(x_n-Ty_n)\|^2 &\leq (1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n L^2\|y_n-\alpha q\|^2\\ &\quad -\gamma_n(1-\gamma_n)\{(1+L)\|x_n-\alpha q\|^2+L(1+L)\|y_n-\alpha q\|^2\}\\ &= (1-\gamma_n)\|x_n-\alpha q\|^2+\gamma_n L^2\|y_n-\alpha q\|^2\\ &\quad -\gamma_n(1-\gamma_n)(1+L)\|x_n-\alpha q\|^2-\gamma_n(1-\gamma_n)L\|y_n-\alpha q\|^2\\ &\quad -\gamma_n L^2\|y_n-\alpha q\|^2+\gamma_n^2L^2\|y_n-\alpha q\|^2\\ &= (1-\gamma_n)\|x_n-\alpha q\|^2-\gamma_n(1-\gamma_n)(1+L)\|x_n-\alpha q\|^2\\ &\quad -[\gamma_n(1-\gamma_n)L-L^2\gamma_n^2]\|y_n-\alpha q\|^2.\end{aligned} \tag{3.19}$$

(3.13) and (3.19) imply

$$\begin{aligned}\|x_n-\alpha q-\gamma_n(x_n-Ty_n)\|^2 &\leq (1-\gamma_n)\|x_n-\alpha q\|^2-\gamma_n(1-\gamma_n)(1+L)\|x_n-\alpha q\|^2\\ &\quad -[\gamma_n(1-\gamma_n)L-L^2\gamma_n^2]\{(1-\beta_n)\|x_n-\alpha q\|^2+\beta_n^2\|x_n-Tx_n\|^2\}\\ &\leq (1-\gamma_n)\|x_n-\alpha q\|^2-\gamma_n L[1-\gamma_n-\gamma_n L]\{(1-\beta_n)\|x_n-\alpha q\|^2\\ &\quad +\beta_n^2\|x_n-Tx_n\|^2\}.\end{aligned} \tag{3.20}$$

By condition $(i), 1-\gamma_n-\gamma_n L>0, \forall n\geq 0$. Consequently,

$$\begin{aligned}\|x_n-\alpha q-\gamma_n(x_n-Ty_n)\|^2 &\leq \|x_n-\alpha q\|^2\\ &\quad -(1-\gamma_n-\gamma_n L)\beta_n^2\gamma_n L\|x_n-Tx_n\|^2.\end{aligned} \tag{3.21}$$

(3.17)and (3.21) imply

$$\begin{aligned}\|x_{n+1}-\alpha q\|^2 &\leq \|x_n-\alpha q\|^2-(1-\gamma_n-\gamma_n L)\beta_n^2\gamma_n L\|x_n-Tx_n\|^2\\ &-2\delta_n\langle x_n, x_{n+1}-\alpha q\rangle.\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a constant $B>0$ such that $-2\langle x_n, x_{n+1}-\alpha q\rangle \leq B$. Thus,

$$\begin{aligned}\|x_{n+1}-\alpha q\|^2 &\leq \|x_n-\alpha q\|^2-(1-\gamma_n-\gamma_n L)\beta_n^2\gamma_n L\|x_n-Tx_n\|^2\\ &\delta_n B.\end{aligned}$$

The last inequality implies that

$$\|x_{n+1}-\alpha q\|^2-\|x_n-\alpha q\|^2+(1-\gamma_n-\gamma_n L)\beta_n^2\gamma_n L\|x_n-Tx_n\|^2 \leq \delta_n B. \tag{3.22}$$

Now, we consider the following two cases:

Case A: Suppose there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-\alpha q\|\}$ is non-increasing. Then, $\{\|x_n-\alpha q\|\}$ is convergent. Clearly, $\|x_{n+1}-\alpha q\|-\|x_n-\alpha q\| \to 0$. In view, of condition (ii) and (3.22), we have $\|x_n-Tx_n\| \to 0$. By Lemma 2.4, it is obvious that $\omega_\omega(x_n) \subset F(T),\ where\, \omega_\omega(x_n)\{x : \exists x_{n_k} \rightharpoonup \alpha x^\star\}$ is the weak limit set of $\{x_n\}$. This implies that the sequence $\{x_n\}$ converges weakly to a fixed point $\alpha x^\star$ of T.

Suppose there exists some subsequences $\{x_{n_k}\}_{k=0}^\infty \subset \{x_n\}_{n=0}^\infty$ such that $x_{n_k} \rightharpoonup \alpha y^\star$ weakly and $\alpha y^\star \neq \alpha x^\star$. Since $\lim_{n\to\infty}\|x_n-\alpha v\|$ exists for $\alpha v \in F(T)$, by virtue of Opial condition on H, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-\alpha x^\star\| &= \lim_{n\to\infty}\|x_{n_j}-\alpha x^\star\| < \lim_{n\to\infty}\|x_{n_j}-\alpha y^\star\| = \lim_{n\to\infty}\|x_{n_k}-\alpha y^\star\|\\ &< \lim_{n\to\infty}\|x_{n_k}-\alpha x^\star\| = \lim_{n\to\infty}\|x_{n_j}-\alpha y^\star\|,\end{aligned}$$

which is a contradiction. Consequently, $\alpha y^\star = \alpha x^\star$. This implies that $\{x_{n_j}\}_{j=0}^\infty$ converges wealy to a common fixed point of T.

Next, we prove that $\{x_n\}_{n=0}^\infty$ converges strongly to $x^{\star/}$ Let $\xi_n = \gamma_n Ty_n + (1-\gamma_n x_n)$. Then, from (3.1), we obtain $x_{n+1} = P_K[\xi_n - \delta_n x_n], n \geq 0$. This implies that

$$\begin{aligned}x_{n+1} &= P_K[\xi_n+\delta_n\xi_n+\delta_n\xi_n-\delta_n x_n\\ &= P_K[(1-\delta_n)\xi_n+\delta_n(\xi_n-x_n)].\end{aligned} \tag{3.23}$$

Observe that

$$\|\xi_n-\alpha x^\star\|^2 = \|x_n-\alpha x^\star-\gamma_n(x_n-Ty_n)\|^2. \tag{3.24}$$

By using the same argument as in (3.20), with $\alpha x^\star = \alpha q$, we get, from (3.24), that

$$\|\xi_n-\alpha x^\star\| = \|x_n-\alpha x^\star\|. \tag{3.25}$$

Again, from (3.1), we obtain

$$\|y_n-x_n\| = \beta_n\|x_n-Tx_n\| \to 0 \ as\ n\to\infty, \beta_n \in (0,1). \tag{3.26}$$

In addition, since T is Lipschitz, it follows that

$$\begin{aligned} \|\xi_n - x_n\| &= \|\gamma_n[(Ty_n - Tx_n) - (x_n - Tx_n)]\| \\ &\le \gamma_n\|Ty_n - Tx_n\| - \gamma_n\|x_n - Tx_n\| \\ &\le \gamma_n L\|y_n - x_n\| + \gamma_n\|x_n - Tx_n\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.27}$$

Now, using (3.23), we get

$$\begin{aligned} \|x_{n+1} - \alpha x^\star\|^2 &\le \|(1-\delta_n)\xi_n + \delta_n(\xi_n - x_n) - \alpha x^\star\|^2 \\ &= \|(1-\delta_n)(\xi_n - \alpha x^\star) + \delta_n(\xi_n - x_n) - \delta_n\alpha x^\star\|^2, \end{aligned}$$

which by Lemma 2.3 yields

$$\begin{aligned} \|x_{n+1} - \alpha x^\star\|^2 &\le \|(1-\delta_n)(\xi_n - \alpha x^\star) + \delta_n(\xi_n - x_n)\|^2 - 2\delta_n\langle \alpha x^\star, x_{n+1} - \alpha x^\star\rangle \\ &= (1-\delta_n)\|\xi_n - \alpha x^\star\|^2 + \delta_n\|\xi_n - x_n\|^2 - \delta_n(1-\delta_n)\|x_n - \alpha x^\star\|^2 \\ &\quad -2\delta_n\langle \alpha x^\star, x_{n+1} - \alpha x^\star\rangle \\ &\le (1-\delta_n)\|\xi_n - \alpha x^\star\|^2 + \|\xi_n - x_n\|^2 - 2\delta_n\langle \alpha x^\star, x_{n+1} - \alpha x^\star\rangle \\ &= (1-\delta_n)\|\xi_n - \alpha x^\star\|^2 - 2\delta_n\langle \alpha x^\star, x_{n+1} - \alpha x^\star\rangle \quad (\text{ by } (3.27)) \qquad (3.28) \\ &\le (1-\delta_n)\|\xi_n - \alpha x^\star\|^2 \qquad (3.29) \end{aligned}$$

(3.29) and Lemma 2.2 imply that $x_n \to \alpha x^\star$ *as* $n \to \infty$.

Case B: Assume that $\{\|x_n - \alpha q\|\}_{n=0}^\infty$ is not a monotonically increasing sequence. Set $V_n = \|x_n - \alpha q\|^2$ and let $\tau : \mathbb{N} \longrightarrow \mathbb{N}$ be a mapping defined by

$$\tau_n = \max\{k \in \mathbb{N} : k \le n, V_n \le V_{n+1}\}, \forall n \ge n_0,$$

for some n_0 large enough. Obviously, $\{\tau_n\}_{n=0}^\infty$ is a nondecreasing sequence given that $\tau_n \to \infty$ *as* $n \to \infty$ and $V_{\tau_n} \le V_{\tau_{n+1}}$ for all $n \ge n_0$. From (3.22),

$$\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \le \frac{\delta_{\tau(n)}B}{(1-\gamma_{\tau(n)} - \gamma_{\tau(n)}L)\beta^2_{\tau(n)}\gamma_{\tau(n)}L} \to 0 \text{ as } n \to \infty. \tag{3.30}$$

Therefore, $\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0$. Using similar argument as Case A above, we conclude that $\{x_{\tau(n)}\} \to \alpha x^\star \ \to \infty$.

From (3.28), we have

$$0 \le \|x_{\tau(n)+1} - \alpha x^\star\|^2 - \|x_{\tau(n)} - \alpha x^\star\|^2 \le \delta_{\tau(n)}[2\langle \alpha x^\star - x_{\tau(n)+1} - \|x_{\tau(n)} - \alpha x^\star\|^2], \tag{3.31}$$

for $\delta_{\tau(n)} \in (0,1)$. Hence, $\lim_{n\to\infty}\|x_{\tau(n)} - \alpha x^\star\|^2 = 0$. This implies that $\lim_{n\to\infty} V_{\tau(n)} = \lim_{n\to\infty} V_{\tau(n)+1} = 0$. In addition, for $n \ge n_{0,}$ it is easy to see that $V_{\tau(n)} = V_{\tau(n)+1}$ if $n \ne \tau(n)$ (i.e., $\tau(n) < n$) because $V_j > V_{j+1}, \ for \ \tau(n) + 1 \le n$. Consequently. we obtain, for all $n \ge n_0$, $0 \le V_{\tau(n)} max\{V_{\tau(n)}, V_{\tau(n)+1}\} = V_{\tau(n)+1}$. Hence, $\lim_{n\to\infty} V_n = 0$. That is, $\{x_n\}_{n=0}^\infty$ converges strongly to $\alpha x^\star$, and this completes the proof.

□

The following corollaries are immediate consequence of Theorem 3.1.

Corollary 3.2. *Let H be a real Hilbert space, K a nonempty closed convex subset of H and $T : K \longrightarrow K$ an L-Lipschitz hemicontractive mapping. For any arbitrary $x_0 \in H$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively as follows:*

$$\begin{cases} x_{n+1} = P_K[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n] \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \, n \geq 1, \end{cases} \tag{3.32}$$

where the sequences $\{\delta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in (0,1)$ satisfy the following conditions:

(i) $0 < \delta \leq \delta_n \leq \beta_n \leq \gamma_n \leq \gamma \leq \dfrac{1-\delta}{1+L^2}$;
(ii) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (3.32) weakly and strongly converges to the fixed point of T.

Corollary 3.3. *Let H be a real Hilbert space, K a nonempty closed convex subset of H and $T : K \longrightarrow K$ is α-demicontractive mapping. For any arbitrary $x_0 \in H$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively as follows:*

$$\begin{cases} x_{n+1} = P_K[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n] \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \, n \geq 1, \end{cases} \tag{3.33}$$

where the sequences $\{\delta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in (0,1)$ satisfy the following conditions:

(i) $0 < \delta \leq \delta_n \leq \beta_n \leq \gamma_n \leq \gamma \leq \dfrac{1-\delta}{1+L^2}$;
(ii) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (3.33) weakly and strongly converges to the fixed point of T.

Corollary 3.4. *Let H be a real Hilbert space, K a nonempty closed convex subset of H and $T : K \longrightarrow K$ is demicontractive mapping. For any arbitrary $x_0 \in H$, define the sequence $\{x_n\}_{n=0}^{\infty}$ iteratively as follows:*

$$\begin{cases} x_{n+1} = P_K[(1-\alpha_n-\gamma_n)x_n + \gamma_n T y_n] \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \, n \geq 1, \end{cases} \tag{3.34}$$

where the sequences $\{\delta_n\}_{n=0}^{\infty}, \{\gamma_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \in (0,1)$ satisfy the following conditions:

(i) $0 < \delta \leq \delta_n \leq \beta_n \leq \gamma_n \leq \gamma \leq \dfrac{1-\delta}{1+L^2}$;
(ii) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{n=0}^{\infty} \delta_n = \infty$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *generated by* (3.34) *weakly and strongly converges to the fixed point of* T*.*

Competing Interest. The authors declare that there is no conflict of interest.

References

[1] F.E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967) 875-882.

[2] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197-228. https://doi.org/10.1016/0022-247X(67)90085-6.

[3] C.E. Chidume, Picards Iteration for nonlinear Lipschitz strong pseudocontractions in uniformly strong Banach spaces, ICTP Preprint, IC/951, (1995) 88. https://www.osti.gov/etdeweb/biblio/194049.

[4] C.E. Chidume, Geometric properties of Banach spaces and nonlinear iterations, Springer-Verlag, London, (2009).

[5] C. Chidume, C. Moore, Fixed point iteration for pseudocontractive maps, Proc. Amer. Math. Soc. 127 (1999) 1163-1170. https://doi.org/10.1090/S0002-9939-99-05050-9.

[6] C.E. Chidume, S.A. Mutangadura, An example of the mann iterative method for lipschitz pseudocontractions, Proc. Amer. Math. Soc. 13 (1974) 2359-2363.

[7] T.L. Hicks, J.D. Kubicek, On the Mann iteration process in a Hilbert space, J. Math. Anal. Appl. 59 (1977) 498-504. https://doi.org/10.1016/0022-247X(77)90076-2.

[8] N. Hussain, A. Rafiq, M.S. Kang, Iteration schemes for two hemicontractive mappings in arbitrary Banach spaces, Int. J. Math. Anal. 7 (2013) 863-871.

[9] D.I. Igbokwe, Construction of fixed points of strictly pseudocontractive mappings of Browder-Petryshn-type in arbitrary Banach spaces, J. Fixed Point Theory Appl. 4 (2004) 137-147.

[10] E.E. Epuke, Approximation of fixed points and solutions of variational inequalities for certain classes of mappings using hybrid iteration scheme, Unpublished M.Sc. Thesis, University of Nigeria, Nsukka, (2010).

[11] L. Qihou, On Naimpally and Singh's open questions, J. Math. Anal. Appl. 124 (1987) 157-164. https://doi.org/10.1016/0022-247X(87)90031-X.

[12] G. Marino, H.-K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007) 336-346. https://doi.org/10.1016/j.jmaa.2006.06.055.

[13] L. Maruster, S. Maruster, Strong convergence of the Mann iteration for α-demicontractive mappings, Math. Computer Model. 54 (2011) 2486-2492. https://doi.org/10.1016/j.mcm.2011.06.006.

[14] M.A. Noor, K.I. Noor, T.M. Rassias, Some aspects of variational inequalities, J. Comput. Appl. Math. 47 (1993) 285-312. https://doi.org/10.1016/0377-0427(93)90058-J.

[15] M.O. Osilike, Strong and weak convergence of Ishikawa iteration methods for a class of nonlinear equations, Bull. Korean Math. Soc. 37 (2000) 153-169.

[16] M.O. Osilike, F.O. Isiogugu, Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces, Nonlinear Anal.: Theory Methods Appl. 74 (2011) 1814-1822. https://doi.org/10.1016/j.na.2010.10.054.

[17] M.O. Osilike, A.C. Onah, Strong convergence of the Ishikawa iteration for Lipschitz α-hemicontractive mappings, Ann. West Univ. Timisoara - Math. Computer Sci. 53 (2015) 151-161. https://doi.org/10.1515/awutm-2015-0008.

[18] L. Wang, An iteration method for nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2007 (2007) 28619. https://doi.org/10.1155/2007/28619.

[19] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308. https://doi.org/10.1006/jmaa.1993.1309.

[20] Y. Yao, Y.-C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal.: Theory Methods Appl. 71 (2009) 4997-5002. https://doi.org/10.1016/j.na.2009.03.075.

[21] H. Zhou, Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 343 (2008) 546-556. https://doi.org/10.1016/j.jmaa.2008.01.045.

[22] H. Zhou, Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces, Nonlinear Anal.: Theory Methods Appl. 70 (2009) 3140-3145. https://doi.org/10.1016/j.na.2008.04.017.

[23] A.B. George, Weak and strong convergence of the Ishikawa iterative sequence to fixed points of Lipschitz pseudo-contractive maps in Hilbert spaces, Adv. Fixed Point Theory, 5 (2015) 147-157.

[24] E.I. Veluhan, Weak and strong convergence algorithm for Lipschitz Pseudocontractive Maps in Hilbert Spaces, Unpublished M.Sc. Thesis, Department of Mathematics, University of Nigeria, Nsukka, (2014).

[25] O.O. Owojori, Some convergence results for fixed point of hemicontractive operators in some Banach spaces, Kragujevac J. Math. 31 (2008) 111-129.

[26] C. Morales, J. Jung, Convergence of paths for pseudo-contractive mappings in Banach spaces, Proc. Amer. Math. Soc. 128 (2000) 3411-3419. https://doi.org/10.1090/S0002-9939-00-05573-8.

[27] I. K. Agwu, D. I. Igbokwe, Hybrid-type iteration scheme for approximating fixed point of Lipschitz α-hemicontractive mappings, Adv. Fixed Point Theory, 10 (2020) 3. https://doi.org/10.28919/afpt/4442.